IV.3 Proof of Lax's Existence Theorem

Let $u_L \in \mathbb{R}^N$ be constant.

Let $\mathcal{U}$ be a neighborhood of u_L, small enough.

For $u \in \mathcal{U}$, define for $\varepsilon \in (-a, a)$

$$\varepsilon \longmapsto T^k_\varepsilon u$$

as the k-wave curve or k-contact discontinuity curve (depending on whether (λ_k, r_k) is genuinely nonlinear or linearly degenerate) emanating from u.

Define further, for $\vec{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_N) \in (-a, a)^N$,

$$T(\vec{\varepsilon}) := T^N_{\varepsilon_N} \dots T^1_{\varepsilon_1} u \quad := T^N_{\varepsilon_N} \circ \dots \circ T^1_{\varepsilon_1} u \ .$$

composition

Then, the Riemann problem for states u_L & u_R has a solution if and only if $\exists\, \vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N)$ such that

$$u_R = T(\vec{\varepsilon})\, u_L \ .$$

By Inverse Function Theorem, such $\vec{\varepsilon}$ exist for any u_R sufficiently close to u_L, since for

$$F(\vec{\varepsilon}) := T(\vec{\varepsilon})\, u_L - u_L$$

we have $F(0) = 0$ & $\underbrace{\left.\frac{\partial F}{\partial \varepsilon_k}\right|_{\vec{\varepsilon}=0} = r_k(u_L)}$

$\Rightarrow \frac{\partial F}{\partial \vec{\varepsilon}}$ invertible, because $\{r_1(u_L), \dots, r_N(u_L)\}$ linearly independent

Enjoy your Christmas break!

Lemma (C^2-contact):

Assume $|r_k(u)| = 1$ and assume shock curve $S_k : \varepsilon \mapsto u(\varepsilon)$ is parameterized by arc-length, $|\dot{u}(\varepsilon)| = 1$, such that $\dot{u}(0) \cdot r_k(u_L) \geq 0$.

Then: The shock speed $s(\varepsilon)$ satisfies $s(0) = \lambda_k(u_L)$ & $\dot{s}(0) = \frac{1}{2} r_k \cdot \nabla\lambda_k\big|_{u_L}$ and the k-shock curve satisfies $\dot{u}(0) = r_k(u_L)$ & $\ddot{u}(0) = r_k \cdot \nabla r_k\big|_{u_L}$.

proof:

- For each ε, $s(\varepsilon)$ and $u(\varepsilon)$ meet the RH-condition

$$s(u - u_L) = f(u) - f(u_L), \qquad \text{(RH)}$$

and by construction of shock curves for previous theorem, we have

$$s(0) = \lambda_k(u_L) \checkmark.$$

- Differentiate (RH) in ε:

$$\Rightarrow \underbrace{s(0)}_{\lambda_k(u_L)} \dot{u}(0) = \underbrace{df\big|_{u_L}}_{=:A(u_L)} \cdot \dot{u}(0) \iff \big(A(u_L) - \lambda_k(u_L)\big)\dot{u}(0) = 0$$

$$\Rightarrow \dot{u}(0) = r_k(u_L) \checkmark \quad \text{(using that } |r_k| = 1 = |\dot{u}| \text{ \& } r_k(u_L) \cdot \dot{u}(0) \geq 0)$$

- Differentiate (RH) twice in ε:

$$\Rightarrow \big(A(u_L) - \lambda_k(u_L)\big)\ddot{u}(0) = \big(2\dot{s}(0) - \dot{A}(0)\big) r_k(u_L) \qquad (1)$$

- Consider eigenvalue problem

$$\big(A(u(\varepsilon)) - \lambda_k(u(\varepsilon))\big) r_k(u(\varepsilon)) = 0$$

$$\overset{\frac{d}{d\varepsilon}\big|_0}{\Longrightarrow} \big(A(u_L) - \lambda_k(u_L)\big) r_k \nabla r_k\big|_{u_L} = \big(r_k \cdot \nabla\lambda_k\big|_{u_L} - \dot{A}(0)\big) \cdot r_k(u_L) \qquad (2)$$

- Let ℓ_k denote left-eigenvector, i.e.

$$\ell_k(u) \cdot \big(A(u) - \lambda_k(u)\big) = 0$$

Then multiply (1) by ℓ_k gives

$$\ell_k \cdot \dot{A}(0) \cdot r_k = 2\dot{s}(0)\, \ell_k \cdot r_k\big|_{u_L}$$

and multiplying (2) by ℓ_k gives

$$\ell_k \cdot \dot{A}(0) \cdot r_k = r_k \cdot \nabla \lambda_k\big|_{u_L}\, \ell_k \cdot r_k\big|_{u_L} .$$

Then, substracting above equations gives

$$\left(2\dot{s}(0) - r_k \cdot \nabla r_k\big|_{u_L}\right) \overbrace{\ell_k \cdot r_k\big|_{u_L}}^{\neq 0} = 0$$

$$\Leftrightarrow \quad \dot{s}(0) = \frac{1}{2} r_k \cdot \nabla r_k\big|_{u_L} \quad \checkmark \tag{3}$$

- Adding (1) & (2) and substituting (3) yields

$$\left(A(u_L) - \lambda_k(u_L)\right)\left(r_k \cdot \nabla r_k\big|_{u_L} - \ddot{u}(0)\right) = 0$$

$\Rightarrow \quad \ddot{u}(0) = r_k \cdot \nabla r_k\big|_{u_L}$, since $\ddot{u}$ & $r \cdot \nabla r_k \perp r_k$ by arc-length parameterization.

∎

Thm:

Let $u_R = u(\varepsilon)$ be a state on the k-shock curve $S_k : \varepsilon \mapsto u(\varepsilon)$, parameterized by arc-length as in previous Lemma.
Then (u_L, u_R) satisfy the "Lax condition"

$$\lambda_k(u_R) < s < \lambda_k(u_L)$$

if and only if $\varepsilon < 0$.

proof:

Define $\phi(\varepsilon) := \lambda_k(\varepsilon) - s(\varepsilon)$

$$\overset{\text{Lemma}}{\underset{\text{above}}{\Longrightarrow}} \quad \phi(0) = 0 \quad \& \quad \dot\phi(0) = \overbrace{\dot u(0)}^{= r_k(u_L)} \cdot \nabla \lambda_k\big|_{u_L} - \overbrace{\dot s(0)}^{= \frac{1}{2} r_k \cdot \nabla\lambda_k|_{u_L},\ \text{(Lemma)}}$$

$$= \frac{1}{2} r_k \cdot \nabla \lambda_k\big|_{u_L} > 0 \quad \text{, by gen. nonlinear}$$

$\Rightarrow$ $\phi(\varepsilon)$ is increasing for ε sufficiently close to $\varepsilon = 0$.

$$\Rightarrow \underbrace{\begin{cases} \lambda_k(u(\varepsilon)) > s > \lambda_k(u_L) & \text{, for } \varepsilon > 0 \\ \lambda_k(u(\varepsilon)) < s < \lambda_k(u_L) & \text{, for } \underline{\varepsilon < 0} \end{cases}}_{\text{Lax conditions}}$$

$\blacksquare$

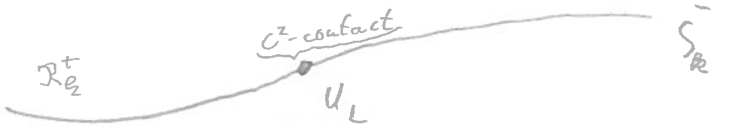
By the previous theorem we find that for states on the shock curves with parameter $\varepsilon > 0$, the corresponding Riemann problem cannot be solved by an <u>admissible</u> (physical) shock wave. We thus only take part of shock curve with negative ε and match at $\varepsilon = 0$ to a rarefaction curve with $\varepsilon \geq 0$:

<u>Def:</u>

Assume (λ_k, r_k) is genuinely nonlinear. Assume $|r_k| = 1$ and the arc-length parameterization of Lemma "C^2-contact" of k-shock curve $\mathcal{S}_k^-: \varepsilon \longmapsto u^-(\varepsilon)$ and for rarefaction curve $\mathcal{R}_k^+: \varepsilon \longmapsto u^+(\varepsilon)$. We define the "k-wave curve" as

$$\varepsilon \longmapsto u(\varepsilon) = \begin{cases} u^-(\varepsilon) \ , \ \varepsilon < 0 & \text{(k-shock curve)} \\ u^+(\varepsilon) \ , \ \varepsilon > 0 & \text{(k-rarefaction curve)} \end{cases}$$

where $\varepsilon \in (-a, a)$ for some $a > 0$.



Thm

Assume (λ_k, r_k) is linearly degenerate. Let $u_L \in \mathbb{R}^N$ constant.

Then there exist a curve $\mathcal{C}: \varepsilon \mapsto u(\varepsilon)$, $u(0) = u_L$, $\forall \varepsilon \in (-a, a)$ for some $a>0$, such that the Riemann problem for $(u_L, u(\varepsilon))$ can be solved by a contact discontinuity with speed $s = \lambda_k(u_L)$ for any $\varepsilon \in (-a,a)$.

The curve is unique up to parameterization.

proof:

Define $\varepsilon \mapsto v(\varepsilon)$ as flow:
$$\begin{cases} \frac{dv}{d\varepsilon} = r_k(v(\varepsilon)) & , \varepsilon \in (-a,a) \\ v(0) = u_L \end{cases}$$

$$\Rightarrow \frac{d}{d\varepsilon} \lambda_k(v(\varepsilon)) = r_k \cdot \nabla\lambda_k \Big|_{v(\varepsilon)} = 0 \quad , \text{by linear degeneracy}$$

$$\Rightarrow \lambda_k(u_L) = \lambda_k(v(\varepsilon)) \quad \forall \varepsilon \in (-a,a) \qquad (*)$$

Now, define
$$u(t,x) := \begin{cases} u_L & , x < \lambda_k(u_L) t + x_0 \\ v(\varepsilon) & , x > \lambda_k(u_L) t + x_0 \end{cases}$$

Then for each $\varepsilon \in (-a,a)$, u solves Riemann problem for $(u_L, v(\varepsilon))$ by a contact discontinuity, since
$$\frac{d}{d\varepsilon}\left(f(v(\varepsilon)) - \lambda_k(u_L)\, v(\varepsilon)\right) = \Big(A(v(\varepsilon)) - \underbrace{\lambda_k(u_L)}^{= \lambda_k(v(\varepsilon)),\ \text{by } (*)}\Big)\, r_k(v(\varepsilon)) = 0$$

$$\Rightarrow \lambda_k(u_L)\left(v(\varepsilon) - u_L\right) = f(v(\varepsilon)) - u_L \qquad \forall \varepsilon > 0$$

$$\Rightarrow \text{RH-conditions}$$

$\blacksquare$