Extremal Kähler Metrics on Toric Lagrangian Fibrations

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Geometria em Lx
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General program:

Understand the geometry of Hamiltonian spaces of proper symplectic groupoids

- Past work with Marius Crainic (Utrecht) and David Martinez-Torres (Madrid):
  - M. Crainic, R.L.F. & D. Martinez-Torres, *Poisson manifolds of compact type*, I, II, III
- Ongoing joint work with Miguel Abreu (IST-Lisbon) and Maarten Mol (U Toronto)
  - R.L.F. & M. Mol, *Kähler metrics and toric Lagrangian fibrations*, `arXiv:2401.02910`
- Ongoing discussions with Daniele Sepe and Camilo Arias Abad (Universidad Nacional de Colombia-Medellin)

Overview

Hamiltonian space of
compact Lie group

$$G \curvearrowright (S, \Omega) \xrightarrow{\mu} \mathfrak{g}^*$$

- Reduction
- Convexity
- D-H measures
- Localization
- Toric actions
- Multiplicity free actions

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Hamiltonian space of
proper symplectic Lie groupoid

$$\begin{array}{ccc} (\mathcal{G}, \omega) & \circlearrowright & (S, \Omega) \\ & \searrow\!\!\searrow & \downarrow \mu \\ & & (M, \pi) \end{array}$$

✓ A. Weinstein *et al.*

✓ N.-T. Zung, *PMCT papers*

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— L. Zwaan

today!

×

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Overview

Hamiltonian space of compact Lie group	Hamiltonian space of proper symplectic Lie groupoid
$G \curvearrowright (S,\Omega) \xrightarrow{\mu} \mathfrak{g}^*$	$(\mathcal{G},\omega) \circlearrowright (S,\Omega)$, $\mathcal{G} \rightrightarrows (M,\pi)$, $\mu: (S,\Omega) \to (M,\pi)$
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• D-H measures	✓ L. Zwaan, *PMCT papers*
• Localization	— L. Zwaan
• Toric actions	today!
• Multiplicity free actions	×
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- One can also consider quasi-symplectic groupoids and/or other categories: contact, complex, GC, . . .

Overview - Motivation

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Classical Problem: (Thom, Berger, Yau)
Given a manifold M, does it carry a **best** Riemannian structure G?

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$$\text{E-L eqs for } \mathcal{F} \quad \Leftrightarrow \quad X_{\mathrm{Scal}_G} \text{ is a Killing vector field} \qquad \text{(E)}$$

- We adopt (E) as definition of extremal, whether S is compact or not.

Example: Any constant scalar curvature Kähler (cscK) metric is extremal.

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- For symplectic toric manifolds this problem has been extensively studied by many people: Guillemin, Abreu, Donaldson, ...
- Allowing for groupoid symmetry one can hope to treat more general symplectic manifolds
- Today: this is true already for the abelian case (= symplectic torus bundles)!

$$\mathbb{T}^4,\ \mathbb{S}^2 \times \mathbb{T}^2,\ \mathbb{S}^2 \mathbin{\tilde{\times}} \mathbb{T}^2, \dots, \mathbb{P}(L_1 \oplus \cdots \oplus L_k),\ S \mathbin{\tilde{\times}} \mathbb{T}^{2n}, \dots$$

0) Review of classical toric case

Compact symplectic toric manifolds:

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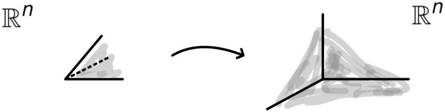

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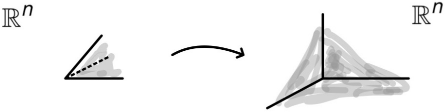

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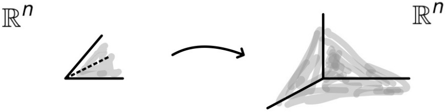

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$$S = \mathbb{C}^d /\!\!/ \mathbb{T}^{d-n} \qquad d = \#\, \textit{facets of } \Delta$$

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Given a $\mathbb{T}^n$-invariant G on S, $\exists!$ metric g on $\mathring{\Delta}$ such that $\mu : (\mathring{S}, G) \to (\mathring{\Delta}, g)$ is a Riemannian submersion

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Donaldson *et al.*: Analytic program to tackle Abreu's equation via K-stability for Delzant polytopes

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  - The only compact surfaces are the torus and the Klein bottle.
  - Markus conjecture (1960s!): $M = \mathbb{R}^n/\Gamma$ with $\Gamma \subset \mathrm{Aff}(n, \mathbb{Z}) := \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

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- A metric g on (M, Λ) is **hessian** if $\mathrm{d}^\nabla g = 0$ (viewing $g \in \Omega^1(M, T^*M)$).

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Proposition

Given a bundle of tori $\mathcal{T} \to M$ with a multiplicative symplectic form $\omega \in \Omega^2(T)$, there exists a unique i.a.s. $\Lambda \subset T^*M$ and a canonical isomorphism

$$\phi : \mathcal{T} \xrightarrow{\simeq} T^*M/\Lambda, \qquad \omega = \phi^*\underline{\omega_{\mathrm{can}}}.$$

2) Hamiltonian spaces of symplectic torus bundles

A Hamiltonian $\mathcal{T}$-space is a symplectic manifold (S, Ω) with an action

$$\begin{array}{ccc} (\mathcal{T}, \omega) & \circlearrowright & (S, \Omega) \\ & \searrow^{p} & \downarrow{\mu} \\ & & M \end{array}$$

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A Hamiltonian $\mathcal{T}$-space is called toric if:

(i) The $\mathcal{T}$-action is effective;

(ii) $\dim(S) = 2 \dim(M)$;

(iii) The map μ has connected fibers and it is proper as map onto its image.

If action is free it is called principal.

Some examples

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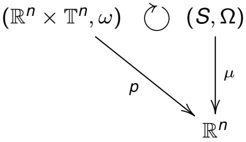

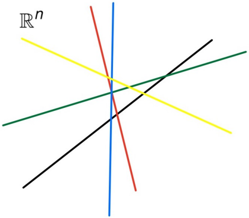

$\mathcal{T}$-Hamiltonian spaces $\equiv$ ordinary $\mathbb{T}^n$-Hamiltonian spaces

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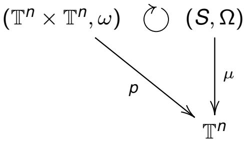

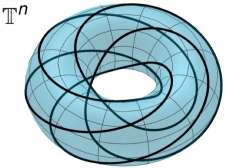

$\mathcal{T}$-Hamiltonian spaces $\equiv$ group-valued $\mathbb{T}^n$-Hamiltonian spaces

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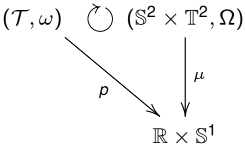

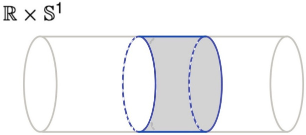

$\Delta = [-1, 1] \times \mathbb{S}^1$

$(\mathbb{S}^2 \times \mathbb{T}^2, \mathrm{pr}^* \omega_{\mathbb{S}^2} + \mathrm{pr}^* \omega_{\mathbb{T}^2})$ **is a toric $\mathcal{T}$-Hamiltonian space**

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$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}\theta_1 + \mathrm{d}x^2 \wedge \mathrm{d}\theta_2, \quad \nabla_{\partial_{x^1}} \mathrm{d}x^2 = \mathrm{d}x^1, \ \nabla_{\partial_{x^i}} \mathrm{d}x^j = 0.$$

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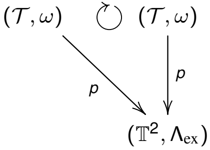

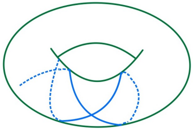

$(\mathcal{T}, \omega)$ **is a principal $\mathcal{T}$-Hamiltonian space (every $\mathcal{T}$ is!)**

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$$\begin{array}{ccc} (\mathcal{T}, \omega) & \circlearrowright & (\mathcal{T}, \omega) \\ & \searrow^{p} & \downarrow p \\ & & (\mathbb{T}^2, \Lambda_{\mathrm{ex}}) \end{array} \qquad\qquad \begin{array}{ccc} (\mathbb{T}^2 \times \mathbb{T}^2, \omega_{\mathrm{can}}) & \circlearrowright & (\mathcal{T}, \omega) \\ & \searrow^{p} & \downarrow \mu \\ & & (\mathbb{T}^2, \Lambda_{\mathrm{st}}) \end{array}$$

$$p(x^1, x^2, \theta_1, \theta_2) = (x^1, x^2) \qquad\qquad \mu(x^1, x^2, \theta_1, \theta_2) = (x^2, \theta^1)$$

$(\mathcal{T}, \omega)$ **is both a** $\mathcal{T}$**-Hamiltonian space and a** $(\mathbb{T}^2 \times \mathbb{T}^2)$**-Hamiltonian space**

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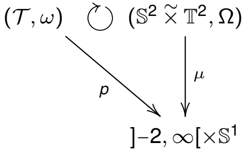

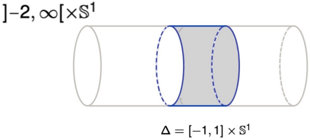

$$\mathbb{S}^2 \tilde{\times} \mathbb{T}^2 := (\mathbb{S}^2 \times \mathbb{R}^2)/\Gamma \quad \text{where} \quad \Gamma : (h, \phi, x, y) \mapsto \begin{cases} (h, \phi - y, x+1, y) \\ (h, \phi, x, y+1) \end{cases}$$

$$\Omega := \mathrm{d}h \wedge (x\mathrm{d}y + \mathrm{d}\phi) + (h+2)\mathrm{d}x \wedge \mathrm{d}y, \qquad \mu(h, \phi, x, y) := (h, x)$$

Non-trivial, orientable $\mathbb{S}^2$-bundle over $\mathbb{T}^2$ is a toric $\mathcal{T}$-Hamiltonian space

3) Characterization of toric Hamiltonian $\mathcal{T}$-spaces

A **singular Lagrangian fibration** is a map $\mu : (S^{2n}, \Omega) \to M^n$ satisfying:

(i) μ has connected fibers and is proper as a map onto its image;

(ii) the fibers of $\mu : S^{\mathrm{reg}} \to M$ are Lagrangian.

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Theorem (RLF & Mol (2024))

For a map $\mu : (S, \Omega) \to M$, the following are equivalent:

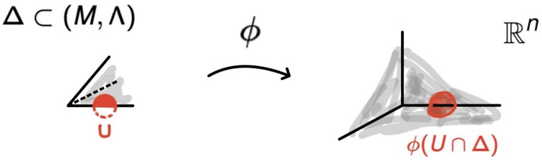
(a) $\mu : (S, \Omega) \to M$ *is a toric Hamiltonian* $\mathcal{T}$*-space;*

(b) $\mu : (S, \Omega) \to M$ *is a toric Lagrangian fibration;*

(c) *For any* $x \in \mu(S)$ *there is a local chart* (U, ϕ) *such that* $\phi \circ \mu : (\mu^{-1}(U), \Omega) \to \mathbb{R}^n$ *is the moment map of a toric* $\mathbb{T}^n$*-action.*

Note: Non-singular Lagrangian fibrations $\leftrightarrow$ principal Hamiltonian $\mathcal{T}$-spaces.

A domain $\Delta \subset (M, \Lambda)$ is called Delzant if $\forall x \in \Delta$ there exists $\mathbb{Z}$-affine chart (U, ϕ) centered at x such that $\phi(U \cap \Delta)$ is an open in $\mathbb{R}^m_k := [0, \infty[^k \times \mathbb{R}^{m-k}$.



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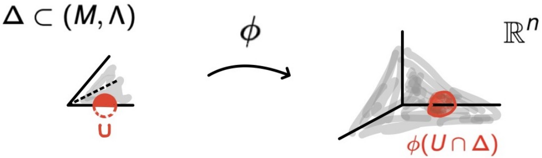

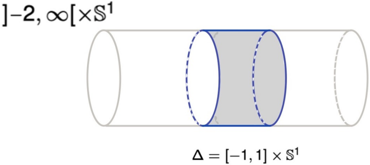

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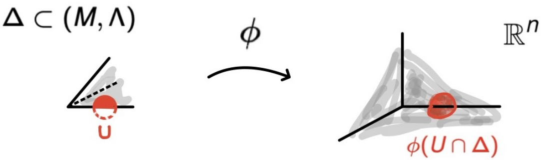

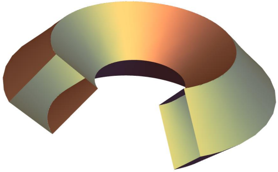

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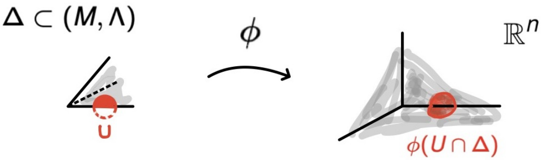

Theorem (Mol (2023))

A toric Hamiltonian $\mathcal{T}$-space $\mu : (S, \Omega) \to M$ is classified by:

(i) $\Delta := \mu(S)$, *a Delzant domain of M;*

(ii) *the Lagrangian-Chern class* $c_1(S, \Omega) \in \check{H}^1(\Delta, \underline{\mathcal{T}_{Lag}})$.

Note: This generalizes both Delzant's classification of toric symplectic manifolds and Duistermaat's classification of (non-singular) Lagrangian fibrations.

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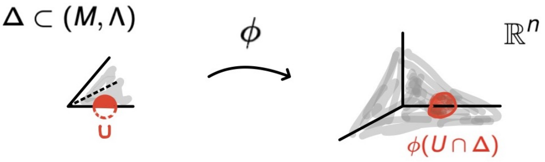

Theorem (RLF & Mol (2024))

A toric Hamiltonian $\mathcal{T}$-space $\mu : (S, \Omega) \to M$ is a classical symplectic toric manifold if and only if $\Delta := \mu(S)$ has trivial affine holonomy.

Corollary

A Hamiltonian $\mathcal{T}$-space whose Delzant domain Δ is 1-connected is a classical symplectic toric manifold. This holds whenever (S, ω) is 1-connected.

4) Invariant Kähler metrics

On a Hamiltonian $\mathcal{T}$-space $\mu : (S, \Omega) \to M$ it makes sense to look at $\mathcal{T}$-invariant geometric structures.

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Given a toric Hamiltonian $\mathcal{T}$-space $\mu : (S, \Omega) \to M$ and letting $\Delta = \mu(S)$, there is a 1:1 correspondence:

$$\left\{ \begin{array}{c} \mathcal{T}\text{-invariant, } \Omega\text{-compatible} \\ \text{Kähler metrics } G \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{l} \bullet \text{ flat elliptic Lagrangian connections} \\ \quad \text{on } S \text{ with zero residue over } \partial\Delta \\ \\ \bullet \text{ hessian hybrid b-metrics on } \Delta \\ \quad \text{with residue on facets equal to} \\ \quad \frac{1}{4\pi}(\text{primitive outward normal}) \end{array} \right\}$$

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$$\bigcup \qquad\qquad\qquad\qquad \bigcup$$

$$\left\{ \begin{array}{c} \text{Extremal} \\ \text{Kähler metrics } G \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{hessian hybrid b-metrics } g \\ \text{with } \mathrm{Scal}_G \text{ an affine function} \end{array} \right\}$$

Remarks:

- The theorem describes the singular behavior of D and g over $\partial\Delta$, in terms of
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- The scalar curvature can be expressed in terms of the hessian metric g:

$$\mathrm{Scal}_G = \mathrm{div}_{\nu_\Lambda}(g^\sharp(\alpha)),$$

  where
  - α is the Koszul form $\nabla_X \nu_g = \alpha(X)\nu_g$;
  - $g^\sharp(\alpha)$ is the mean curvature vector field of the the μ-fibers (Gonçalo-Rosa).

5) Extremal affine function and K-stability

Theorem (Abreu, RLF & Mol)

If $\Delta \subset (M, \Lambda)$ *is a compact Delzant domain there exists a unique affine function* $s_\Delta \in \mathrm{Aff}(\Delta)$ *such that*

$$\int_\Delta s_\Delta v\, \mu_\Lambda = 2\pi \int_{\partial\Delta} v\, \mu_{\partial\Lambda}, \quad \forall v \in \mathrm{Aff}(\Delta).$$

If a toric fibration $\mu : (X, \Omega) \to (M, \Lambda)$ *with* $\Delta = \mu(X)$ *admits a compatible invariant extremal Kähler metric G, then* $\mathrm{Scal}_G = s_\Delta \circ \mu$.

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Corollary

Any extremal Kähler metric on a compact non-singular Lagrangian fibration has zero scalar curvature.

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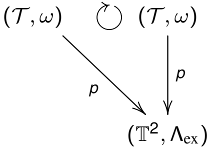
  allows to define notions of K-stability for Delzant domains.
- For a Delzant polytope $\Delta \subset \mathbb{R}^n$ all this recovers results of Donaldson and others.

Examples revisited

Kodaira-Thurston manifold: $(\mathcal{T}, \omega)$



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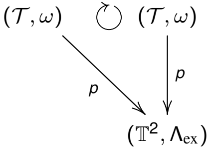

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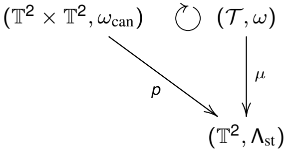

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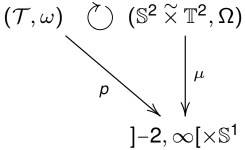
Non-trivial, orientable $\mathbb{S}^2$ bundle over $\mathbb{T}^2$



$$\Delta = [-1, +1] \times \mathbb{S}^1$$
$$\Lambda = \mathbb{Z}\{\mathrm{d}h, (h+2)\mathrm{d}x + x\mathrm{d}h\}$$

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$$\begin{array}{ccc} (\mathcal{T}, \omega) & \circlearrowright & (\mathbb{S}^2 \mathbin{\widetilde{\times}} \mathbb{T}^2, \Omega) \\ & {}_{p}\searrow & \downarrow{\scriptstyle \mu} \\ & &]-2, \infty[\times \mathbb{S}^1 \end{array}$$

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- Elliptic Lagrangian connections: $D_a := \left\langle \partial_x + a(\partial_y - x\partial_\phi), \partial_h \right\rangle$
- hessian hybrid b-metrics: $g = \underbrace{\left(\frac{1}{2\pi(1-h^2)} + f(h) \right)}_{\frac{1}{\tau(h)}} \mathrm{d}h^2 + (h+2)c\,\mathrm{d}x^2$

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Scalar curvature

$$\mathrm{Scal}_G = -\frac{1}{2}\tau''(h) - \frac{\tau'(h)}{h+2}$$

is an affine function if and only if

$$\tau(h) = -\frac{4\pi}{11}\left(2(h+2)^3 - 5(h+2)^2 - 15 + \frac{18}{h+2}\right).$$

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$$(\mathcal{T}, \omega) \circlearrowright (\mathbb{S}^2 \mathbin{\tilde{\times}} \mathbb{T}^2, \Omega)$$

$$p : (\mathcal{T}, \omega) \to]-2, \infty[\times \mathbb{S}^1, \qquad \mu : (\mathbb{S}^2 \mathbin{\tilde{\times}} \mathbb{T}^2, \Omega) \to]-2, \infty[\times \mathbb{S}^1$$

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Scalar curvature

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$\Rightarrow$ extremal Kähler metric

Non-trivial, orientable $\mathbb{S}^2$ bundle over $\mathbb{T}^2$

$$
\begin{array}{ccc}
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& \searrow^{p} & \downarrow^{\mu} \\
& &]-2, \infty[\times \mathbb{S}^1
\end{array}
$$

$$
\begin{aligned}
\Delta &= [-1, +1] \times \mathbb{S}^1 \\
\Lambda &= \mathbb{Z}\{\mathrm{d}h, (h+2)\mathrm{d}x + x\mathrm{d}h\}
\end{aligned}
$$

Using the elliptic Lagrangian connection, one finds the extremal Kähler metric:

$$
(\mathbb{S}^2 \mathbin{\widetilde{\times}} \mathbb{T}^2, J) \simeq \mathbb{P}(L_0 \oplus L_1), \quad G = \frac{1}{\tau(h)}\mathrm{d}h^2 + \tau(h)\theta^2 + (h+2)g_{\mathbb{T}^2}
$$

where $\theta := (x\mathrm{d}y + \mathrm{d}\phi)$ and $g_{\mathbb{T}^2}$ is the flat metric on the torus given by

$$
g_{\mathbb{T}^2} = \frac{1}{c}\big((a^2 + c^2)\mathrm{d}x^2 - 2a\mathrm{d}x\mathrm{d}y + \mathrm{d}y^2\big).
$$

This type of extremal metrics were obtained by Apostolov *et al.* by a more complicated approach (”semi-simple principal toric fibrations”)

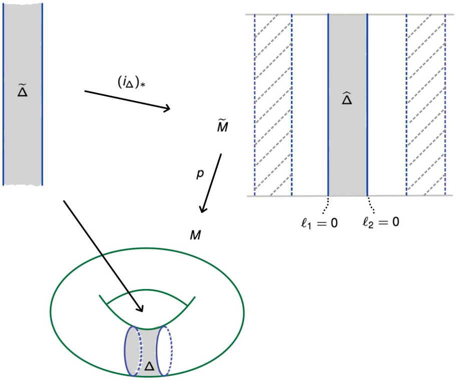
6) A Delzant type construction

A Delzant domain $\Delta \subset (M, \Lambda)$ is of finite type if

$$\widehat{\Delta} = \bigcap_{i=1}^{d} \{\ell_i \leq 0\}$$

w/ $\ell_1, \ldots, \ell_d$ affine functions on the universal covering space $(\widetilde{M}, \widetilde{\Lambda})$ and $\widehat{\Delta}$ image of

$$(i_\Delta)_* : \widetilde{\Delta} \to \widetilde{M}.$$



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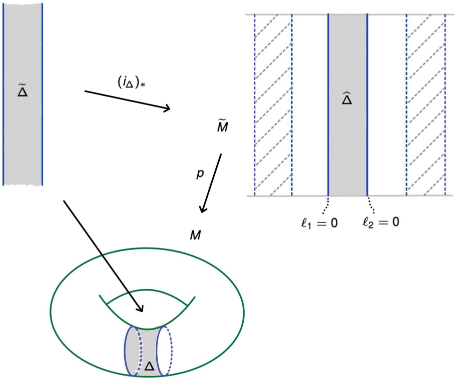

- **Conjecture.** Every compact Delzant domain is of finite type.

Theorem (RLF & Mol (2024))

Let $\Delta \subset M$ be a finite type Delzant domain with d facets. The toric $\mathcal{T}$-space with moment map image Δ and trivial Lagrangian-Chern class can be realized as a symplectic quotient

$$\big((p^*\mathcal{T} \times \mathbb{C}^d) /\!\!/ (\Gamma \ltimes \mathbb{T}^d), \omega_{\text{red}}\big), \qquad (\star)$$

where $p : \tilde{M} \to M$ and Γ is the image of $(i_\Delta)_* : \pi_1(\Delta) \to \pi_1(M)$.

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Corollary

If (M, Λ, g) is an integral affine hessian manifold, $\Delta \subset M$ is of finite type, the symplectic quotient $(\star)$ has a Kähler metric whose induced hessian hybrid b-metric on Δ is

$$g_\Delta = g + \mathrm{Hess}_\Lambda(\phi), \quad \textit{where} \quad \phi = -\frac{1}{4\pi}\sum_{i=1}^{d} \ell_i \log|\ell_i| \in C^\infty(\mathring{\Delta}).$$

Some ongoing/planned related work:

- K-stability for Delzant domains
- Ehrhart polynomial and Euler-Maclaurin formulas for strong integral Delzant domains
- contact $\mathcal{T}$-spaces and invariant Sasakian metrics;
- generalized complex $\mathcal{T}$-spaces and generalized Kähler structures
- non-abelian case, i.e., multiplicity free actions of proper symplectic groupoids (spherical varieties, F. Knop & I. Losev)

$$\begin{array}{ccc} (\mathcal{G},\omega) & \circlearrowright & (S,\Omega) \\ & \searrow\!\searrow & \downarrow \mu \\ & & (M,\pi) \end{array}$$

Obrigado!