(Non)local minimal surfaces and phase transitions

Serena Dipierro & Enrico Valdinoci

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The Allen-Cahn equation

Critical points of the short-range energy functional give rise to the equation

$$\Delta u(x) = W'(u(x)).$$

The model case reduces to

$$-\Delta u = u - u^3,$$

which is known as the Allen-Cahn equation.

This equation indeed produces the stationary states of an evolution equation describing the phase separation in multi-component alloy systems.

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The Allen-Cahn equation

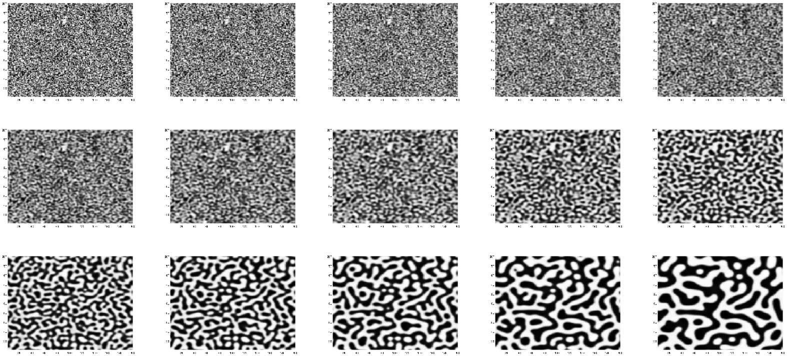
Moreover, solutions of the Allen-Cahn equation are also stationary states of the so-called Cahn-Hilliard equation

$$\partial_t u = \Delta \left(u^3 - u - \Delta u \right),$$

which was introduced to represent the process of spontaneous phase separation in a binary fluid.

The Allen-Cahn equation

The success of the Cahn-Hilliard equation in describing spontaneous phase separation with a tendency of similar phases to cluster together is indeed quite perceptible:



The limit interface

We now come back to the singular perturbation problem for short-range interactions, producing a “rescaled version” of the Allen-Cahn equation of the form

$$\varepsilon^2 \Delta u(x) = W'(u(x)).$$

To appreciate the theory of Γ-convergence which describes the limit as $\varepsilon \searrow 0$, let us consider a functional of the form $\mathcal{F}_\varepsilon$ and let us try to discuss a convenient meaning for a suitable convergence of $\mathcal{F}_\varepsilon$ to some $\mathcal{F}$.

Notice that a pointwise convergence could be out of reach, because a singular perturbation problem may drastically change the structure of the limit functional as well as its natural domain of definition, therefore a different notion of convergence is called for.

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In particular, to make the theory serviceable, it is desirable to keep the notion of local energy minimizers in the limit: namely,

if u_ε is a local minimizer for $\mathcal{F}_\varepsilon$
and $u_\varepsilon \to u$ in some topology X as $\varepsilon \searrow 0$,
this functional notion of convergence should entail
that u is a local minimizer for $\mathcal{F}$.

To this extent, the limit functional $\mathcal{F}$ may be considered as an effective energy and the choice of the topology X can be possibly made "loose enough" to ensure compactness of the minimizers beforehand (choosing a "too strong" topology X produces the pitfall that minimizers may not converge!).

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It is also desirable that

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With these remarks in mind, it is not too difficult to "guess" what an "appropriate" notion of functional convergence should be. To this end, we distinguish between the lower limit and the upper limit.

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For the lower limit we take inspiration from the classical Fatou's Lemma (after all, it is sensible that a good functional convergence turns out to be compatible with the classical scenarios) in which one considers the very special case of u_ε being a sequence of nonnegative measurable functions converging pointwise to u,
takes $\mathcal{F}_\varepsilon(v) := \mathcal{F}(v) := \int_{\mathbb{R}^n} v(x)\,dx$ and writes that

$$\liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} u_\varepsilon(x)\,dx \geq \int_{\mathbb{R}^n} u(x)\,dx = \mathcal{F}(u).$$

Hence, a natural requirement for a general notion of functional convergence is that

$$\text{whenever } u_\varepsilon \to u \text{ in } X, \quad \liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}(u).$$

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Let us now consider an upper limit condition regarding a minimizer $u_\varepsilon^\star$ for $\mathcal{F}_\varepsilon$. Then, for every competitor u_ε for $u_\varepsilon^\star$,

$$\mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathcal{F}_\varepsilon(u_\varepsilon^\star)$$

and to maintain minimizers in the limit, we aim at showing that

$$\mathcal{F}(u) \geq \mathcal{F}(u^\star).$$

For this, if u_ε is any sequence converging to u in X, we know that

$$\limsup_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(u_\varepsilon^\star) \geq \mathcal{F}(u^\star).$$

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Therefore it suffices to find one, possibly very special sequence u_ε converging to u in X for which

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This special sequence making the job is sometimes called recovery sequence.
Thus, a natural upper limit condition consists in

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These two sided limit conditions are often accompanied by a compactness assumption under a bounded energy requirement, such that

$$\text{if } \sup_{\varepsilon \in (0,1)} \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty,$$

then there exists a subsequence $u_{\varepsilon'}$ converging in X as $\varepsilon' \searrow 0$.

When the two sided limit conditions and the compactness conditions are met, then one says that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$.

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The limit interface

One of the chief achievements of the Γ-convergence theory consists precisely in the correct limit assessment of the singular perturbation problem posed by the Allen-Cahn equation:

Theorem (Modica-Mortola 1977)

The functional

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(\frac{\varepsilon\, |\nabla u(x)|^2}{2} + \frac{W(u(x))}{\varepsilon} \right) dx$$

Γ*-converges as* $\varepsilon \searrow 0$ *to*

$$\mathcal{F}(u) := \begin{cases} c\,\mathrm{Per}(E,\Omega) & \text{if } u = \chi_E - \chi_{\mathbb{R}^n \setminus E} \\ & \text{for some set } E \text{ of finite perimeter,} \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$c := \int_{-1}^{1} \sqrt{2W(r)}\, dr.$$

The limit interface

A useful variant of this consists in a “geometric” convergence results for the level sets of the minimizers, stating, roughly speaking, that if u_ε is a minimizer, then its level sets approach locally uniformly the limit interface:

The limit interface

Theorem (Caffarelli-Córdoba 1995)

Assume that u_ε is a local minimizer for the functional $\mathcal{F}_\varepsilon$ in the ball $B_{1+\varepsilon}$. Then:

- *There exists $C > 0$ such that $\mathcal{F}_\varepsilon(u_\varepsilon, B_1) \leq C$.*
- *Up to a subsequence, $u_\varepsilon \to \chi_E - \chi_{\mathbb{R}^n \setminus E}$ as $\varepsilon \searrow 0$ in $L^1(B_1)$ and the set E has locally minimal perimeter in B_1.*
- *Given ϑ_1, $\vartheta_2 \in (-1, 1)$, if $u_\varepsilon(0) > \vartheta_1$, then*

$$\Big|\{u_\varepsilon > \vartheta_2\} \cap B_r\Big| \geq cr^n,$$

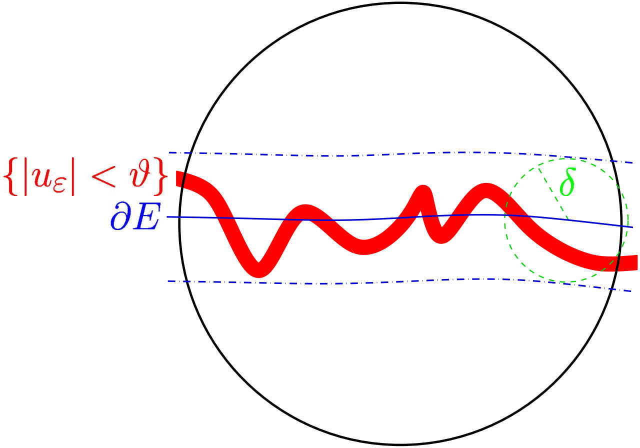
  as long as $r \in (0, 1]$ and $\varepsilon \in (0, c_\star\, r]$.
- *Similarly, given ϑ_1, $\vartheta_2 \in (-1, 1)$, if $u_\varepsilon(0) < \vartheta_1$, then*

$$\Big|\{u_\varepsilon < \vartheta_2\} \cap B_r\Big| \geq cr^n.$$

- *The set $\{|u_\varepsilon| < \vartheta\}$ approaches ∂E locally uniformly as $\varepsilon \searrow 0$: given $r_0 \in (0, 1)$ and $\delta > 0$ there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$,*

$$\{|u_\varepsilon| < \vartheta\} \cap B_{r_0} \subseteq \bigcup_{x \in \partial E} B_\delta(x).$$

The limit interface



The limit interface

An important consequence is that the interface of a phase transition behaves "like a codimension one" set in terms of density estimates: given $\vartheta \in (0,1)$, if $u_\varepsilon(0) \in (-\vartheta, \vartheta)$, then, when $r \in (0,1]$ and $\varepsilon \in (0, c_\star r]$,

$$\Big|\{|u_\varepsilon| < \vartheta\} \cap B_r\Big| \le C\varepsilon r^{n-1}$$

$$\text{and} \qquad \min\Big\{\Big|\{u_\varepsilon > \vartheta\} \cap B_r\Big|,\ \Big|\{u_\varepsilon < -\vartheta\} \cap B_r\Big|\Big\} \ge cr^n.$$

To check these, one deduces from the theorem that $\Big|\{u_\varepsilon > \vartheta\} \cap B_r\Big| \ge cr^n$ and $\Big|\{u_\varepsilon < -\vartheta\} \cap B_r\Big| \ge cr^n$, leading to the second inequality.

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Additionally, setting $\varepsilon' := \frac{\varepsilon}{r}$ and $v_{\varepsilon'}(x) := u_\varepsilon(rx)$, we have that $v_{\varepsilon'}$ is a local minimizer of the functional $\mathcal{F}_{\varepsilon'}$ in the ball $B_{\frac{1+\varepsilon}{r}} \supseteq B_{1+\varepsilon'}$ and so

$$C \geq \mathcal{F}_{\varepsilon'}(v_{\varepsilon'}, B_1) = \int_{B_1} \left(\frac{\varepsilon' \, r^2 \, |\nabla u_\varepsilon(rx)|^2}{2} + \frac{W(u_\varepsilon(rx))}{\varepsilon'} \right) dx$$

$$= \frac{1}{r^{n-1}} \int_{B_r} \left(\frac{\varepsilon \, |\nabla u_\varepsilon(y)|^2}{2} + \frac{W(u_\varepsilon(y))}{\varepsilon} \right) dy.$$

In particular,

$$\begin{aligned} C &\geq \frac{1}{\varepsilon r^{n-1}} \int_{\{|u_\varepsilon|<\vartheta\}\cap B_r} W(u_\varepsilon(y))\, dy \\ &\geq \frac{1}{\varepsilon r^{n-1}} \min_{[-\vartheta,\vartheta]} W \left| \{|u_\varepsilon| < \vartheta\} \cap B_r \right|, \end{aligned}$$

from which we obtain the first inequality.

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It is also worth pointing out that these inequalities are essentially optimal: e.g., given $\vartheta \in (0,1)$, if $u_\varepsilon(0) \in (-\vartheta, \vartheta)$, then, when $r \in (0,1]$ and $\varepsilon \in (0, c_\star\, r]$,

$$\Big|\{|u_\varepsilon| < \vartheta\} \cap B_r\Big| \geq c_o\, \varepsilon r^{n-1}.$$

To check this, we define

$$\widetilde{u}_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } u_\varepsilon(x) \in (-\vartheta, \vartheta), \\ \vartheta & \text{if } u_\varepsilon(x) \in [\vartheta, +\infty), \\ -\vartheta & \text{if } u_\varepsilon(x) \in (-\infty, -\vartheta] \end{cases}$$

and we let μ be the average of $\widetilde{u}_\varepsilon$ in B_r.

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Hence, supposing $\mu \le 0$ (the other cases being similar) then

$$\int_{B_r} |\widetilde{u}_\varepsilon(x) - \mu|\,dx \ge \int_{\{\widetilde{u}_\varepsilon \ge \vartheta\}\cap B_r} (\widetilde{u}_\varepsilon(x) - \mu)\,dx$$

$$\ge \vartheta \left|\{\widetilde{u}_\varepsilon \ge \vartheta\} \cap B_r\right| = \vartheta \left|\{u_\varepsilon \ge \vartheta\} \cap B_r\right| \ge c\vartheta r^n.$$

Thus, by Poincaré Inequality,

$$\int_{B_r} |\nabla \widetilde{u}_\varepsilon(x)|\,dx \ge \frac{c_1}{r} \int_{B_r} |\widetilde{u}_\varepsilon(x) - \mu|\,dx \ge c_2 r^{n-1}.$$

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Furthermore, using the Cauchy-Schwarz inequality, for every $\Lambda > 0$,

$$\begin{aligned}
\int_{B_r} |\nabla \widetilde{u}_\varepsilon(x)|\, dx &= \int_{B_r} |\nabla u_\varepsilon(x)|\, \chi_{\{|u_\varepsilon| \le \vartheta\}}(x)\, dx \\
&\le \frac{1}{2} \int_{B_r} \left(\frac{|\nabla u_\varepsilon(x)|^2}{\Lambda} + \Lambda \chi^2_{\{|u_\varepsilon| \le \vartheta\}}(x) \right) dx \\
&\le \frac{C r^{n-1}}{\varepsilon \Lambda} + \frac{\Lambda}{2} \Big| \{|u_\varepsilon| \le \vartheta\} \cap B_r \Big|
\end{aligned}$$

As a consequence,

$$c_2 r^{n-1} \le \frac{C r^{n-1}}{\varepsilon \Lambda} + \frac{\Lambda}{2} \Big| \{|u_\varepsilon| \le \vartheta\} \cap B_r \Big|.$$

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$$\begin{aligned}\int_{B_r} |\nabla \widetilde{u}_\varepsilon(x)|\,dx &= \int_{B_r} |\nabla u_\varepsilon(x)|\,\chi_{\{|u_\varepsilon|\leq\vartheta\}}(x)\,dx \\ &\leq \frac{1}{2}\int_{B_r}\left(\frac{|\nabla u_\varepsilon(x)|^2}{\Lambda} + \Lambda\chi^2_{\{|u_\varepsilon|\leq\vartheta\}}(x)\right)dx \\ &\leq \frac{Cr^{n-1}}{\varepsilon\Lambda} + \frac{\Lambda}{2}\Big|\{|u_\varepsilon|\leq\vartheta\}\cap B_r\Big|\end{aligned}$$

As a consequence,

$$c_2 r^{n-1} \leq \frac{Cr^{n-1}}{\varepsilon\Lambda} + \frac{\Lambda}{2}\Big|\{|u_\varepsilon|\leq\vartheta\}\cap B_r\Big|.$$

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Therefore, choosing $\Lambda := \frac{2C}{\varepsilon c_2}$,

$$\frac{c_2 r^{n-1}}{2} \leq \frac{C}{\varepsilon c_2} \Big| \{ |u_\varepsilon| \leq \vartheta \} \cap B_r \Big|,$$

as desired.

The limit interface

Another interesting consequence of the previous geometric constructions is a clean ball condition: namely, looking at a ball centered at the interface, one can also find balls of comparable size in either side of the interface (hence the interface is not “spread out” here and there).

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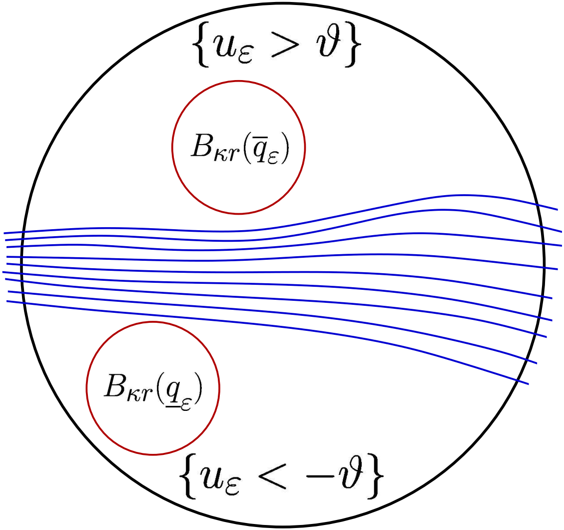

The limit interface

Theorem (Caffarelli-Córdoba 1995)

If $\vartheta \in (0,1)$, $r \in (0,1]$, $\varepsilon \in (0, c_\star r]$ *and* $|u_\varepsilon(0)| < \vartheta$ *then there exist* $\kappa \in (0,1)$, *depending only on* n, W *and* ϑ, *and points* $\underline{q}_\varepsilon$ *and* $\overline{q}_\varepsilon$ *such that*

$$B_{\kappa r}(\underline{q}_\varepsilon) \subseteq \{u_\varepsilon < -\vartheta\} \cap B_r \qquad \text{and} \qquad B_{\kappa r}(\overline{q}_\varepsilon) \subseteq \{u_\varepsilon > \vartheta\} \cap B_r.$$

The limit interface

To prove this, given $\kappa \in \left(0, \frac{1}{100}\right)$, we have that

$$\{u_\varepsilon < \vartheta\} \cap B_{r/20} \subseteq \bigcup_{p \in \{u_\varepsilon < \vartheta\} \cap B_{r/20}} B_{2\kappa r}(p).$$

By the Vitali Covering Lemma, we can extract a family of disjoint balls $\left\{B_{2\kappa r}(p_j)\right\}_{j \in \mathcal{N}}$, for some at most countable set of indexes $\mathcal{N}$, such that

$$\bigcup_{p \in \{u_\varepsilon < \vartheta\} \cap B_{r/20}} B_{2\kappa r}(p) \subseteq \bigcup_{j \in \mathcal{N}} B_{10\kappa r}(p_j).$$

We know that

$$\left|\{u_\varepsilon < \vartheta\} \cap B_{r/20}\right| \geq c r^n$$

up to renaming c, and consequently

$$c r^n \leq \left| \bigcup_{p \in \{u_\varepsilon < \vartheta\} \cap B_{r/20}} B_{2\kappa r}(p) \right| \leq \left| \bigcup_{j \in \mathcal{N}} B_{10\kappa r}(p_j) \right| \leq \sum_{j \in \mathcal{N}} |B_{10\kappa r}(p_j)| = \kappa^n r^n |B_{10}| \# \mathcal{N},$$

yielding that

$$\# \mathcal{N} \geq \frac{\widetilde{c}}{\kappa^n},$$

for some $\widetilde{c} > 0$.

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$$c r^n \leq \left| \bigcup_{p \in \{u_\varepsilon < \vartheta\} \cap B_{r/20}} B_{2\kappa r}(p) \right| \leq \left| \bigcup_{j \in \mathcal{N}} B_{10\kappa r}(p_j) \right| \leq \sum_{j \in \mathcal{N}} |B_{10\kappa r}(p_j)| = \kappa^n r^n |B_{10}| \# \mathcal{N},$$

yielding that

$$\# \mathcal{N} \geq \frac{\widetilde{c}}{\kappa^n},$$

for some $\widetilde{c} > 0$.

The limit interface

To prove this, given $\kappa \in \left(0, \frac{1}{100}\right)$, we have that

$$\{u_\varepsilon < \vartheta\} \cap B_{r/20} \subseteq \bigcup_{p \in \{u_\varepsilon < \vartheta\} \cap B_{r/20}} B_{2\kappa r}(p).$$

By the Vitali Covering Lemma, we can extract a family of disjoint balls $\left\{B_{2\kappa r}(p_j)\right\}_{j \in \mathcal{N}}$, for some at most countable set of indexes $\mathcal{N}$, such that

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Now, let $\widetilde{\mathcal{N}}$ denote the indexes $j \in \mathcal{N}$ for which $B_{\kappa r}(p_j) \cap \{|u_\varepsilon| \leq \vartheta\} \neq \varnothing$. Accordingly, for each $j \in \widetilde{\mathcal{N}}$, let us pick a point $\zeta_j \in B_{\kappa r}(p_j) \cap \{|u_\varepsilon| \leq \vartheta\}$. We stress that if $x \in B_{\kappa r}(\zeta_j)$ then $|x - p_j| \leq |x - \zeta_j| + |\zeta_j - p_j| < 2\kappa r$ and therefore

$$B_{\kappa r}(\zeta_j) \subseteq B_{2\kappa r}(p_j).$$

We also note that

$$\Big|\{|u_\varepsilon| < \vartheta\} \cap B_{\kappa r}(\zeta_j)\Big| \geq c_o\, \varepsilon \kappa^{n-1} r^{n-1}.$$

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Hence, we find that

$$\begin{aligned}
c_o\, \varepsilon \kappa^{n-1} r^{n-1}\, \#\widetilde{\mathcal{N}} &\leq \sum_{j\in\widetilde{\mathcal{N}}} \left|\{|u_\varepsilon| < \vartheta\} \cap B_{\kappa r}(\zeta_j)\right| \\
&\leq \sum_{j\in\widetilde{\mathcal{N}}} \left|\{|u_\varepsilon| < \vartheta\} \cap B_{2\kappa r}(p_j)\right| \\
&= \left|\{|u_\varepsilon| < \vartheta\} \cap \left(\bigcup_{j\in\widetilde{\mathcal{N}}} B_{2\kappa r}(p_j)\right)\right| \\
&\leq |\{|u_\varepsilon| < \vartheta\} \cap B_r|\,.
\end{aligned}$$

Therefore

$$c_o\, \varepsilon \kappa^{n-1} r^{n-1}\, \#\widetilde{\mathcal{N}} \leq C \varepsilon r^{n-1}$$

and, as a consequence,

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The limit interface

Comparing with the above, we deduce that, if κ is conveniently small,

$$\#\left(\mathcal{N} \setminus \widetilde{\mathcal{N}}\right) \geq \frac{\widetilde{c}}{2\kappa^n} > 0.$$

In particular, we can pick $j_\star \in \mathcal{N} \setminus \widetilde{\mathcal{N}}$, yielding that

$$B_{\kappa r}(p_{j_\star}) \cap \{|u_\varepsilon| \leq \vartheta\} = \varnothing.$$

Since $u_\varepsilon(p_{j_\star}) \in \{u_\varepsilon < \vartheta\}$, we conclude that $B_{\kappa r}(p_{j_\star}) \subseteq \{u_\varepsilon \leq -\vartheta\}$, as desired.

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De Giorgi's Conjecture

The link between Bernstein's problem and the limit interfaces of phase transition models (as described by the Γ-convergence theory) was possibly an inspiring motivation for Ennio De Giorgi to state one of his most famous conjectures.

Given that, at a large scale, the level sets of "good" solutions of the Allen-Cahn equation approach perimeter minimizing surfaces and given that minimal graphs reduce to hyperplanes in dimension $n \leq 8$ (according to Bernstein's problem), would it be possible that level sets of "good" global solutions of the Allen-Cahn equation are already hyperplanes?

Since level sets corresponding to different values of the solution cannot intersect, this would say that all the level sets are in fact parallel hyperplanes and therefore the solution only depends on the distance to one of these hyperplanes (in particular, the solution would be a function depending only on one Euclidean variable).

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In all this heuristic discussion, we have been vague about what a "good" solution precisely is: in a sense, besides boundedness and regularity assumptions, a natural hypothesis would be to require that the solution is a local minimizer; furthermore, to fall within the range of application of Bernstein's problem, it would be desirable to know that the limit minimal surface has a graphical structure and for this some monotonicity assumption on the solution could be helpful (since, at least locally, it would entail a graphical structure of the level set via Implicit Function Theorem).

It would be however desirable to keep the number of assumptions to the minimum and possibly to confine them to assumptions of "geometric" type: in this spirit, one may be tempted to remove the minimality assumption (which is instead of "variational" and "energetic" type) and focus mainly on a monotonicity assumption (roughly speaking, after all, maybe monotonicity is already an indication of some "weak" form of minimality since it avoids oscillations that increase energy).

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De Giorgi's Conjecture

A precise notion of this is given by the observation that monotonicity implies stability: namely, if u is a solution of

$$\Delta u = W'(u)$$

such that $\partial_n u > 0$ in some domain $\Omega \subseteq \mathbb{R}^n$, then, for all $\phi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega \Big(|\nabla \phi(x)|^2 + W''(u(x))\, \phi^2(x) \Big)\, dx \geq 0.$$

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Indeed, under the monotonicity assumption it is fair to define $\psi := \frac{\phi^2}{\partial_n u}$ and infer that

$$\int_\Omega \Big(|\nabla\phi(x)|^2 + W''(u(x))\,\phi^2(x)\Big)\,dx = \int_\Omega \left(|\nabla\phi(x)|^2 + \partial_n\Big(W'(u(x))\Big)\,\frac{\phi^2(x)}{\partial_n u(x)}\right)\,dx$$

$$= \int_\Omega \Big(|\nabla\phi(x)|^2 + \partial_n\Big(\Delta u(x)\Big)\,\psi(x)\Big)\,dx$$

$$= \int_\Omega \left(\left|\nabla\Big(\sqrt{\psi(x)}\,\sqrt{\partial_n u(x)}\Big)\right|^2 - \nabla\partial_n u(x)\cdot\nabla\psi(x)\right)dx$$

$$= \int_\Omega \left(\left|\frac{\sqrt{\partial_n u(x)}\,\nabla\psi(x)}{2\sqrt{\psi(x)}} + \frac{\sqrt{\psi(x)}\,\nabla\partial_n u(x)}{2\sqrt{\partial_n u(x)}}\right|^2 - \nabla\partial_n u(x)\cdot\nabla\psi(x)\right)\,dx$$

$$= \int_\Omega \left(\frac{\partial_n u(x)\,|\nabla\psi(x)|^2}{4\psi(x)} + \frac{\psi(x)\,|\nabla\partial_n u(x)|^2}{4\partial_n u(x)} - \frac{1}{2}\nabla\partial_n u(x)\cdot\nabla\psi(x)\right)\,dx$$

$$= \int_\Omega \left|\frac{\sqrt{\partial_n u(x)}\,\nabla\psi(x)}{2\sqrt{\psi(x)}} - \frac{\sqrt{\psi(x)}\,\nabla\partial_n u(x)}{2\sqrt{\partial_n u(x)}}\right|^2\,dx \geq 0,$$

which is the stability condition.

De Giorgi's Conjecture

Conjecture (De Giorgi 1979)

Let $n \leq 8$ and $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a global solution of the Allen-Cahn equation

$$-\Delta u = u - u^3$$

such that

$$\partial_n u(x) > 0 \quad \textit{for every } x \in \mathbb{R}^n.$$

Is it true that u is one-dimensional, i.e. that there exist $u_0 : \mathbb{R} \to \mathbb{R}$ and $\omega \in \partial B_1$ such that $u(x) = u_0(\omega \cdot x)$ for all $x \in \mathbb{R}^n$?

This conjecture has been proven for $n \in \{2, 3\}$ [Ghoussoub-Gui 1998, Berestycki-Caffarelli-Nirenberg 1997, Ambrosio-Cabré 2000, Alberti-Ambrosio-Cabré 2001] and an example of global, bounded and monotone solution of the Allen-Cahn equation which is not one-dimensional has been constructed in dimension $n \geq 9$ [del Pino-Kowalczyk-Wei 2011].

De Giorgi's Conjecture

In dimension $n \in \{4, \ldots, 8\}$ the conjecture is open, but known to hold under an additional assumption on the profiles of the solution at infinity. Namely, since u is bounded and monotone in the direction of e_n, one can define, for all $x' \in \mathbb{R}^{n-1}$,

$$\overline{u}(x') := \lim_{x_n \to +\infty} u(x', x_n) \qquad \text{and} \qquad \underline{u}(x') := \lim_{x_n \to -\infty} u(x', x_n).$$

In this setting, it has been proved [Savin 2009] that the conjecture holds true under the additional assumption

$$\overline{u}(x') = -\underline{u}(x') = 1 \qquad \text{for every } x' \in \mathbb{R}^{n-1}.$$

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