Hydrodynamic Limits

IST, CAMGSD
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1) [Cercignani, Illner, Pulvirenti] The mathematical theory of dilute gases. (Springer, 1994)

2) [Saint-Raymond] Hydrodynamic limits of the Boltzmann equation. (Springer, 2009)

3) [A., Saint-Raymond] From the Vlasov-Maxwell-Boltzmann system to incompressible viscous electro-magneto-hydrodynamics, Vol. 1. (EMS, 2019)

Foundations of the Boltzmann equation

<u>Maxwellian velocity distribution</u> (Maxwell's original argument) (~1860)

Stationary velocity distribution: $M(v) \geq 0$, $M : \mathbb{R}^3 \to \mathbb{R}$

Independence of velocity components: $M(v) = f(v_1) f(v_2) f(v_3)$

same function (isotropy of space)

Spherical symmetry: $M(v) = \phi(|v|^2)$

$$\Rightarrow \quad \phi(|v|^2) = f(v_1) f(v_2) f(v_3)$$

$$\frac{\partial}{\partial v_1} \rightsquigarrow \quad 2v_1 \, \phi'(|v|^2) = f'(v_1) f(v_2) f(v_3)$$

$$\Rightarrow 2v_1 \frac{\phi'(|v|^2)}{\phi(|v|^2)} = \frac{f'(v_1)}{f(v_1)} \Rightarrow \frac{\phi'(|v|^2)}{\phi(|v|^2)} \text{ and } \frac{f'(v_1)}{2v_1 f(v_1)} \text{ must be constant}$$

$$\Rightarrow \quad \phi(r) = A e^{-\alpha r}, \quad A \geq 0, \ \alpha > 0.$$

$$M(v) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v|^2}{2\theta}}, \quad \begin{array}{l} \rho > 0 \text{ is the density} \\ \theta > 0 \text{ is the temperature.} \end{array}$$

<u>Non-equilibrium statistical mechanics</u>

Particle number density $f(t,x,v) > 0$, $t \in \mathbb{R}^+$ (time), $x \in \Omega$ (space), $v \in \mathbb{R}^d$ (velocity), $d \geq 2$.

$\Omega = \mathbb{T}^d$ (or $\mathbb{R}^d$)

In the absence of collisions or external field: $\partial_t f + v \cdot \nabla_x f = 0$

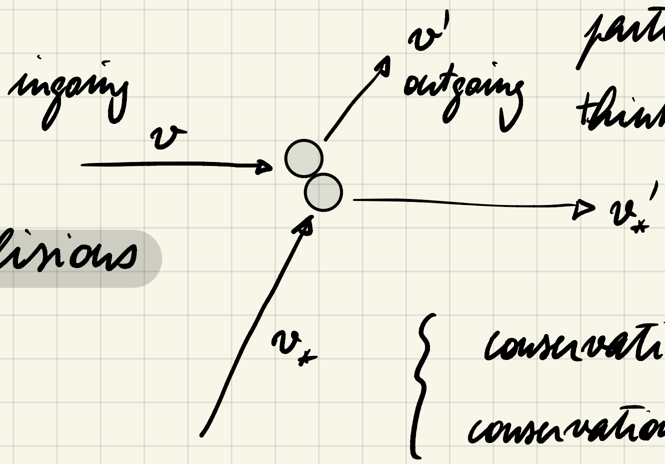
(free transport eq.)

$$\Rightarrow \quad f(t,x,v) = f_0(x-tv, v).$$

With external field: $\quad \partial_t f + v.\nabla_x f + F(t,x).\nabla_v f = 0$ (Vlasov eq.)

With collisions: $\quad \partial_t f + v.\nabla_x f = C(f)$

<u>Geometry of collisions (hard spheres):</u>



particles have same mass

think of particles as point-particles

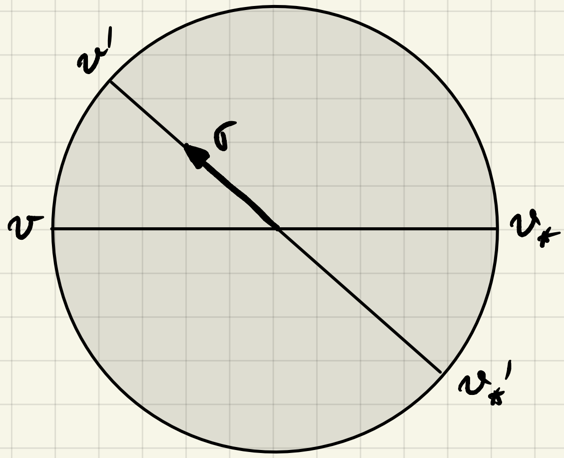
elastic collisions

localization in time and space

$$\begin{cases} \text{conservation of momentum } v+v_* = v'+v_*' \\ \text{conservation of energy } |v|^2+|v_*|^2 = |v'|^2+|v_*'|^2 \end{cases}$$

$$\Rightarrow \begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma \\ v_*' = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma \end{cases}$$

$\sigma \in \mathbb{S}^{d-1}$



<u>Collision invariants:</u>

Suppose $\phi : \mathbb{R}^d \to \mathbb{R}$ is measurable.

One can show:

$$\phi(v) + \phi(v_*) = \phi(v') + \phi(v_*') \iff \phi(v) = A + B\cdot v + C|v|^2$$

$$A, C \in \mathbb{R}, \; B \in \mathbb{R}^d$$

$$\iff \phi \in \operatorname{span}\{1, v_1, \dots, v_d, |v|^2\}.$$

The Boltzmann collision operator: $\quad C(f) = C^+(f) - C^-(f)$

loss term $$C^-(f) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} F(v,v_*)\, b\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\underbrace{\frac{v'-v_*'}{|v'-v_*'|}}_{=\sigma}\right) d\sigma\, dv_*$$

ingoing velocities

2-particle number density

binary collisions

statistical weight $b \geq 0$

$$f(v) = \int F(v,v_*)\,dv_*$$

molecular chaos $$C^-(f) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)f(v_*)\, b\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma\, dv_*$$

microreversibility of collisions $\quad F(v,v_*) = F(v',v_*')$

post-collisional $\quad$ pre-collisional

Gain term

$$C^+(f) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} F(v',v_*')\, b\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma\, dv_*$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v')f(v_*')\, b\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) d\sigma\, dv_*$$

The Boltzmann equation:

$$\boxed{(\partial_t + v\cdot\nabla_x) f(t,x,v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (f'f_*' - f f_*)\, b(v-v_*,\sigma)\, dv_*\, d\sigma}$$

$b(v-v_*,\sigma) = b\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)$ is the cross-section or collision kernel.

Initial data: $f(0,x,v) = f_0(x,v)$ $\quad$ Notation: $f_* = f(v_*)$, $f' = f(v')$, $f_*' = f(v_*')$

Collision symmetries:

$$(v, v_*, \sigma) \longmapsto \left(v', v_*', \frac{v-v_*}{|v-v_*|}\right)$$

$$(v, v_*, \sigma) \longmapsto (v_*, v, -\sigma)$$

are volume-preserving involutions on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$.

$$\Rightarrow \int_{\mathbb{R}^d} C(f)(v)\,\phi(v)\,dv = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \frac{1}{4}\left(f'f_*' - ff_*\right)\left(\phi + \phi_* - \phi' - \phi_*'\right) b\,dv\,dv_*\,d\sigma$$

$$\Rightarrow \boxed{\int_{\mathbb{R}^d} C(f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0}$$

Macroscopic conservation laws

$$\begin{cases} \partial_t \int f\,dv + \operatorname{div} \int f\,v\,dv = 0 & \text{(mass)} \\ \partial_t \int f v\,dv + \operatorname{div} \int f\,v\otimes v\,dv = 0 & \text{(momentum)} \\ \partial_t \int f\,\frac{v^2}{2}\,dv + \operatorname{div} \int f\,\frac{v^2}{2}\,v\,dv = 0 & \text{(energy)} \end{cases}$$

macroscopic quantities: $\rho = \int f\,dv$ (density), $\rho u = \int f\,v\,dv$ (bulk velocity),

$$\rho\theta = \int f\,\frac{|v-u|^2}{d}\,dv \quad \text{(temperature)}$$

$$\mathbb{P} = \int f\,(v-u)\otimes(v-u)\,dv \quad \text{(stress tensor)}$$

$$q = \int f\,\frac{1}{2}|v-u|^2(v-u)\,dv \quad \text{(heat flux vector)}$$

$$\Rightarrow \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho\, u \otimes u + P) = 0 \\ \partial_t\left(\frac{1}{2}\rho u^2 + \frac{d}{2}\rho\theta\right) + \operatorname{div}\left(\left(\frac{1}{2}\rho u^2 + \frac{d}{2}\rho\theta\right)u + Pu + q\right) = 0 \end{cases}$$

Equilibrium states

Note that $\displaystyle\int_{\mathbb{R}^d} C(f)(v)\log f(v)\,dv = \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \overbrace{(f'f'_* - ff_*)\log\left(\frac{ff_*}{f'f'_*}\right)}^{\leq 0} b\,dv\,dv_*\,d\sigma$

$C(f) = 0 \iff ff_* = f'f'_* \iff \log f$ is a collision invariant

$\iff f$ is a Maxwellian $\boxed{M(v) = \dfrac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}}$

If $f = M$ is a Maxwellian distribution, then $P = \rho\theta\,\mathrm{Id}$ and $q = 0$.

Conservation laws become: (it's the ideal gas law, $PV = nRT$)

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho\, u \otimes u) + \nabla(\rho\theta) = 0 \\ \partial_t\left(\frac{1}{2}\rho u^2 + \frac{d}{2}\rho\theta\right) + \operatorname{div}\left(\left(\frac{1}{2}\rho u^2 + \frac{d+2}{2}\rho\theta\right)u\right) = 0 \end{cases} \quad \left(\begin{array}{l}\text{compressible Euler} \\ \text{equations}\end{array}\right)$$

The H-theorem:

$$\frac{d}{dt}\underbrace{\int f\log f\,dx\,dv}_{= H(f)(t)} + \frac{1}{4}\int \underbrace{(ff_* - f'f'_*)\log\left(\frac{ff_*}{f'f'_*}\right)}_{\geq 0} b\,dx\,dv\,dv_*\,d\sigma = 0$$

entropy dissipation $\mathcal{D}(f)$

$\Rightarrow$ The entropy $H(f)(t)$ decays monotonically in time.

In general, one cannot hope for more than an entropy inequality:

$$H(f)(t) + \int_0^t D(f)(s)\,ds \leq H(f_0), \quad \forall\, t \geq 0.$$

$D(f) \in L^1(\mathbb{R}^+)$ suggests that $f f_* - f' f_*' \to 0$ as $t \to \infty$.

$\Rightarrow$ $f \to$ Maxwellian state as $t \to \infty$.

Remark: The H-theorem establishes the irreversibility of the Boltzmann equation.