(3−ε)-dimensional TQFTs for derived quantum group representations

w/ A. Czenky $\hbar = \bar{\hbar}$

paper I: Next week? (cochain TQFT)
paper II: who knows (derived TQFT)

Theorem I: [CN] From any (fin.) modular ⊗ cat $\mathcal{A}$, we can construct a 3-dim (anom) TQFT w/ values in Ch(Vect)

Theorem II [N]: From $\mathcal{A}$ as before, and $\mathcal{D} = \mathcal{D}(\mathcal{A})$ its derived ∞ cat, we have a 3-dim (anom) TQFT ... w/ values in $\widetilde{Vect}$
the ∞-cat of dg/homotopical vector spaces.

- Background (math/phys)
- Theorem I
- Theorem II
- Conclusion?

- Main BG

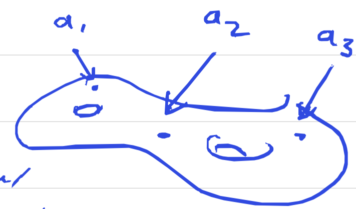
In De Renzi, Gainutdinov, Geer, Patureau-Mirand, Runkel [DGGPR 22] the authors construct a 3-dim (anomalous) TQFT

$$Z_{\mathcal{A}} : \underbrace{\mathrm{Bord}^{adm}_{\mathcal{A}}} \longrightarrow \mathrm{Vect}$$

$\mathrm{Bord}_{3,2}(\mathcal{A})^{adm}$

from any finite modular $\otimes$-cat $\mathcal{A}$.

objects are marked surfaces



morphisms are 3-dim bordisms w/ embedded, $\mathcal{A}$-labeled, ribbon graphs

admissible = same projectivity constraint in dim 3.

We have $Z_{\mathcal{A}}(\Sigma_{\vec{a}}) = \mathrm{Hom}_{\mathcal{A}}(1, E^{\otimes g} \otimes a)$

and $Z_{\mathcal{A}}(M)$ = "renormalized" Lyubashenko inv.

Why marked bordisms? CFT, belongs to 4d.
What's the point? Nonsemisimple WRT.

- Physics suggestion.

Phys Creutzig-Dimofte-Garner-Geer [CDGG24]
suggest that there should be a type of "derived"
TQFT (*)

Z_{Der} $\mathrm{Bord}^{nc}_{\mathcal{D}} \longrightarrow \mathrm{Vect}$ valued in co-ct of Lten vect. spaces

for $\mathcal{D} = \mathcal{D}(\mathcal{A})$ [specifically $\mathcal{A} = \ldots$]

"w/ flat connection"

We expect here

$$Z_{\mathcal{D}}(T^2) = HH^*(\mathcal{A}) = \mathrm{Maps}_{\mathcal{A}}(\mathbb{1}, E)$$

$$Z_{\mathcal{D}}(S^2) = H^*(\mathcal{A}, \mathbb{1}) = \mathrm{Maps}_{\mathcal{A}}(\mathbb{1}, \mathbb{1}).$$

Career for same twisting ... in 4-d (3-d?)

Mid Term Goal —

Conj: For finite mod's $\mathcal{A}$ w/ automorphism group $H = \mathrm{Aut}^{\otimes}(\mathcal{A})$ there exist 3-d Theory (*) for $\mathcal{D}(\mathcal{A})$ w/ H-local systems.

We want to approach [Conj] via DGGPR a starting point.

- A cochain valued theory [Creutzig · N]

Take $\mathrm{Bord}^{adm}_{Ch} = \Big\{$ $Ch(\mathcal{A})$-marked surfaces and (admissible) ribbon bordisms.

Cochain DGGPR

Thm [C-N]: There is a symm. mon. functor

$$Z^*_{\mathcal{A}} : \mathrm{Bord}^{adm}_{Ch} \to \underbrace{Ch(\mathrm{Vect})}_{}$$

symmetry has signs

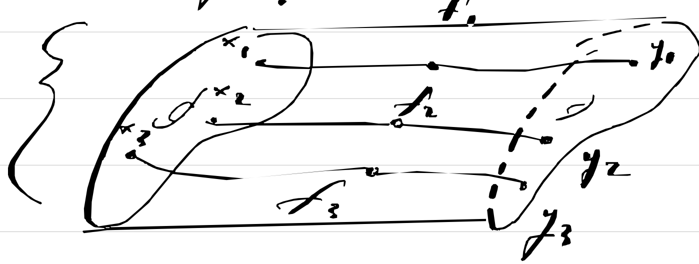
w/ • $Z^*_{\mathcal{A}}(\Sigma^{\bullet}_{x}) \cong \mathrm{Hom}^*_{\mathcal{A}}(1, E^{\otimes g} \otimes x)$

$$\to \mathrm{Hom}^{-1}_{\mathcal{A}}(1, 4) \xrightarrow{d_{Hom}} \mathrm{Hom}^{0}_{\mathcal{A}}(1, E^{g} \otimes x) \to \mathrm{Hom}^{1}_{\mathcal{A}}(1, E \ldots$$

• $Z^*_{\mathcal{A}}(M) = \sum'_{n \in \mathbb{Z}} \pm Z_{\mathcal{A}}(M_n)$.

$$\begin{tikzcd} \text{Abelian theory} \arrow[dr] & & \boxed{??} \arrow[dr] & \\ & {} \arrow[ur, phantom] & & \text{Derived theory} \\ & \text{Cochain theory} \arrow[uu] & & \end{tikzcd}$$

In Bord_{ch}^{adm} we have an obj class of homotopy equiv.



w/ all f_i homotopy equiv in $Ch(\mathcal{A})$.

Lets call these alg. homotopy equivalences, and take

$W_{alg} \subseteq \mathrm{Bord}_{ch}^{adm}$ the class of all alg htop equiv.

Take $W_{K} \subseteq Ch(\mathrm{Vect})$ the coll of all (cochn) htop. equiv.

Fact: $Ch(\mathcal{A})[\mathrm{Htop}^{-1}] = K(\mathcal{A})$ as a cat

$D(\mathcal{A}) = K(\mathcal{A})[\mathrm{Qiso}^{-1}] \hookleftarrow K(\mathcal{A})$.

Thm [C-NJ]: The cochain thy $Z_{\mathcal{A}}^*$ preserves homotopy equiv.

$$Z_{\mathcal{A}}^*(W_{alg}) \subseteq W_K.$$

- To ∞ categories

By second Thm [G-N] we see that $Z^*_{\mathcal{A}}$ localizes to produce a symm. monoidal fun

$$Z^*_{\mathcal{A}}[W] : \mathrm{Bord}^{adm}_{Ch}[W^{-1}] \to Ch(Vect)[W]$$

between symm monoidal ∞-cats. The localization $Ch(Vect)[W] = Vect$ already.

The ∞-cat $\mathrm{Bord}^{adm}_{Ch}[W^{-1}]$ is harder to calculate, but up to some approximation it's an ∞-cat of ribbon bordisms w/ labels in the homotopy ∞-cat $K(\mathcal{A})$.

Long story short, this is the bridge which connects our cochain theory to this proposed derived theory

$$\text{Cochain Theory} \xrightarrow{\text{localize}} \text{Derived Theory}$$

Ultimately

Conj: The derived theory $Z_{\mathcal{D}}$ extends down to dim 1, and has

$$Z_{\mathcal{D}}(S^1) = \mathcal{D}(\mathcal{A}).$$

Construction of Z^*? What can $Z_{\mathcal{D}}$ say about $\mathcal{A}$? Introducing local systems?

- Flat Conn.

Alg the presence of flat conn. should be implied by the existence of a canonical H-crossed extension for $\mathcal{A}$ equipped w/ its algebraic H-action

$$H \curvearrowright \mathcal{A}.$$

Our main example of interest here is

$$H = G^{\mathrm{ham}} \curvearrowright (\mathrm{Rep}_q G)_{\mathrm{small}} = \mathcal{A}.$$

Conj [Etingof]: For a connected group H acting (faithfully?) on a modular $\otimes$-cat $\mathcal{A}$, there exists a universal H-crossed extension

$$\mathcal{A} \longrightarrow \mathcal{A}_H.$$

[i.e. all obstructions vanish]

the correct ext in the quantum group case should be "something like" the De Concini-Kac alg.

Now, when we talk about introducing flat-conn we're proposing that each state space

$$\begin{tikzcd} Z_D(\Sigma) \arrow[r] \arrow[d] & \tilde{Z}_D(\Sigma) \arrow[d] \\ * \arrow[r, "1"] & \mathrm{LocSys}_H(\Sigma) \end{tikzcd}$$

should deform to a sheaf over H-local systems on Σ, and the 3-mfld invariants should extend to functions

$$Z_D(M) : \mathrm{LocSys}_H(M) \to \mathbb{C}.$$

This kind of structure is predicted both physically [Cr-Di-Ga-Geer, Gaiotto, Kapustin-Witten], and it's full-dimen analog should be producible (in an illegible form) a la P. Kinnear.