The symplectic geometry of the three-body problem

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Book: *The symplectic geometry of the three-body problem* (**Springer Nature**), arXiv:2101.04438.

Restricted three-body problem

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Setup. Three objects: Earth (E), Moon (M), Satellite (S) with masses m_E, m_M, m_S, under gravitational interaction.

Classical assumptions:

1. **(Restricted)** $m_S = 0$, i.e. S is *negligible.*
2. **(Circular)** The *primaries* E and M move in circles around their center of mass.
3. **(Planar)** S moves in the plane containing E and M.

Spatial case: drop the planar assumption.

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Spatial case: drop the planar assumption.

Goal: Study motion of S.

Spatial circular restricted three-body problem

In rotating coordinates where $E = (\mu, 0, 0), M = (-1 + \mu, 0, 0)$ are fixed, the Hamiltonian is autonomous and so is conserved:

$$H : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}$$

$$H(q, p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q - M\|} - \frac{1-\mu}{\|q - E\|} + p_1 q_2 - p_2 q_1,$$

where we normalize so that $m_E + m_M = 1$, and $\mu = m_M$.

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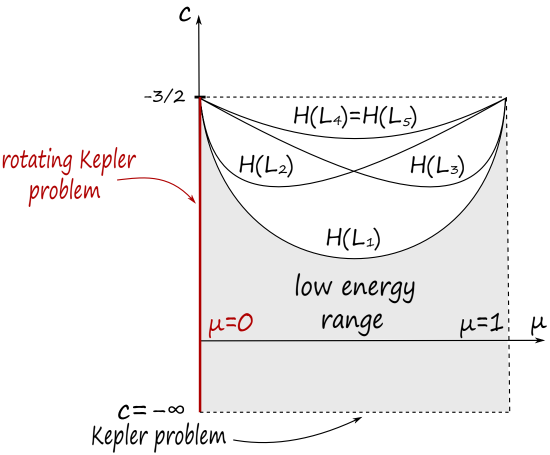
Two parameters: μ, and $H = c$ Jacobi constant.

Lagrangian points

H has five critical points: $L_1, \ldots, L_5$ called *Lagrangians*.



The critical values of H.

Integrable limit cases

If $\mu = 0 \rightsquigarrow H = K + L$, where

$$K(q,p) = \frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}$$

is the *Kepler energy* (two-body problem), and

$$L = p_1 q_2 - p_2 q_1$$

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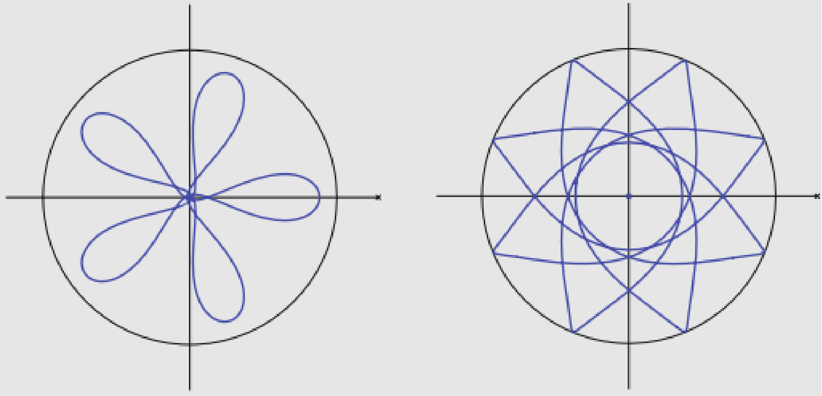
If $T(K) = \frac{\pi}{2(-K)^{3/2}}$ is the period of a Kepler ellipse of energy $K < 0$ (Kepler's 3rd law), then closed orbits iff *resonance condition*:

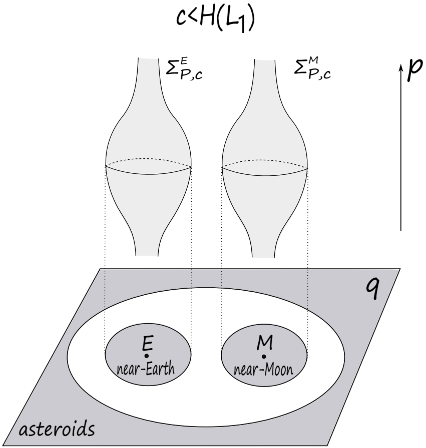
$$T(K) = \frac{a}{b}2\pi, \text{ for some } a, b \in \mathbb{Z}.$$

Periodic orbits in the rotating Kepler problem



Some orbits with different resonance.

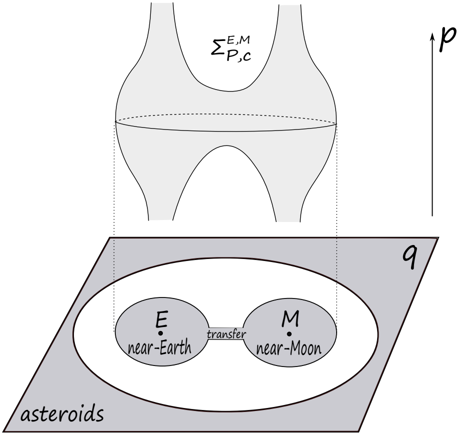
Low energy Hill regions



Morse theory in the three-body problem.

Low energy Hill regions

$c \in (H(L_1), H(L_1) + \varepsilon)$



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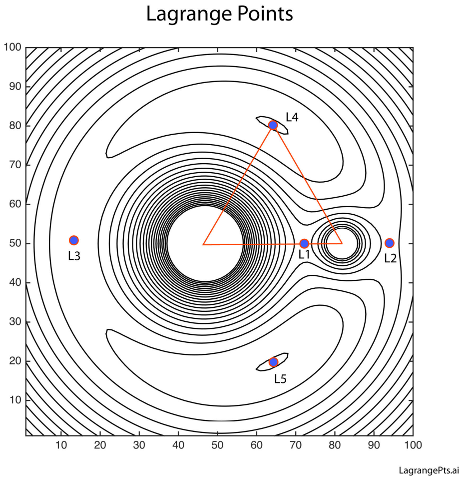
Level sets of potential



The Lagrange points and the level sets of the potential. The Euler points L_1, L_2, L_3 are collinear and unstable, the Lagrange points L_4, L_5 give equilateral triangles and are stable.

Moser regularization

H is singular at *collisions* ($q = E$ ó $q = M \rightsquigarrow p = \infty$).

Moser regularization, near E or M:

$$(q, p) \overset{\text{switch}}{\longmapsto} (-p, q) \overset{\text{stereo. proj.}}{\longmapsto} (\xi, \eta) \in T^*S^3$$

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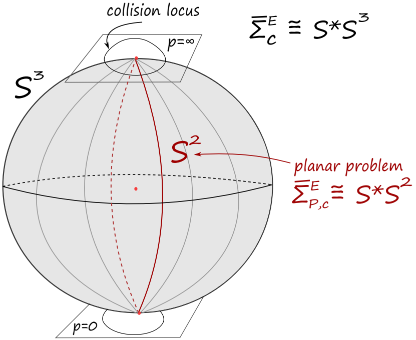
$\rightsquigarrow$ regularized Hamiltonian $Q : T^*S^3 \to \mathbb{R}$, with level set $Q^{-1}(0) = \overline{\Sigma}_c^E \cong S^*S^3 = S^3 \times S^2$.



Contact manifolds

Contact manifold: $(M^{2n-1}, \xi = \ker \alpha)$, α 1-form, satisfying

Contact condition: $\alpha \wedge d\alpha^{n-1} > 0$.

$\alpha =$ *contact form*.

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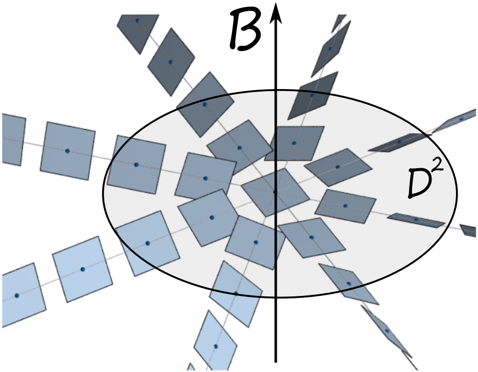
Example: $(\mathbb{R}^3, \xi_{std} = \ker \alpha_{std} = \ker(dz + r^2 d\theta))$.

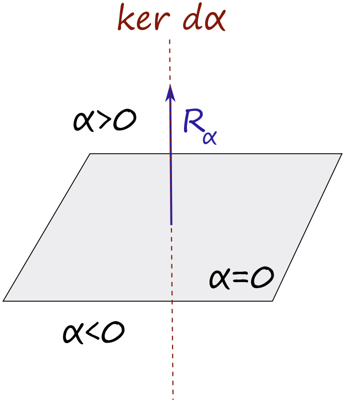


Reeb dynamics

α contact form $\rightsquigarrow$ Reeb field R_α, defined by:

(I) $d\alpha(R_\alpha, \cdot) = 0$,

(II) $\alpha(R_\alpha) = 1$.



Standard example

Q manifold, $(T^*Q, \omega_{std}) =$ *phase-space*.

$$(q, p) = (\text{position,momenta}) \in T^*Q,$$

$$\omega_{std} = dq \wedge dp = d\lambda_{std}$$

standard symplectic form, with

$$\lambda_{std} = -pdq,$$ **standard Liouville form**.

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If (Q, g) Riemannian,

$$S^*Q = \{(q, p) \in T^*Q : |p| = 1\}.$$

Then

$$(S^*Q, \xi_{std} = \ker \lambda_{std}|_{S^*Q})$$

is contact.

Contact geometry of the three-body problem

$\overline{\Sigma}_c^E$, $\overline{\Sigma}_c^M$ bounded energy components for $c < H(L_1)$, $\overline{\Sigma}_c^{E,M}$ connected sum bounded component, $c \in (H(L_1), H(L_2))$. Similarly, $\overline{\Sigma}_{P,c}^E$, $\overline{\Sigma}_{P,c}^M$ and $\overline{\Sigma}_{P,c}^{E,M}$ for planar problem.

Theorem ([AFvKP] (planar problem), [CJK] (spatial problem))

We have

$$\overline{\Sigma}_c^E \cong \overline{\Sigma}_c^M \cong (S^*S^3, \xi_{std}), \text{ if } c < H(L_1),$$

$$\overline{\Sigma}_{P,c}^E \cong \overline{\Sigma}_{P,c}^M \cong (S^*S^2, \xi_{std}), \text{ if } c < H(L_1),$$

and

$$\overline{\Sigma}_c^{E,M} \cong (S^*S^3, \xi_{std}) \# (S^*S^3, \xi_{std}), \text{ if } c \in (H(L_1), H(L_1) + \epsilon).$$

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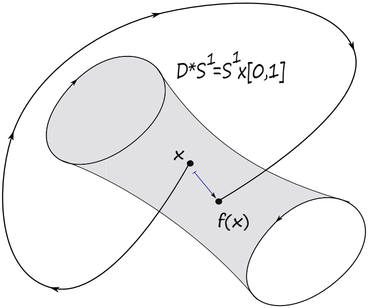
In all above cases, the planar problem is a codimension-2 contact submanifold of the spatial problem. □

Poincaré-Birkhoff and the planar problem

To find closed orbits in the **planar** problem, Poincaré's approach is:

(1) Global surface of section for the dynamics;

(2) Fixed point theorem for the return map.



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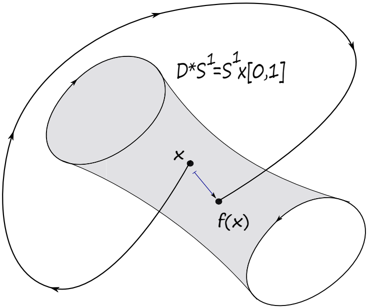

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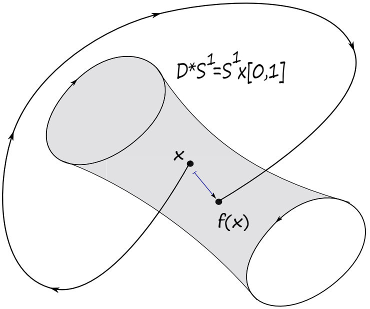

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Goal: Generalize this approach to the **spatial** problem.

Step 1: Global hypersurfaces of section

Open book decompositions

An **OBD** on M is a fibration

$$\pi : M \backslash B \to S^1,$$

with $B \subset M$ codim-2, and
$\pi(b, r, \theta) = \theta$ on collar $B \times \mathbb{D}^2$.

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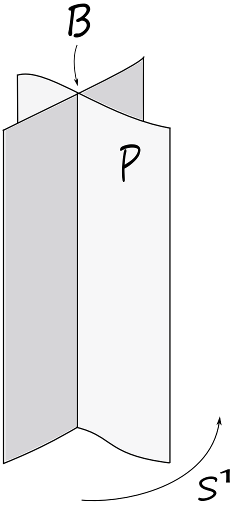
Notation: $M = \mathbf{OB}(P, \phi)$.

- $P = \overline{\pi^{-1}(pt)} =$ *page*;
- $B = \partial P =$ *binding*;
- $\phi : P \xrightarrow{\cong} P$ *monodromy*, $\phi|_B = id$.



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$\varphi_t : M \to M$ flow, then π is **adapted to the dynamics** if B is invariant, and orbits are transverse to the interior of all pages.

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Each page P is a **global hypersurface of section**, i.e.

- P is codimension-1;
- $B = \partial P$ is invariant;
- orbits in $M \backslash B$ meet interior of pages transversely in bounded future and past time.

$\rightsquigarrow$ Poincaré return map $f : \mathrm{int}(P) \to \mathrm{int}(P)$.

Step 1: Open books in the spatial three-body problem

$\overline{\Sigma}_c = H^{-1}(c)$ bounded regularized energy surface in the spatial CR3BP.

Theorem (M–van Koert)

For $\mu \in [0, 1]$,

$$\overline{\Sigma}_c = \begin{cases} \boldsymbol{OB}(\mathbb{D}^* S^2, \tau^2), & c < H(L_1), \\ \boldsymbol{OB}(\mathbb{D}^* S^2 \natural \mathbb{D}^* S^2, \tau_1^2 \circ \tau_2^2), & c \in (H(L_1), H(L_1) + \epsilon), \end{cases},$$

adapted to the dynamics.

- $\tau, \tau_i =$ *Dehn-Seidel twist.*
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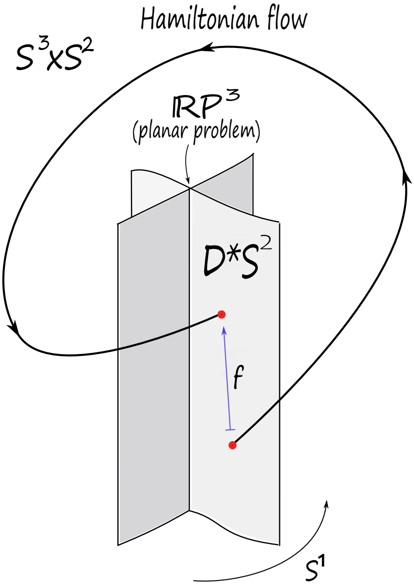
This reduces the dynamics to that of the return map, a Hamiltonian map of $\mathbb{D}^* S^2$. The section is **non-perturbative**, and **explicit** (good for numerics).

Open books



Basic idea

Let $B = \{p_3 = q_3 = 0\}$ (planar problem). Define

$$\pi(q,p) = \frac{q_3 + ip_3}{\|q_3 + ip_3\|} \in S^1, \; d\pi = \frac{p_3 dq_3 - q_3 dp_3}{p_3^2 + q_3^2}.$$

Then

$$d\pi(X_H) = \frac{p_3^2 + q_3^2 \cdot \left(\frac{1-\mu}{\|q-E\|^3} + \frac{\mu}{\|q-M\|^3} \right)}{p_3^2 + q_3^2} > 0,$$

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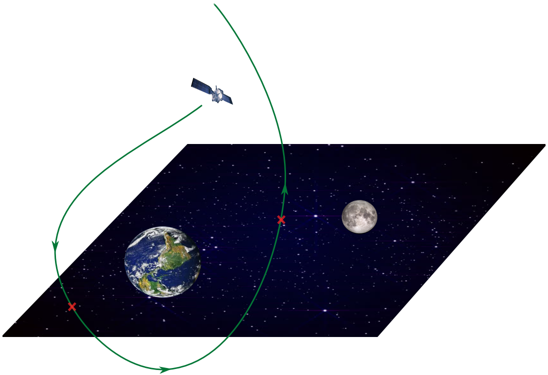
if $p_3^2 + q_3^2 \neq 0$, and the numerator vanishes only on B.

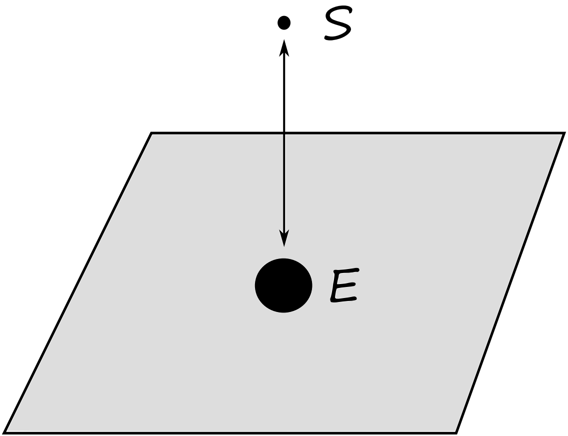
Problem: It does **not** extend to the collision locus $q = E, q = M$.

Physical interpretation



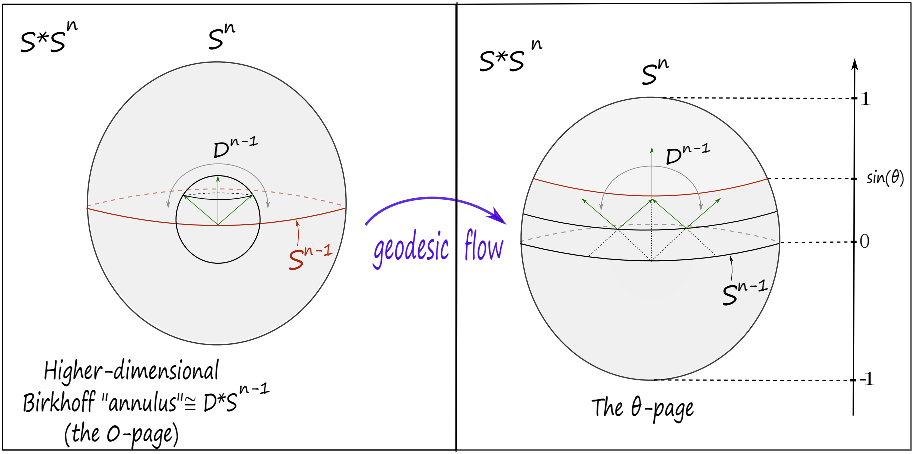
The fiber over $\pi/2$ corresponds to $q_3 = 0$, $p_3 > 0$, and the spatial orbits of S are transverse to the plane containing E, M away from collisions.

Polar orbits



Polar orbits prevent transversality on the collision locus.

The geodesic open book



The geodesic open book for S^*S^n.

Return map

Theorem (M.–van Koert)

For every $\mu \in [0,1]$, $c < H(L_1)$, and page P, the return map f extends smoothly to the boundary $B = \partial P$, and in the interior it is an exact symplectomorphism

$$f = f_{c,\mu} : (int(P), \omega) \to (int(P), \omega),$$

where $\omega = d\alpha|_P$, $\alpha = \alpha_{\mu,c}$ ambient contact form. Moreover, f is Hamiltonian in the interior, and the Hamiltonian isotopy extends smoothly to the boundary.

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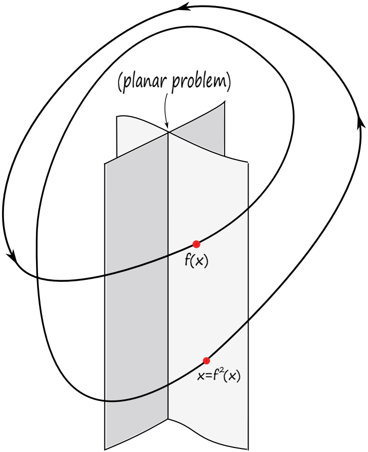
Problem: ω degenerates at the boundary.

Step 2: Fixed-point theory of Hamiltonian twist maps

{**spatial** orbits} $\longleftrightarrow$ {**interior** periodic points}.



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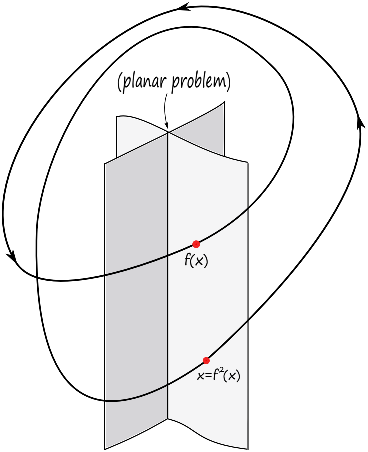
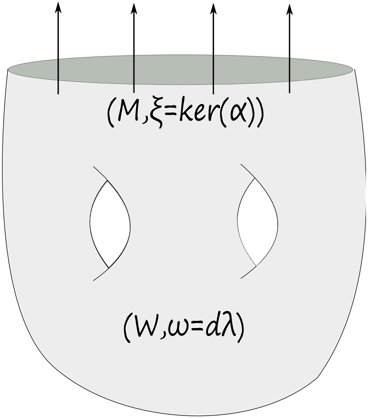


Goal: Find infinitely many *interior* periodic points.

Liouville domains

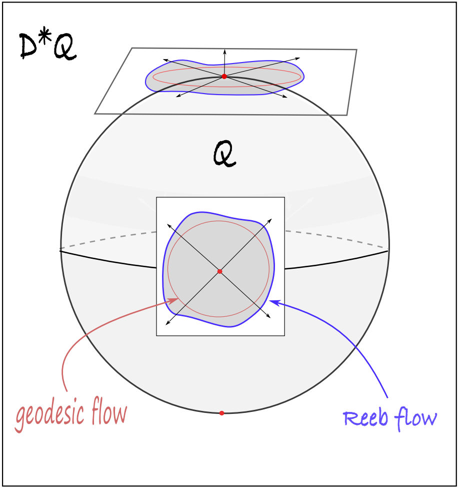
A **Liouville domain** is an *exact* symplectic manifold $(W, \omega = d\lambda)$ with contact boundary $(M = \partial W, \xi = \ker \alpha)$, $\alpha = \lambda|_M$.



A Liouville domain.

Example: fiberwise star-shaped domains

Example: A domain $W \subset T^*Q$ is **fiberwise star-shaped** if the radial vector field $p\partial_p$ is transverse to ∂W. Then $(W, \omega_{std}|_W)$ is a Liouville domain, $W \cong D^*Q$.



A trade-off

We can choose one of the following setups:

(A) The return map extends smoothly to the boundary, but the symplectic form degenerates at the boundary; or

(B) The symplectic form is non-degenerate at the boundary, but the return map extends only continuously to the boundary.

The setups are equivalent, via conjugation with a **square root** map (a diffeomorphism in the interior which is only continuous at the boundary).

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In what follows, we choose (B), i.e. the section is a Liouville domain.

Remark: The same phenomenon occurs in the setting of billiards.

C^0-Hamiltonian twist maps

Let $f : (W, \omega) \to (W, \omega)$ be a diffeomorphism on a Liouville domain, and let α be the contact form at $B = \partial W$.

Definition

f is a C^0-**Hamiltonian twist map** if:

- **(Hamiltonian)** $f|_{\text{int}(W)} = \phi_H^1$ is generated by a C^1 Hamiltonian $H_t : \text{int}(W) \to \mathbb{R}$;
- **(Extension)** Both f and the Hamiltonian H_t admit C^0 extensions to the boundary, but not necessarily C^1 extensions; and
- **(Weakened Twist Condition = WTC)** Near the boundary B, the generating Hamiltonian vector field satisfies

$$h_t := \partial_r H_t = \alpha(X_{H_t}) > 0,$$

and $h_t \to +\infty$ as we approach B.

Here, $r\partial_r$ = Liouville v.f. near B. We say that H_t is *infinitely wrapping*.

A generalized Poincaré–Birkhoff theorem

Theorem (Limoge–M. ’25, after M.–van Koert ’20)

Let $f : (W, \omega) \to (W, \omega)$ be a C^0-Hamiltonian twist map. Assume:

- ***(fixed points)*** *f has finitely many fixed points;*
- ***(First Chern class)*** *$c_1(W) = 0$ if $\dim W \geqslant 4$;*
- ***(Symplectic cohomology)*** *$SH^\bullet(W)$ is non-zero in infinitely many degrees.*

Then f has simple interior periodic points of arbitrarily large minimal period.

A few remarks

- A higher-dimensional generalization of the classical Poincaré-Birkhoff theorem, in the spirit of the Conley conjecture.
- A significant improvement from the previous result with van Koert (e.g. twist condition is now open, no index positivity is required, and regularity is relaxed to C^1).

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- We are working on checking the WTC for the integrable case (RKP).
- This opens up a poorly explored line of research: Hamiltonian dynamics on Liouville domains.

Pseudo-holomorphic dynamics

pseudoholomorphic dynamics $\subset$ symplectic dynamics

Pseudoholomorphic dynamics $=$ dynamics on moduli spaces of pseudoholomorphic curves.

Construction: *(pseudo)holomorphic shadow* of the original dynamics.

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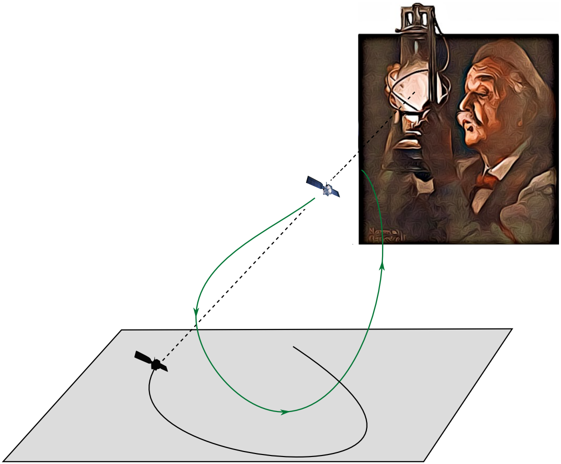
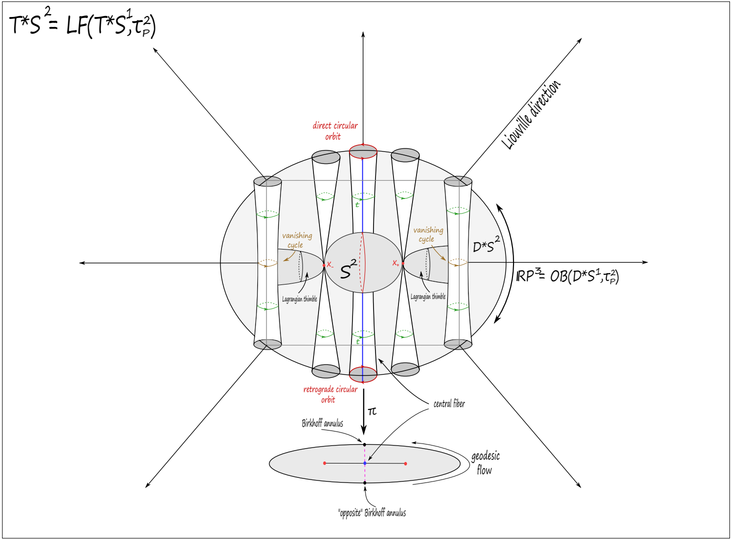


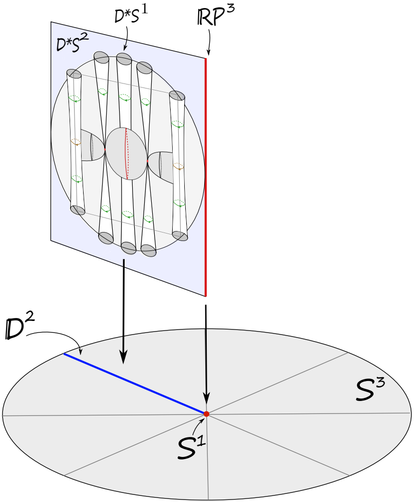
Philosophy: To shed some light on a higher-dimensional problem, try first to look at the shadow that your lantern is producing!

Lefschetz fibration



Topological observation: The section $\mathbb{D}^*S^2$ admits a Lefschetz fibration with annuli fibers.

Leaf space is S^3



The moduli space of fibers (i.e. the leaf space) is $S^3 = \textbf{OB}(\mathbb{D}^2, \mathbb{1})$.

Pseudo-holomorphic foliations in the CR3BP

Let $\alpha = \alpha_{\mu,c}$ contact form giving the CR3BP. We say that (μ, c) lie in the convexity range if the *Levi–Civita regularization* of planar problem is a convex $S^3 \subset \mathbb{R}^4$.

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If (μ, c) in the convexity range, there is a pseudo-holomorphic foliation on the level set S^*S^3 near the Earth or Moon, such that $\omega = d\alpha$ is an area form on each annuli.

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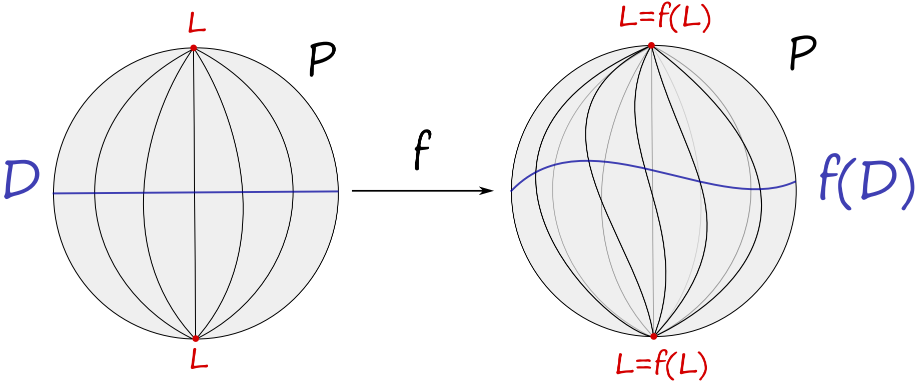
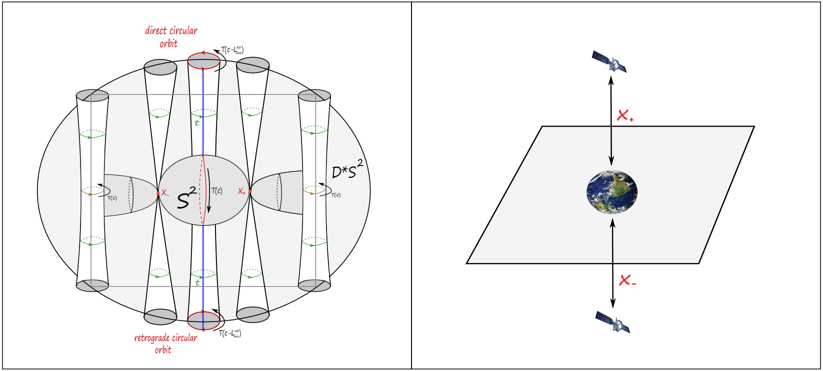
As the return map $f : \mathbb{D}^*S^2 \to \mathbb{D}^*S^2$ preserves ω, it sends a symplectic annulus to another symplectic annulus with the same boundary (direct/retrograde planar orbits), and same symplectic area (the sum of the period of these orbits).

Return map



The return map *f* in general does **not** preserve the foliation.

Integrable case $\mu = 0$.



If $\mu = 0 \rightsquigarrow$ f-invariant foliation, f is a **classical** twist map on the fibers with variable rotation angle $T(K) = \frac{\pi}{2(-K)^{3/2}}$ (Kepler's 3rd law), and the nodal Lefschetz singularities are fixed points (the polar orbits).

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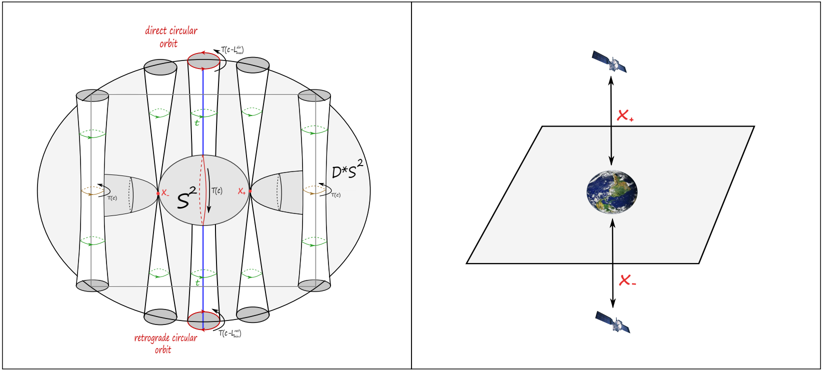

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What happens when we perturb, i.e. $\mu \sim 0$? How does the dynamics interact with the foliation?

Pseudo-holomorphic dynamics

Theorem (M., Contact structures and Reeb dynamics on moduli)

There is an induced contact structure and Reeb flow on the leaf space of the foliation, which is $(S^3, \xi_{std}) = \boldsymbol{OB}(\mathbb{D}^2, id)$.

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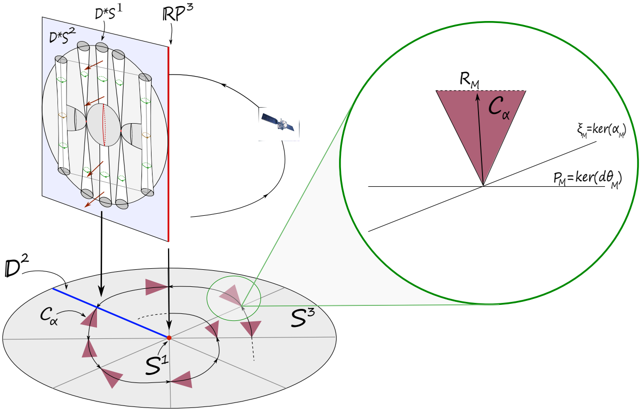
Fiberwise integration:

$$(\alpha_{\mathcal{M}})_u(v) = \int_u \alpha_z(v(z))dz,$$

with $dz = d\alpha|_u$.

The shadowing cone



The shadowing cone is obtained by projecting the flow lines. Orbits of the flow project to orbits of the cone. The “shadow” Reeb flow is the average direction in the cone (“center of mass”).

Holomorphic shadow

Denote **Reeb**$(P, \phi) = \{$Contact forms adapted to **OB**$(P, \phi)\}$. The *holomorphic shadow map* is obtained by taking the shadow:

$$\mathbf{HS} : \mathbf{Reeb}(\mathbb{D}^* S^2, \tau^2) \to \mathbf{Reeb}(\mathbb{D}^2, \mathbb{1})$$

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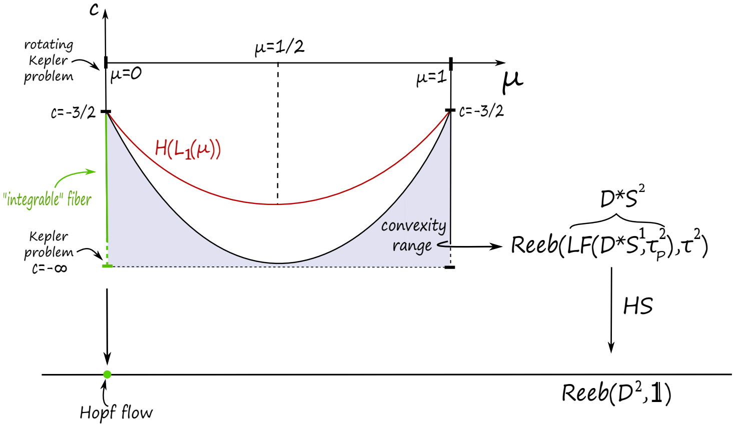
New program: Try to "lift" knowledge from dynamics on S^3.

Case of three-body problem

If (μ, c) in convexity range, $\rightsquigarrow \alpha_{\mu,c} \in$ **Reeb**$(\mathbb{D}^* S^2, \tau^2)$.



Dynamical applications

Definition

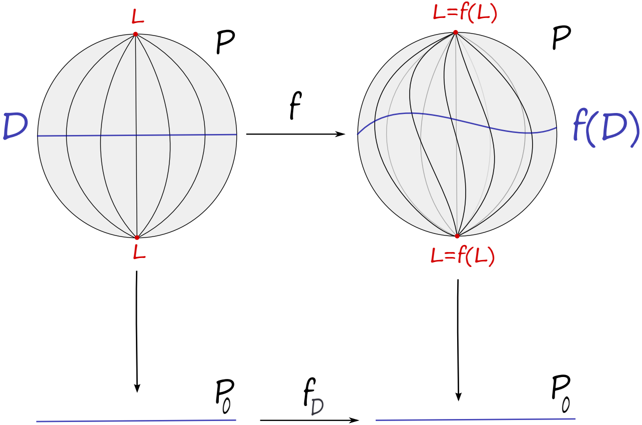
Let P be a page, and $f : \text{int}(P) \to \text{int}(P)$ a return map. A *fiber-wise* k-recurrent point is $x \in \text{int}(P)$ such that $f^k(\mathcal{M}_x) \cap \mathcal{M}_x \neq \emptyset$.

This is a “symplectic version” of a leaf-wise intersection.

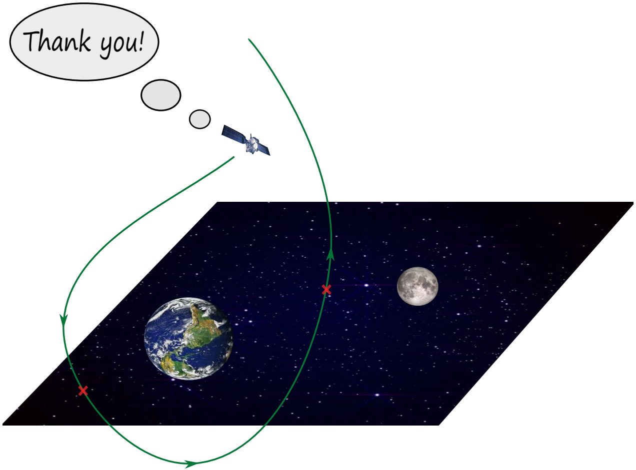
Theorem (M.)

In the SCR3BP, for every k, one can find sufficently small perturbations of the integrable cases which admit infinitely many fiber-wise k-recurrent points.

Idea of proof: symplectic tomographies



We induce maps $f_{\mathcal{D}} : \mathrm{int}(D^2) \to \mathrm{int}(D^2)$ for every symplectic disk section of the LF. These are the identity for the integrable case. These preserve area for near integrable cases, and hence Brouwer applies.


Thank you!

References I

P. Albers, J. Fish, U. Frauenfelder, H. Hofer, O. van Koert.
Global surfaces of section in the planar restricted 3-body problem.
Arch. Ration. Mech. Anal. 204(1) (2012), 273–284.

Albers, Peter; Frauenfelder, Urs; van Koert, Otto; Paternain, Gabriel P.
Contact geometry of the restricted three-body problem.
Comm. Pure Appl. Math. 65 (2012), no. 2, 229–263.

Wanki Cho, Hyojin Jung, Geonwoo Kim.
The contact geometry of the spatial circular restricted 3-body problem.
Abh. Math. Semin. Univ. Hambg. 90 (2020), no. 2, 161–181.

References II

C. Conley.
On Some New Long Periodic Solutions of the Plane Restricted Three Body Problem.
Comm. Pure Appl. Math. 16 (1963), 449–467.

U. Hryniewicz and Pedro A. S. Salomão.
Elliptic bindings for dynamically convex Reeb flows on the real projective three-space.
Calc. Var. Partial Differential Equations 55 (2016), no. 2, Art. 43, 57 pp.

U. Hryniewicz, Pedro A. S. Salomão and K. Wysocki.
Genus zero global surfaces of section for Reeb flows and a result of Birkhoff.
arXiv:1912.01078.

References III

Koh, Dayung; Anderson, Rodney L.; Bermejo-Moreno, Ivan.
Cell-mapping orbit search for mission design at ocean worlds using parallel computing.
The Journal of the Astronautical Sciences, Volume 68, Issue 1, p.172-196.

McGehee, Richard Paul.
Some homoclinic orbits for the restricted three-body problem.
Thesis (Ph.D.), The University of Wisconsin - Madison. ProQuest LLC, Ann Arbor, MI, 1969. 63 pp.0.

A. Moreno.
pseudo-holomorphic dynamics in the spatial restricted three-body problem.
Preprint arXiv:2011.06568.

References IV

A. Moreno.
Contact geometry in the restricted three-body problem -a survey-.
JFPTA, Viterbo 60 edition.

Urs Frauenfelder, Agustin Moreno.
On GIT quotients of the symplectic group, stability and bifurcations of symmetric orbits.
Preprint arXiv:2109.09147.

A. Moreno, O. van Koert.
A generalized Poincaré-Birkhoff theorem.
JFPTA, Viterbo 60 edition.

A. Moreno, O. van Koert.
Global hypersurfaces of section in the spatial restricted three-body problem.
Preprint arXiv:2011.10386.

References V

Poincaré, H.
Sur un théoreme de géométrie.
Rend. Circ. Matem. Palermo 33, 375–407 (1912).