Renormalized solutions:

Conservation laws & H-thm give an a priori bound on the solutions:

$$f(t,x,v)(1+|v|^2),\ f\log(2+f) \in L^\infty_t L^1_{x,v}.$$

$$\Leftrightarrow\ f \in L^\infty(dt; L^1((1+v^2)\,dx\,dv)) \cap L^\infty(dt; L\log L\,(dx\,dv)).$$

Problem: $C(f)$ is quadratic. In x-variable, we need L^2-bound.

Idea: renormalization $\beta : \mathbb{R}^+ \to \mathbb{R}^+$, $|\beta'(z)| \leq \dfrac{C}{1+z}$, $|\beta(z)| \leq C\log(2+z)$.

$$(\partial_t + v\cdot\nabla_x)\beta(f) = \underbrace{\beta'(f)\,C(f)}_{\text{sublinear}} \qquad \text{(in the sense of distributions)}$$

Loss term estimate: $|\beta'(f)C^-(f)| = |\beta'(f) f \int f_* b\, dv_*\, d\sigma| \in L^1_{loc}(dt\,dx\,dv)$

Gain term estimate: $\displaystyle\int f'f'_* \, b \, dv_* d\sigma = \int_{\{f'f'_* \geq K ff_*\}} + \int_{\{f'f'_* < K ff_*\}}$ $\quad K>1$

$$\leq \frac{1}{\log K} \underbrace{\int_{\{f'f'_* \geq K ff_*\}} (f'f'_* - ff_*) \log\left(\frac{f'f'_*}{ff_*}\right) b}_{\text{entropy dissipation}} + K \underbrace{\int f f_* \, b \, dv_* d\sigma}_{\text{loss term}}$$

DiPerna-Lions Thm: (1989, 1991)

Suppose $b(z,\sigma) \in L^1_{loc}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and $\displaystyle\lim_{|v|\to\infty} \frac{1}{|v|^2} \int_{K \times \mathbb{S}^{d-1}} b(v-v_*,\sigma)\, dv_* d\sigma = 0$ $\quad \forall K \subset \mathbb{R}^d$ compact.

Take $f_0 \in L^1((1+v^2)\, \underbrace{dx}_{x\in\mathbb{T}^d} dv) \cap L\log L(dx dv)$, $f_0 \geq 0$.

Then there exists a renormalized solution $f \in L^\infty(dt; L^1((1+v^2) dx dv)) \cap L^\infty(dt; L\log L(dx dv))$ satisfying the entropy inequality.

Moreover,
$$\begin{cases} \partial_t \int f \, dv + \operatorname{div} \int f v \, dv = 0 \\ \frac{d}{dt} \iint f v \, dx dv = 0 \\ \iint f |v|^2 dx dv \leq \iint f_0 |v|^2 dx dv, \ \forall t > 0. \end{cases}$$

Key steps of proof: - Approximate equation (truncating collision integral).

- Weak compactness of approximating sequence (Dunford-Pettis criterion).
- Strong compactness in x (averaging lemmas).
- Strong compactness properties of gain term $C^+(f)$.
- Stability of Boltzmann equation

Approximate equation:

$$\begin{cases} (\partial_t + v\cdot\nabla_x) f^n = \displaystyle\int_{\mathbb{R}^d\times \mathbb{S}^{d-1}} (f^{n\prime} f_*^{n\prime} - f^n f_*^n) \dfrac{b^n \, dv_* \, d\sigma}{1 + \frac{1}{n}\int_{\mathbb{R}^d} f^n \, dv} \\ \\ f^n(t=0) = f_0^n \text{ (smooth truncation)} \end{cases}$$

smooth, compactly supported away from $\left\{ \left| \frac{v-v_*}{|v-v_*|}\cdot\sigma \right| = 1 \right\}$

sublinear

Can be solved in Schwartz's space $C_t^\infty S_{x,v}$.

Weak compactness

Recall the Dunford–Pettis weak compact criterion.

A family $\{g_n\} \subset L^1(\mathbb{R}^d, dx)$ is weakly relatively compact if and only if it is 1) bounded in L^1.

2) equi-integrable, i.e., $\forall \varepsilon > 0, \exists \delta > 0$, such that $|A| < \delta \Rightarrow \sup_n \int_A |g_n| \, dx < \varepsilon$.

3) tight, i.e., $\forall \varepsilon > 0, \exists R > 0$, such that
$$\sup_n \int_{\{|x|>R\}} |g_n| \, dx < \varepsilon.$$

In the context of the Boltzmann equation:

1) conservation of mass provides uniform boundedness in L^1.
2) entropy bound gives equi-integrability.
3) conservation of energy yields tightness.

Up to extraction $f^n \xrightarrow[\text{weak}]{} f$ in $L^1_{loc}(dt; L^1(dx\,dv))$.

Strong compactness, averaging lemmas

Averaging lemma in L^2: Suppose $f, g \in L^2(\mathbb{R}^d)$ and $v \cdot \nabla_x f(x,v) = g(x,v)$

$$\Rightarrow \quad i v \cdot \xi \hat{f}(\xi, v) = \hat{g}(\xi, v)$$

$$\Rightarrow \left| \int_{B(0,R)} \hat{f}(\xi, v)\, dv \right| \le \int_{B(0,R) \cap \{|v\cdot\xi| \le 1\}} |\hat{f}|\, dv + \int_{B(0,R)\cap\{|v\cdot\xi|>1\}} \frac{|\hat{g}|}{|v\cdot\xi|}\, dv$$

$$\le \|\hat{f}\|_{L^2_v} \left|\{|v\cdot\xi| \le 1\}\right|^{1/2} + \|\hat{g}\|_{L^2_v} \left(\int_{B(0,R)\cap\{|v\cdot\xi|>1\}} \frac{1}{|v\cdot\xi|^2}\, dv \right)^{1/2}$$

$$\lesssim |\xi|^{-1/2} \left(\|\hat{f}\|_{L^2_v} + \|\hat{g}\|_{L^2_v} \right).$$

$$\Rightarrow \int_{B(0,R)} f(x,v)\, dv \in H^{1/2}(\mathbb{R}^d).$$

In the context of Boltzmann eq.: $\forall \varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f^n(t,x,v)\varphi(v)\, dv \xrightarrow[\text{strong}]{} \int_{\mathbb{R}^d} f(t,x,v)\varphi(v)\, dv$$

At this stage, one can show, with some work, that

$$\frac{C^\pm(f^n)\ \varphi(t,x)}{1+\delta \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f^n(v_*) b(v-v_*,\sigma)\, dv_* d\sigma} \xrightarrow[\text{weak}]{} \frac{C^\pm(f)\ \varphi(t,x)}{1+\delta \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f(v_*) b(v-v_*,\sigma)\, dv_* d\sigma}$$

in $L^1(dt\,dx\,dv)$, $\forall \delta > 0$, $\forall \varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$.

Strong compactness of $C^+(f)$ (Lions, 1994; Bouchut, Desvillettes, 1998)

$$A(\xi)=\int_{\mathbb{R}^d} e^{-i\xi\cdot v}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} F(v',v'_*)\,\beta\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\,dv_*\,d\sigma\,dv=\int e^{-i\xi\cdot\left(\frac{v+v_*}{2}+\frac{|v-v_*|\sigma}{2}\right)}F(v,v_*)\,\beta\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\,dv\,dv_*\,d\sigma$$

$$=\int e^{-i\,\xi\cdot\frac{v+v_*}{2}-i\,\frac{|\xi|}{2}(v-v_*)\cdot\sigma}F(v,v_*)\,\beta\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\,dv\,dv_*\,d\sigma$$

$\uparrow$ symmetry in $\sigma\in\mathbb{S}^{d-1}$ to exchange $\frac{v-v_*}{|v-v_*|}$ and $\frac{\xi}{|\xi|}$

$$=\int \hat{F}\Big(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma,\ \frac{\xi}{2}-\frac{|\xi|}{2}\sigma\Big)\,\beta\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\,d\sigma$$

if $F(v,v_*)=f(v)g(v_*)$

$$\Rightarrow |A(\xi)|\le\Big(\int\Big|\hat{f}\Big(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\Big)\Big|^2 d\sigma\Big)^{1/2}\|g\|_{L^1}\ \underbrace{\Big(\int\Big|\beta\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big|^2 d\sigma\Big)^{1/2}}_{<\infty}$$

for suitable β.
with support inside $(-1,1)$

SIDEBAR

change of variable

$$\eta=\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\ ;\quad \xi=2\eta-\frac{|\eta|^2}{\eta\cdot\sigma}\sigma$$

$$|\eta|^2=\frac{|\xi|^2}{2}+\frac{|\xi|\,\xi\cdot\sigma}{2}\qquad \eta\cdot\sigma=\frac{\xi\cdot\sigma}{2}+\frac{|\xi|}{2}\ge 0$$

$$\Rightarrow |\eta|^2=\frac{|\xi|^{\cancel{2}}}{\cancel{2}}+|\xi|\Big(\eta\cdot\sigma-\frac{|\xi|}{\cancel{2}}\Big)=|\xi|(\eta\cdot\sigma)\Rightarrow|\xi|=\frac{|\eta|^2}{\eta\cdot\sigma}\Rightarrow\xi=2\eta-\frac{|\eta|^2}{\eta\cdot\sigma}\sigma$$

$$2\Big(\frac{\eta}{|\eta|}\cdot\sigma\Big)^2=\frac{\xi}{|\xi|}\cdot\sigma+1$$

$$\Rightarrow \frac{\xi}{2}-\frac{|\xi|}{2}\sigma=\eta-\frac{|\eta|^2}{2(\eta\cdot\sigma)}\sigma-\frac{|\eta|^2}{2(\eta\cdot\sigma)}\sigma$$

$$\Big|\eta-\frac{|\xi|}{2}\sigma\Big|^2=\cancel{\eta^2}+\frac{\xi^2}{4}-\cancel{|\xi|\eta\cdot\sigma}=\frac{\xi^2}{4}\Rightarrow\Big|\eta-\frac{|\xi|}{2}\sigma\Big|=\frac{|\xi|}{2}$$

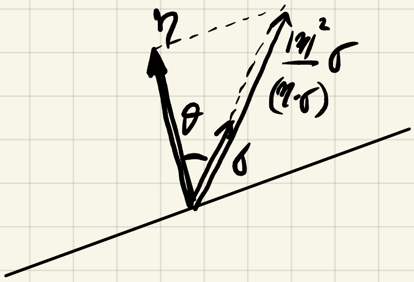
$$\int_{\mathbb{R}^d}h\Big(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\Big)d\xi=\int_0^\infty dr\,r^{d-1}\int_{\mathbb{S}^{d-1}}d\mu\,h\Big(\frac{r}{2}(\mu+\sigma)\Big)=\int_0^\infty dr\,r^{d-1}\int_0^\pi d\theta\,\sin^{d-2}\theta\int_{\mathbb{S}^{d-2}_{\perp\sigma}}d\tilde{\mu}\,h$$

$$=\int_0^\infty dr\,\underset{r=|\xi|}{r^{d-1}}\int_0^{\frac{\pi}{2}}d\theta\,2^{d-1}\sin^{d-2}\theta\cos^{d-2}\theta\int_{\mathbb{S}^{d-2}_{\perp\sigma}}d\tilde{\mu}\,h=\int_0^\infty dr\,\underset{r=|\eta|}{r^{d-1}}\int_0^{\frac{\pi}{2}}d\theta\,\frac{2\sin^{d-2}\theta}{\cos^2\theta}\int_{\mathbb{S}^{d-2}_{\perp\sigma}}d\tilde{\mu}\,h$$

$$=\int_{\mathbb{R}^d\cap\{\eta\cdot\sigma\ge 0\}}h(\eta)\,\frac{2^{d-1}}{\Big(\frac{\eta}{|\eta|}\cdot\sigma\Big)^2}\,d\eta$$

$\leftarrow$ use support of β to control sing.

$$\Rightarrow \int |A(\xi)|^2 \, d\xi \lesssim \iiint_{\{\eta\cdot\sigma>0\}} \left|\hat F\left(\eta, \eta - \frac{|\eta|^2}{(\eta\cdot\sigma)}\sigma\right)\right|^2 d\eta \, d\sigma$$



$$= \int_{\mathbb{R}^d} d\eta \int_0^{\frac{\pi}{2}} d\theta \sin^{d-2}\theta \int_{S^{d-2}\perp\eta} d\tilde\sigma \left|\hat F(\eta, -|\eta|\tan\theta\,\tilde\sigma)\right|^2$$

$$= \int_{\mathbb{R}^d} d\eta \int_0^\infty dr \left(\frac{r^2}{1+r^2}\right)^{\frac{d-2}{2}} \frac{1}{1+r^2} \int_{S^{d-2}\perp\eta} d\tilde\sigma \left|\hat F(\eta, |\eta| r\tilde\sigma)\right|^2$$

gives regularization by $\frac{d-1}{2}$ derivatives $\sim \frac{1}{|\xi|^{d-1}}$

$$= \int_{\mathbb{R}^d} d\eta \int_{\mathbb{R}^{d-1}\perp\eta} dz \, \frac{1}{(1+|z|^2)^{d/2}} \left|\hat F(\eta, |\eta| z)\right|^2 \le \int_{\mathbb{R}^d} d\eta \int_{\mathbb{R}^{d-1}\perp\eta} dz \, \frac{1}{|\eta|^{d-1}} \left|\hat F(\eta, z)\right|^2$$

$$= \int_{\mathbb{R}^d} d\eta \int_{\mathbb{R}^d} dz \, \frac{1}{|\eta|^{d-1}} \frac{\eta}{|\eta|}\cdot\nabla_z \left(|\hat F(\eta, z)|^2\right)$$

produces weight in velocity. Proceed as in Bouchut–Desvillettes.

All in all, one can show an estimate

$$\|C^+(f)\|_{\dot H^{\frac{d-1}{2}}} \lesssim \|f(1+|v|^2)^{\alpha/2}\|^2_{L^2}$$ for a suitable kernel b and weight $(1+|v|^2)^{\alpha/2}$, $\alpha \ge 1$.

As for the construction of renormalized solutions, the compactness of gain term can be improved to

$$\frac{C^+(f^n)\,\varphi(t,x)}{1+\delta \int_{\mathbb{R}^d\times S^{d-1}} f^n(v_*)\, b(v-v_*,\sigma)\, dv_*\, d\sigma} \xrightarrow[\text{strong}]{} \frac{C^+(f)\,\varphi(t,x)}{1+\delta \int_{\mathbb{R}^d\times S^{d-1}} f(v_*)\, b(v-v_*,\sigma)\, dv_*\, d\sigma}$$

in $L^1(dt\,dx\,dv)$ and almost everywhere, $\forall \delta>0$, $\forall \varphi \in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d)$.