Stability of the Boltzmann equation

Take renormalization $\beta_\delta(z) = \frac{z}{1+\delta z}$ (concave), $\beta_\delta'(z) = \frac{1}{(1+\delta z)^2}$ (convex)

$$\left.\begin{aligned} &\beta_\delta(f^n) \overset{*}{\rightharpoonup} \beta_\delta \leq \beta_\delta(f) \\ &\beta_\delta'(f^n) \overset{*}{\rightharpoonup} h_\delta \geq \beta_\delta'(f) \\ &\beta_\delta'(f^n) f^n = \frac{f^n}{(1+\delta f^n)^2} = \beta_\delta(f^n)(1-\delta\beta_\delta(f^n)) \overset{*}{\rightharpoonup} g_\delta \leq \beta_\delta(1-\delta(\beta_\delta) \end{aligned}\right] \text{in weak}^* \text{ top. of } L^\infty_{loc}(dt\,dx\,dv)$$

$$(\partial_t + v.\nabla_x)\beta_\delta(f^n) = \beta_\delta'(f^n) C(f^n) \xrightarrow{\mathcal{D}'} (\partial_t + v.\nabla_x)\beta_\delta = h_\delta C^+(f) - g_\delta f * \left[\int b(\cdot,\sigma)d\sigma\right]$$

One can show that $\beta_\delta, g_\delta \to f$ as $\delta \to 0$ in $L^1_{loc}(dt\,dx\,dv)$
and $h_\delta \to 1$ as $\delta \to 0$ in $L^p_{loc}(dt\,dx\,dv)$, $\forall 1 \leq p < \infty$.

Take an admissible renormalization $\beta(z)$

$$\Rightarrow (\partial_t + v.\nabla_x)\beta(\beta_\delta) = \beta'(\beta_\delta) h_\delta C^+(f) - \beta'(\beta_\delta) g_\delta f * \left[\int b(\cdot,\sigma)d\sigma\right]$$

Let $\delta \to 0 \Rightarrow (\partial_t + v.\nabla_x)\beta(f) = \beta'(f) C^+(f) - \beta'(f) f \; f * \left[\int b(\cdot,\sigma)d\sigma\right]$

Deriving Macroscopic Equations from the Boltzmann Equation

Nondimensional form of the Boltzmann equation

$$St\, \partial_t f + v \cdot \nabla_x f = \frac{1}{Kn} C(f)$$

Kn = Knudsen number ~ mean free path
St = Strouhal number ~ time scaling
Hydrodynamic regimes happen when $Kn \to 0$. Think of it as a continuum limit.
Further consider a global normalized equilibrium state

$$M(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{v^2}{2}}$$

and its fluctuations $f = M(1 + Ma\, g)$.
Ma = Mach number ~ ratio of bulk velocity to speed of sound.

Compressible Euler regime

$$(\partial_t + v \cdot \nabla_x) f_\varepsilon = \frac{1}{\varepsilon} C(f_\varepsilon).$$

$$\varepsilon \to 0 \Rightarrow f_\varepsilon \to f \text{ and } C(f) = 0 \Rightarrow f = \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}$$

Conservation laws of mass, momentum and energy become the compressible Euler system in the limit:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla(\rho\theta) = 0 \\ \partial_t\left(\frac{1}{2}\rho u^2 + \frac{d}{2}\rho\theta\right) + \operatorname{div}\left(\left(\frac{1}{2}\rho u^2 + \frac{d+2}{2}\rho\theta\right)u\right) = 0 \end{cases}$$

Note: von Kármán relation (physics principle) $Kn \sim \frac{Ma}{Re}$ (Re is the Reynolds number) tells us that $Re \to \infty$ as $Kn \to 0$ if $Ma \sim 1$.

$\Rightarrow$ compressible Navier-Stokes cannot be derived with such hydrodynamic limits.

Acoustic waves regime

$$(\partial_t + v.\nabla_x) f_\varepsilon = \frac{1}{\varepsilon} C(f_\varepsilon) \ , \ f_\varepsilon = M(1+\varepsilon^m g_\varepsilon) \ , m>0.$$

$$g_\varepsilon \longrightarrow g = \rho + v \cdot u + \left(\frac{v^2}{2} - \frac{d}{2}\right)\theta$$ ($\in$ kernel of linearized Boltzmann operator)

$$\begin{cases} \partial_t \rho + \operatorname{div} u = 0 \\ \partial_t u + \nabla(\rho+\theta) = 0 \\ \frac{d}{2}\partial_t(\rho+\theta) + \frac{d+2}{2}\operatorname{div} u = 0 \Rightarrow \frac{d}{2}\partial_t \theta + \operatorname{div} u = 0 \end{cases}$$

Incompressible regimes (Bardos, Golse, Levermore, 1991)

$$(\varepsilon \partial_t + v.\nabla_x) f_\varepsilon = \frac{1}{\varepsilon^q} C(f_\varepsilon) \ , \ f_\varepsilon = M(1+\varepsilon^m g_\varepsilon)$$

- $q=1, m=1$, incompressible Navier-Stokes-Fourier system (details later)
- $q>1, m=1$, incompressible Euler-Fourier system
- $q=1, m>1$, incompressible Stokes-Fourier system.

equation for fluctuations:

$$(\varepsilon\partial_t + v.\nabla_x) g_\varepsilon = \frac{1}{\varepsilon^q} \underbrace{\int (g_\varepsilon' + g_{\varepsilon *}' - g_\varepsilon - g_{\varepsilon *}) M_* \, b \, dv_* d\sigma}_{\overset{\shortparallel}{-\mathcal{L}(g_\varepsilon)}} + \frac{1}{\varepsilon^{q-m}} \underbrace{\int (g_\varepsilon' g_{\varepsilon *}' - g_\varepsilon g_{\varepsilon *}) M_* b \, dv_* d\sigma}_{\overset{\shortparallel}{Q(g_\varepsilon)}}$$

$$\overset{\shortparallel}{\frac{1}{\varepsilon^{q+m} M} C(f_\varepsilon)}$$

Suppose $g_\varepsilon \longrightarrow g$.

infinitesimal Maxwellian

Order $\frac{1}{\varepsilon^q}$: $\mathcal{L}(g)=0 \Rightarrow g'+g'_*=g+g_* \Rightarrow \boxed{g=\rho+v\cdot u+\left(\frac{v^2}{2}-\frac{d}{2}\right)\theta}$

$\varepsilon\to 0$ in moment equations: $\int (v\cdot\nabla_x g)\begin{pmatrix}1\\ v\\ v^2\end{pmatrix} M\,dv=0$

$\Rightarrow \boxed{\operatorname{div} u=0}$ and $\boxed{\nabla(\rho+\theta)=0}$

- Evolution equations. Difficulty: fluxes are singular.

Note: $\mathcal{L}: L^2(M\,dv)\longrightarrow L^2(M\,dv)$ (if $b\equiv 1$, otherwise need to use weights in velocity)

$\mathcal{L}=\mathrm{Id}-(\text{compact})$ if $b\equiv 1$

is self-adjoint and Fredholm of index zero on $L^2(M\,dv)$.

$\Rightarrow$ $\operatorname{Ran}\mathcal{L}$ is exactly the orthogonal complement of $\operatorname{Ker}\mathcal{L}$.

Moreover $\operatorname{Ker}\mathcal{L}=\operatorname{span}\{1,v_1,\dots,v_d,|v|^2\}$.

Introduce the hydrodynamic projection Πg_ε of g_ε onto $\operatorname{Ker}\mathcal{L}$:

$$\Pi g_\varepsilon=\rho_\varepsilon+u_\varepsilon\cdot v+\theta_\varepsilon\left(\frac{v^2}{2}-\frac{d}{2}\right)$$

where $\rho_\varepsilon=\int_{\mathbb{R}^d} g_\varepsilon M\,dv$, $u_\varepsilon=\int_{\mathbb{R}^d} g_\varepsilon v M\,dv$, $\theta_\varepsilon=\int_{\mathbb{R}^d} g_\varepsilon\left(\frac{v^2}{d}-1\right)M\,dv$

$$\Rightarrow \begin{cases} dv & \partial_t\rho_\varepsilon+\frac{1}{\varepsilon}\operatorname{div}u_\varepsilon=0\\ v\,dv & \partial_t u_\varepsilon+\frac{1}{\varepsilon}\nabla(\rho_\varepsilon+\theta_\varepsilon)=-\frac{1}{\varepsilon}\operatorname{div}\int_{\mathbb{R}^d} g_\varepsilon\Phi\, M\,dv\\ \frac{v^2}{2}dv & \frac{d}{2}\partial_t(\rho_\varepsilon+\theta_\varepsilon)+\frac{d+2}{2\varepsilon}\operatorname{div}u_\varepsilon=-\frac{1}{\varepsilon}\operatorname{div}\int_{\mathbb{R}^d} g_\varepsilon\Psi\, M\,dv. \end{cases}$$

where $\phi(v) = v \otimes v - \frac{|v|^2}{d} \mathrm{Id}$ and $\Psi(v) = \left(\frac{v^2}{2} - \frac{d+2}{2}\right) v$.

(Note that $\varepsilon \to 0 \Rightarrow \frac{d}{dt} \int_{\mathbb{T}^d} (\rho + \theta)\, dx = 0 \Rightarrow \boxed{\rho + \theta = 0}$ if true initially.)

Observe that $\phi, \Psi \in \operatorname{Ran} \mathcal{L}$ (because $\perp \operatorname{Ker} \mathcal{L}$).

$\Rightarrow \exists\, \tilde{\Phi}, \tilde{\Psi} \in L^2(M\,dv)$ such that $\mathcal{L}\tilde{\Phi} = \phi$ and $\mathcal{L}\tilde{\Psi} = \Psi$.

$$\Rightarrow \begin{cases} \partial_t u_\varepsilon + \frac{1}{\varepsilon} \nabla(\rho_\varepsilon + \theta_\varepsilon) = -\frac{1}{\varepsilon} \operatorname{div} \int_{\mathbb{R}^d} (\mathcal{L} g_\varepsilon) \tilde{\Phi}\, M\,dv \\ \partial_t \left(\frac{d}{2}\theta_\varepsilon - \rho_\varepsilon\right) = -\frac{1}{\varepsilon} \operatorname{div} \int_{\mathbb{R}^d} (\mathcal{L} g_\varepsilon) \tilde{\Psi}\, M\,dv \end{cases}$$

Use equation for g_ε to write $\frac{1}{\varepsilon} \mathcal{L} g_\varepsilon = \varepsilon^{m-1} Q(g_\varepsilon) - \varepsilon^{q-1}(\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon)$

From now on, suppose $m=1$, $q=1$ and $d=3$.

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathcal{L}(g_\varepsilon) = Q(g) - v \cdot \nabla_x g = \frac{1}{2} \mathcal{L}(g^2) - \operatorname{div}(vg)$$

$$\underset{\substack{\text{careful} \\ \text{computation}}}{=} \frac{1}{2} u^t \mathcal{L}(\phi) u + \theta u \cdot \mathcal{L}(\Psi) + \frac{1}{2}\theta^2 \mathcal{L}\left(\frac{|v|^4}{4}\right) - \operatorname{div}\left((\rho + \theta) v + \frac{v^2}{3} u + \phi u + \theta \Psi\right)$$

$$= \frac{1}{2} u^t \mathcal{L}(\phi) u + \theta u \cdot \mathcal{L}(\Psi) + \frac{1}{2}\theta^2 \mathcal{L}\left(\frac{|v|^4}{4}\right) - \operatorname{div}(\phi u + \theta \Psi)$$

It can be shown that $\tilde{\Phi}(v) = \alpha(|v|)\phi(v)$ and $\tilde{\Psi}(v) = \beta(|v|)\Psi(v)$.

(symmetry argument)

$$\Rightarrow \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^3} (\mathcal{L} g_\varepsilon) \tilde{\Phi}\, M\,dv = u \otimes u - \frac{|u|^2}{3} \mathrm{Id} - \mu(\nabla_x u + \nabla_x^t u)$$

and $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^3} (\mathcal{L} g_\varepsilon) \tilde{\Psi}\, M\,dv = \frac{5}{2}\theta u - \frac{5}{2} \kappa \nabla_x \theta$$

with $\mu = \frac{1}{10} \int_{\mathbb{R}^3} \phi : \tilde{\Phi}\, M\,dv$ and $\kappa = \frac{2}{15} \int_{\mathbb{R}^3} \Psi \cdot \tilde{\Psi}\, M\,dv$.

Let $P : L^2(dx) \to L^2(dx)$ denote the Leray projector onto divergence free vector fields, i.e., $P = \frac{\mathrm{rot}\,\mathrm{rot}}{-\Delta} = \mathrm{Id} + \frac{\nabla \mathrm{div}}{-\Delta}$.

$$\Rightarrow \begin{cases} \partial_t u + P(u \cdot \nabla_x u) - \mu \Delta_x u = 0 & , \ \mathrm{div}\, u = 0 \\ \partial_t \theta + u \cdot \nabla_x \theta - k \Delta_x \theta = 0 & , \ \rho + \theta = 0 \end{cases}$$

The incompressible Navier-Stokes-Fourier system.

<u>Theorem</u>: (Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond, A.; 1991 - 2012)

Consider: • A physically relevant collision kernel which derives from an inverse power potential. For example: $b(v-v_*, \sigma) = |v - v_*|^\gamma b_0(\cos\theta)$, $-3 < \gamma \leq 1$

$$b_0(\cos\theta) \sim \frac{1}{\theta^{2+\nu}}, \quad 0 < \nu < 2.$$

• Renormalized solutions $f_\varepsilon = M(1 + \varepsilon g_\varepsilon) \in L^\infty_t L^1_{x,v}$ of $(\varepsilon \partial_t + v \cdot \nabla_x) g_\varepsilon = -\frac{1}{\varepsilon} \mathcal{L}(g_\varepsilon) + Q(g_\varepsilon)$ with well-prepared initial data.

Then, as $\varepsilon \to 0$:

- g_ε is weakly relatively compact in $L^1_{loc}(dt; L^1((1+v^2)M\,dx\,dv))$
- each limit point g of g_ε satisfies $g = v \cdot u + \left(\frac{v^2}{2} - \frac{d+2}{2}\right)\theta$ where $(u, \theta) \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ is a Leray solution of the incompressible Navier-Stokes-Fourier system.

Mathematical difficulties

1) Lack of a priori estimates! We only have the entropy inequality.

2) Renormalized solutions: $g_\varepsilon \in L^1_{loc}$ (at most $L\log L$) such that

$$\left(\varepsilon\partial_t + v\cdot\nabla_x\right)\frac{g_\varepsilon}{2+\varepsilon g_\varepsilon} = \frac{-2}{\varepsilon(2+\varepsilon g_\varepsilon)^2}\mathcal{L}(g_\varepsilon) + \frac{2}{(2+\varepsilon g_\varepsilon)^2}Q(g_\varepsilon)$$

3) Macroscopic conservation laws are not known to hold for renormalized solutions.

4) Time compactness: there are oscillations! (Use compensated compactness to filter out oscillations.)

5) Space compactness: there is some compactness thanks to velocity averaging lemmas. Not enough! Moreover, we need the nonlinear weak compactness estimate:

$$\frac{g_\varepsilon^2}{2+\varepsilon g_\varepsilon} \text{ is weakly compact in } L^1_{loc}.$$

(control concentrations)