The symplectic geometry of the restricted three-body problem

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Lisbon

Practical aspects and applications

Astrodynamics and space missions

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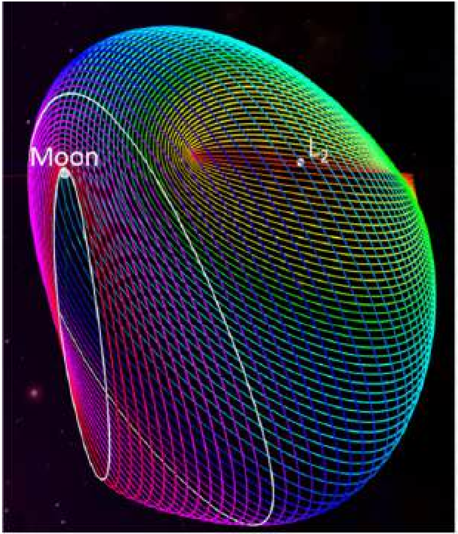
- They come in families (varying the energy) connected through each other through *bifurcations*, codified in a graph of “options” for a putative mission.
- There is an infinitude of periodic orbits, many of which are used for real missions.
- The basic families in the CR3BP can be used to construct nominal orbits which are much more complex, using high-fidelity (*ephemeris*) models deforming the CR3BP.

Halo orbits



Halo orbits for Earth–Moon.

Current context: systems of interest

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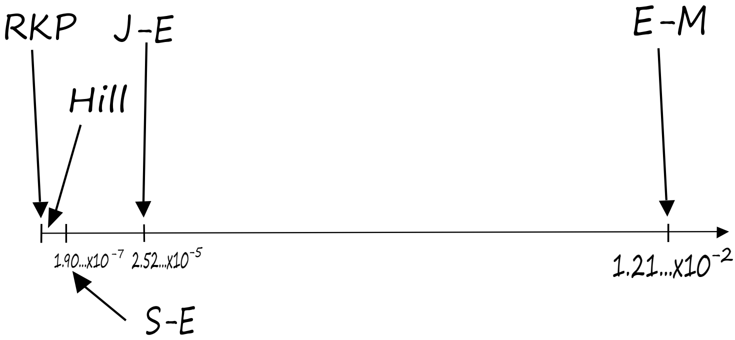
In the context of deep space exploration, NASA's ARTEMIS program (e.g. Gateway, halo orbits) is interested in:

- Earth–Moon.

Systems of interest



Values of μ for different systems of interest.

Symplectic Data Analysis

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Symplectic Data Analysis (SDA).

What follows is based on work with:

- Urs Frauenfelder (Augbsurg)
- Dan Scheeres (aerospace engineer, CO Boulder)
- Otto van Koert (Seoul)
- Cengiz Aydin (Heidelberg)
- Bhanu Kumar (Michigan)
- Dayung Koh (JPL-NASA: Gateway, Europa Clipper, Psyche)

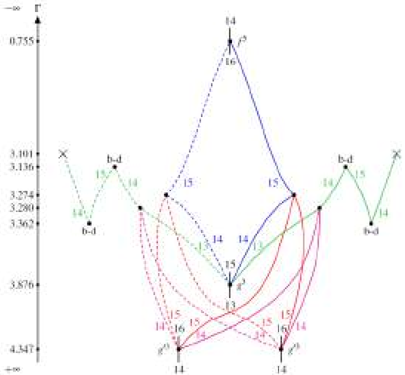
The symplectic toolkit

With Urs Frauenfelder and Cengiz Aydin, we developed a "**symplectic toolkit**" for engineers (that made it to *Quanta Magazine*):

(1) **The B-signs**: A $\pm$ sign, helps predict/obstruct orbits and bifurcations.
(2) **Global topological methods:** for bifurcations and stability.
(3) **Conley-Zehnder index**: organizes families.
(4) **Floer numbers**: A test to detect potentially overlooked families.

Bifurcation graphs

The symplectic toolkit helps organize and find orbits in a systematic way.



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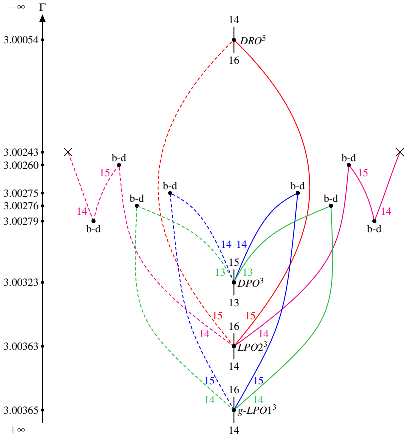
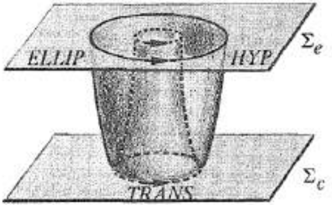


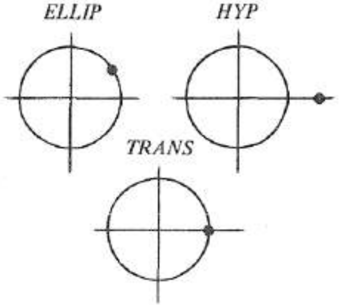
Jupiter-Europa.

Bifurcations

Monodromy matrix: symplectic matrix (linearization of the flow along an orbit).

Bifurcation: e-value 1 is crossed in a family.

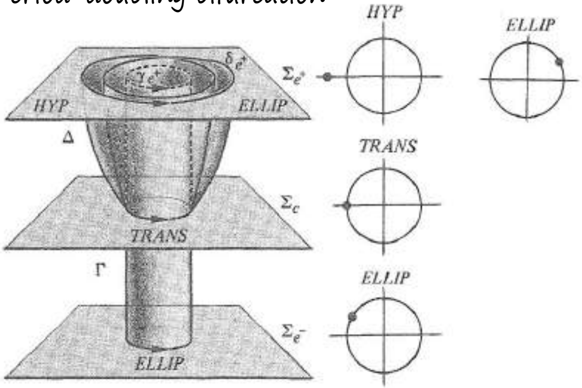




Creation or birth/death.

Bifurcations

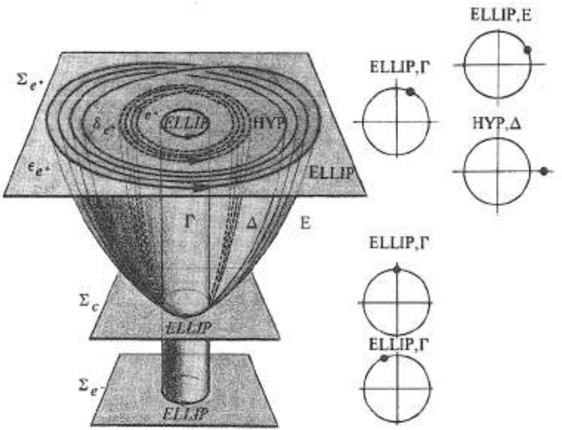
Period-doubling bifurcation



"Foundations of Mechanics", Abraham-Marsden.

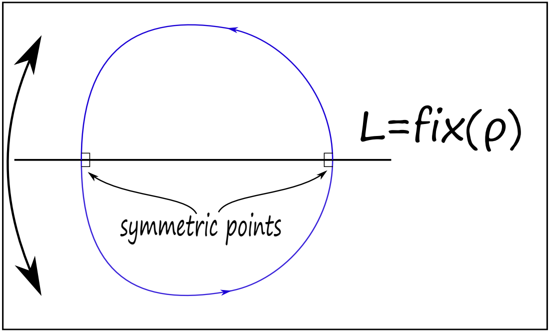
Period-doubling bifurcation or subtle division.

Bifurcations



Emission, or k-fold bifurcation ($k = 4$).

Symmetries



Symmetric periodic orbits.

Wonenburger matrices

The monodromy matrix of a symmetric orbit at a **symmetric point** is a **Wonenburger** matrix:

$$M = M_{A,B,C} = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \in Sp(2n), \qquad (1)$$

where

$$B = B^T, \quad C = C^T, \quad AB = BA^T, \quad A^T C = CA, \quad A^2 - BC = \mathbb{1}. \qquad (2)$$

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The eigenvalues of M are determined by those of the first block A.

Lemma

- *λ e-val of M $\rightsquigarrow$ its **stability index** $a(\lambda) = \frac{1}{2}(\lambda + 1/\lambda)$ e-val of A.*
- *a e-val of A $\rightsquigarrow$ $\lambda(a) = a + \sqrt{a^2 - 1}$ e-val of M.*

Toolkit

Global topological methods

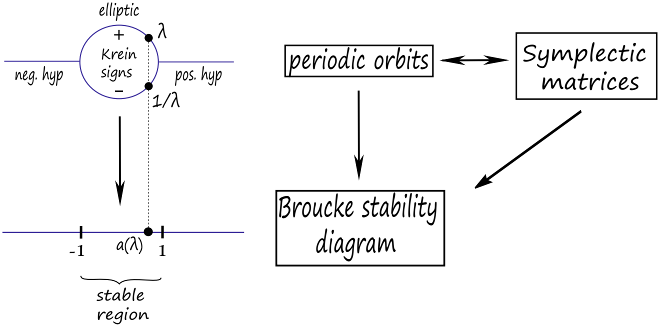
These methods encode:

- Bifurcations;
- stability;
- eigenvalue configurations;
- obstructions to existence of regular families;
- *B-signs*,

in a visual and resource-efficient way.

Broucke's stability diagram: 2D

Let $\boxed{n=2}$, λ eigenvalue of monodromy, with **stability index** $a(\lambda) = \frac{1}{2}(\lambda + 1/\lambda)$.



If we know that two points lie in different components, then one should expect bifurcations in any path between them.

Broucke's stability diagrams: 3D

Let $\boxed{n=3}$. Given $M_{A,B,C}$, its **stability point** is $p = (\mathrm{tr}(A), \det(A)) \in \mathbb{R}^2$.

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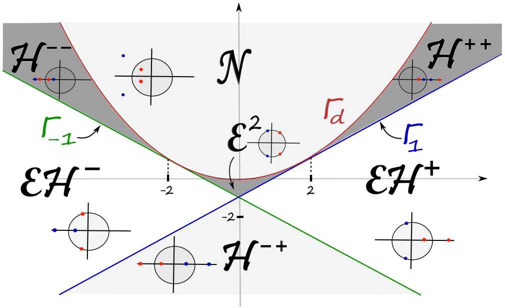

- $\Gamma_{\pm 1}$ = e-val ± 1.
- Γ_d = double e-val.
- $\mathcal{E}^2$ = doubly elliptic (**stable region**).
- etc.

Bifurcations in the Broucke diagram

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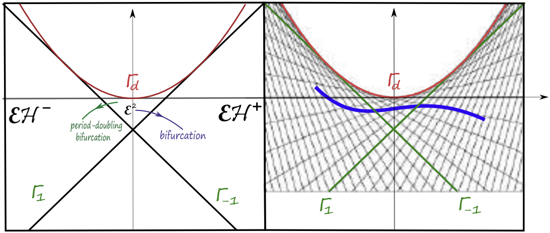

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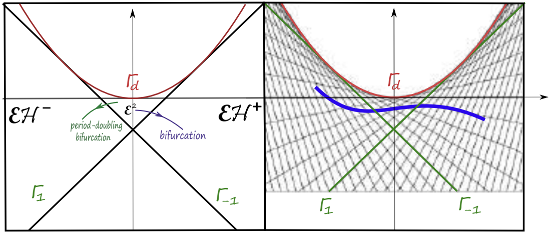

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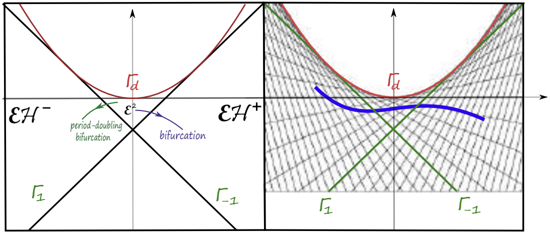

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If we know that two points lie in different components, then one should expect bifurcations in any path between them.

B-signs

Let x be a symmetric orbit with monodromy

$$M = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}.$$

Assume λ simple, elliptic or hyperbolic e-value. Let v satisfy $A^T v = a(\lambda) \cdot v$. The **B-sign** of λ is

$$\epsilon(\lambda) = \text{sign}(v^T B v) = \pm.$$

- $\boxed{n=2}$ two B-signs, one for each symmetric point.
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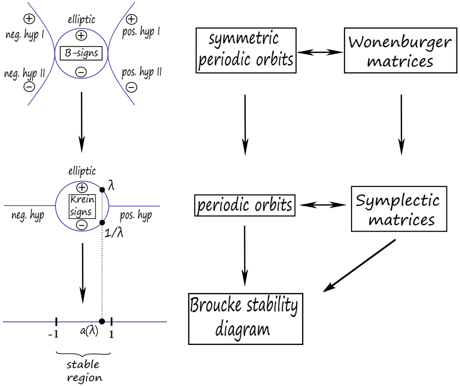
Theorem (Frauenfelder–M. ’23)

A planar symmetric orbit is negative hyperbolic iff the B-signs of its two symmetric points differ.

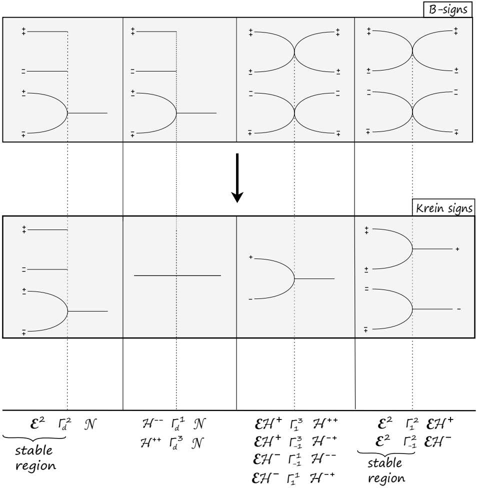
Global topological methods: GIT sequence, 2D

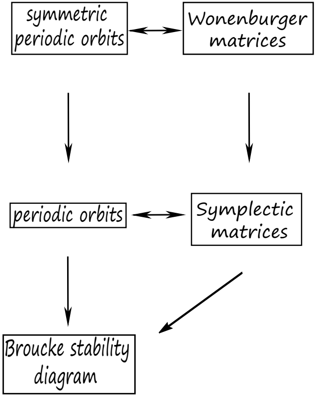
GIT sequence = refinement of Broucke diagram for **symmetric** orbits.



- *B*-signs "separate" hyperbolic branches, for symmetric orbits.
- If two points lie in the same component, but *B*-signs differ, one should *also* expect bifurcation in any path joining them.

Global topological methods: GIT sequence, 3D





The branches are two-dimensional, and come together where we cross from one region to another. If the *B*-signs are not mixed, then a stable orbit cannot cross from $\mathcal{E}^2$ to $\mathcal{N}$ (Krein–Moser theorem).

Conley–Zehnder index

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- CZ-index stays constant if no bifurcation occurs;
- Then helps understand which families of orbits connect to which.
- It can be thought of as an intersection number with the *Maslov cycle*, consisting of symplectic matrices with 1 as an eigenvalue.

Conley–Zehnder index

n=2 x planar orbit with (reduced) monodromy M_x, x^k k-fold cover.

- **Elliptic case:**

$$M_x \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

Then

$$\mu_{CZ}(x^k) = 1 + 2 \cdot \lfloor k \cdot \theta/2\pi \rfloor$$

In particular, it is odd, and jumps by $\pm$ 2 if the e-val 1 is crossed.

- **Hyperbolic case:**

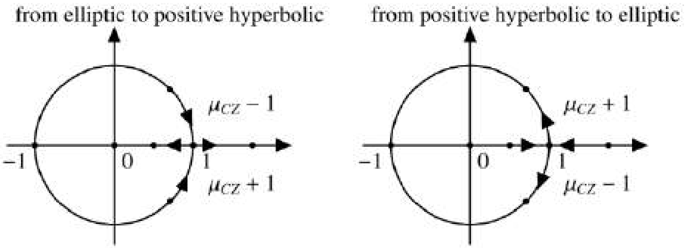
$$M_x \sim \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix},$$

Then

$$\mu_{CZ}(x^k) = k \cdot n,$$

where linearized flow rotates eigenspaces by angle $\frac{\pi n t}{T}$, with n even/odd if x pos./neg. hyp.

CZ-jumps



μ_{CZ} jumps by ± 1 when crossing 1, according to direction of bifurcation. If it stays elliptic, the jump is by ± 2.

Conley–Zehnder index

$\boxed{n=3}$, $x \subset \mathbb{R}^2$ planar orbit in 3BP, then

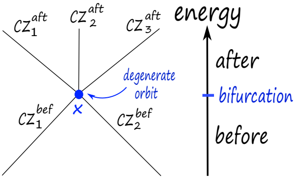
$$\mu_{CZ}(x) = \mu^p_{CZ}(x) + \mu^s_{CZ}(x),$$

a spatial and a planar index.

- Planar to planar bifurcations correspond to jumps in μ^p_{CZ}.
- Planar to spatial bifurcations correspond to jumps of μ^s_{CZ}.

The jump is determined by the *B*-sign. The CZ-index may be computed directly (numerically or via above formulas), or via deformation of known cases.

Floer numerical invariants



Given a bifurcation at x, the *Floer number* of x is

$$\chi(x) = \sum_i (-1)^{CZ_i^{bef}} = \sum_j (-1)^{CZ_j^{aft}}.$$

The sum on the LHS is over (**good**) orbits *before* bifurcation, and RHS is over (**good**) orbits *after* bifurcation.

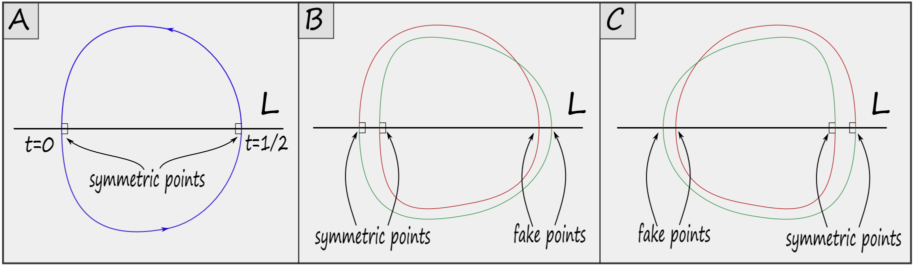
Invariance

The fact that the sums agree before and after –*invariance*– follows from very deep results in **Floer theory** from symplectic geometry.

In Memoriam Andreas Floer, 1956-1991.

The Floer number can be used as a **test**: if the sums do *not* agree, we know the algorithm missed an orbit.

Example: symmetric period doubling bifurcation



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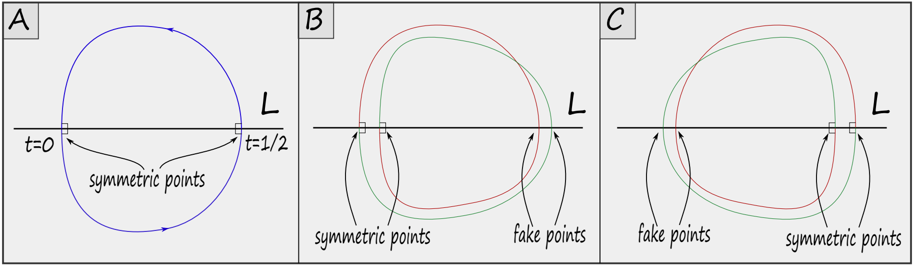

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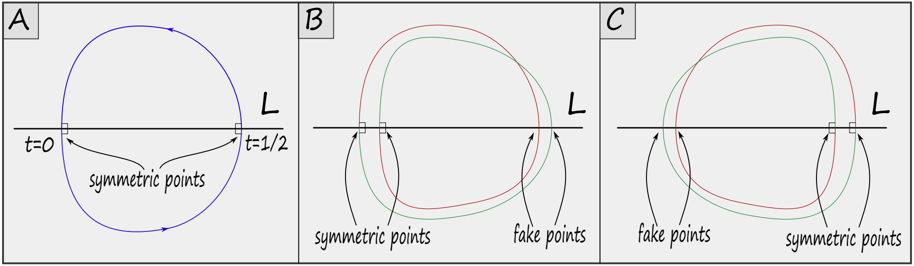

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- A priori there could be two bifurcations for each symmetric point (B or C).
- Invariance of $\chi(x^2)$ implies only one can happen (note x^2 is *bad*).
- Bifurcation happens at the symmetric point in which the B-sign does *not* jump.

Summary of toolkit

(1) **The B-signs**: a $\pm$ sign associated to elliptic/hyperbolic orbits, which helps predict bifurcations.

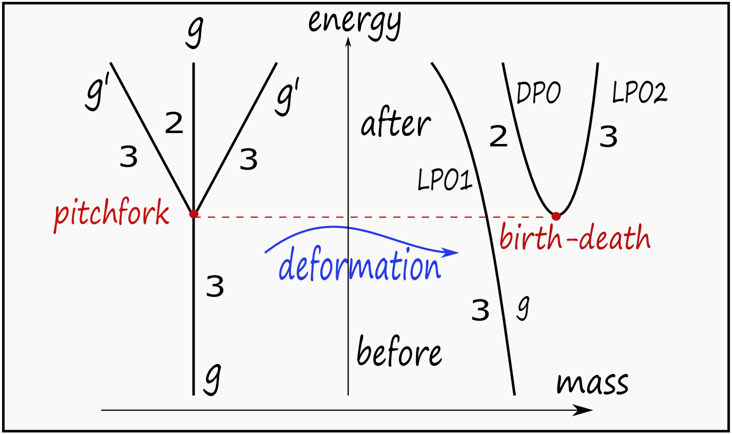
(2) **Global topological methods:** the *GIT-sequence*, a topological refinement of *Broucke's stability diagram*, which encodes bifurcations and stability of orbits.

(3) **Conley-Zehnder indices:** an integer associated to a (non-degenerate) orbit which only jumps at bifurcation, and so predicts which families connect to which.

(4) **Floer numerical invariants:** numerical counts of orbits that stay the same before and after a bifurcation, and so help predict existence of orbits.

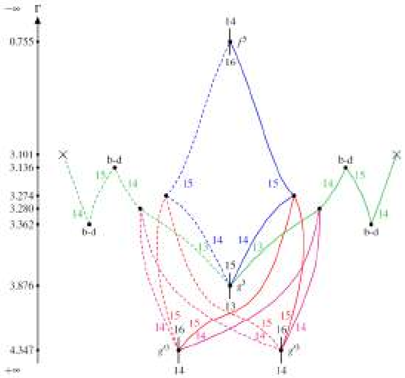
Numerical work

Example: pitchfork bifurcation



Hill's problem has plenty of symmetry: a (nno-generic) pitchfork bifurcation (Hénon) deforms to a generic bifurcation in Jupiter–Europa. Birth-death branch is hard to detect. The Floer number is easy to compute.

Bifurcation graphs



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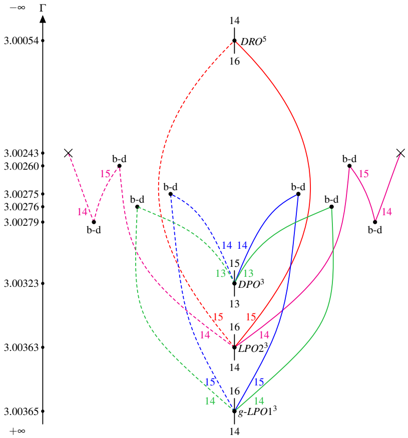
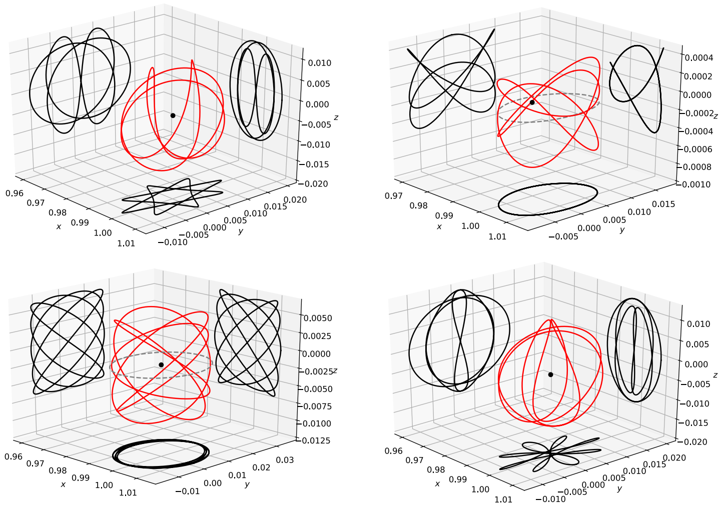
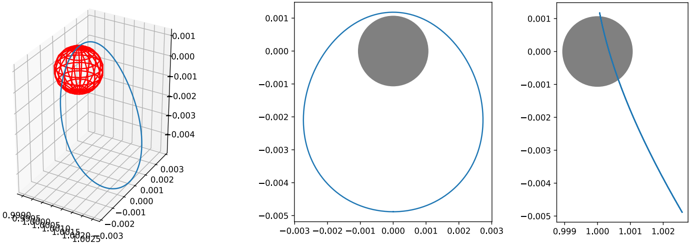


Jupiter-Europa.

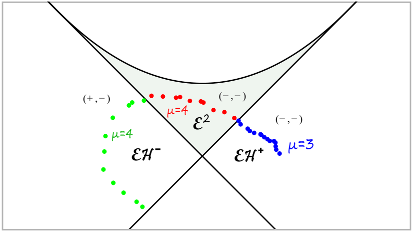


Jupiter-Europa: Spatial connection between retrograde planar orbit to direct planar orbit, CZ-index 15.

Halo orbits in Saturn-Enceladus



Halo orbit, altitude = 29km, CZ = 4 (Enceladus Orbilander, NASA).



GIT plot of Halo orbits with CZ-index.

Remarks

- Otto van Koert produced a Python **CZ-index calculator** for CR3BP (available in GitHub); Bhanu Kumar has a MATLAB version.

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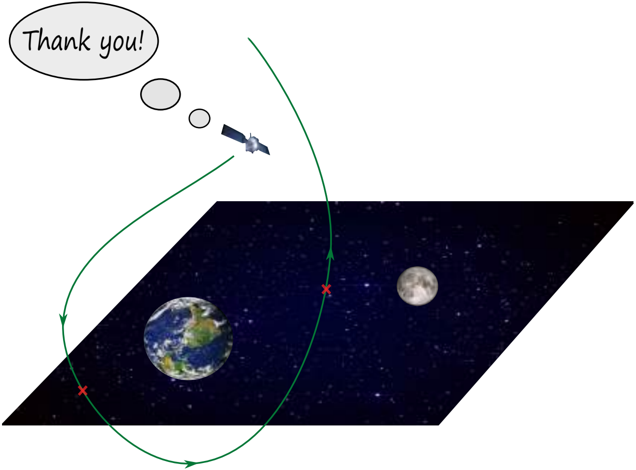
- With Dan Scheeres and Gavin Brown, we applied these tools to the HR4BP, a higher-fidelity, time-dependent deformation of the CR3BP (Astrodynamics Specialists Conference 2024).


Thank you!

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