Topological symmetries and their gaugings in 2d CFT and 3d TFT

Ingo Runkel (Univ. Hamburg)

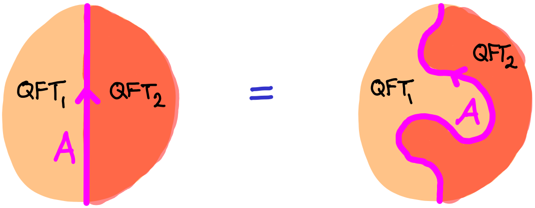
- topological line defects in 2d QFT
  - → pivotal fusion categories $\mathcal{F}$
  - → restrictions on 2d QFT

- gauging topol. defects in 2d and 3d

2d CFT

3d TFT

Topological defects



This talk : 2d (mostly QFT) and 3d (mostly TFT)

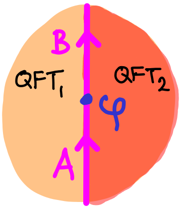
Much recent activity : topol. phases of matter, higher dim. QFTs, lattice models

McGreevy, '22, Kong, Zhang '22, Cordova, Dumitrescu, Intriligator, Shao '22, Schäfer-Nameki '23
Brennan, Hong, '23, Bhardwaj, Bottini, Fraser-Taliente, Gladden, Gould, Platschorre, Tillim, '23
Shao, '23, Carqueville, Del Zotto, Runkel '23

Topological defects in 2d QFT

Davydov, Kong, IR '11

Topological defects for 2d QFTs form a bicategory with two-sided adjoints.

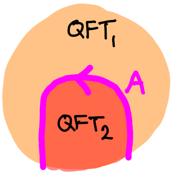


objects : 2d QFTs on surface patches

1-morph. : topol. defect lines

2-morph.: topol. junction fields

Adjoints:

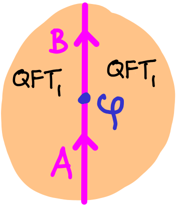


etc.

... topol. defects in 2d

In particular:

Topol. endo-defects form a pivotal monoidal category $\mathcal{E}(QFT_1)$



$\mathcal{F}$: pivotal fusion category

finitely semisimple

Q : 2d QFT

We say Q has symmetry $\mathcal{F}$ if there is a (ℂ-linear) pivotal monoidal functor $\mathcal{F} \xrightarrow{\ f\ } \mathcal{E}(Q)$.

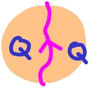
⚠ A topol. sym. need not be invertible.

. . . topol. defects in 2d

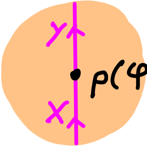
$$\mathcal{F} \xrightarrow{\ \rho\ } \mathcal{E}(Q)$$

pivotal fusion cat.

pivotal mon. functor

topol. defects 

- ρ is automatically faithful (injective on morphisms)
- ρ may or may not be full (surjective on morphisms)

$\mathrm{Hom}_{\mathcal{F}}(X,Y) \ni \varphi \longmapsto$ 

$\rho(\varphi)$ topol. point defect

... topol. defects in ~~2d~~ 3d

Algebraic framework in 3d:

- tricategories with adjoints — Carqueville, Meusburger, Schaumann '16
- fusion 2-categories — Douglas, Reutter '18

Pivotal fusion categories

In 2d QFT

finite group symmetry $\rightsquigarrow$ pivotal fusion cat. of topol. defects

E.g.: G – finite group

Vect_G – G-graded fin. dim. v.sp.

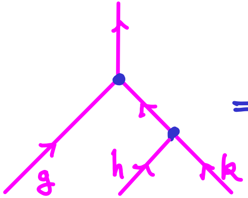
$\mathbb{C}_g$, $g \in G$ – simple objects

$\otimes$: $\mathbb{C}_g \otimes \mathbb{C}_h \simeq \mathbb{C}_{gh}$

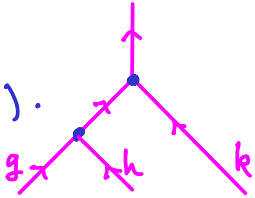
... pivotal fusion cat.

associator

$\omega \in Z^3(G, \mathbb{C}^\times)$



$= \omega(g,h,k) \cdot$

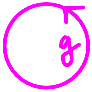


up to $\otimes$ equiv (keeping obj. fixed) $[\omega] \in H^3(G, \mathbb{C}^\times)$

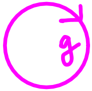
pivotal structure

$d : G \longrightarrow \mathbb{C}^\times$ group hom.

$(\Leftrightarrow d \in H^1(G, \mathbb{C}^\times))$



$= d(g)$



$= d(g^{-1})$

... pivotal fusion cat.

E.g. $G = \mathbb{Z}_2$ $\quad H^3(\mathbb{Z}_2, \mathbb{C}^\times) \simeq \mathbb{Z}_2$

$\{d : \mathbb{Z}_2 \to \mathbb{C}^\times\} \simeq \mathbb{Z}_2$ $\Big\}$ 4 choices

Write additively: $\mathbb{Z}_2 \simeq \{0, 1\}$

	$\omega(1,1,1) = 1$	$\omega(1,1,1) = -1$
$d(1) = 1$	$\mathrm{Vect}_{\mathbb{Z}_2}$	$\mathrm{Vect}_{\mathbb{Z}_2}^{\omega}$
$d(1) = -1$	SVect	SVect^{ω}

... pivotal fusion cat.

E.g. $G = \mathbb{Z}_2$ $\quad H^3(\mathbb{Z}_2, \mathbb{C}^\times) \simeq \mathbb{Z}_2$

$\{d : \mathbb{Z}_2 \to \mathbb{C}^\times\} \simeq \mathbb{Z}_2$ } 4 choices

Write multiplicatively : $\mathbb{Z}_2 \simeq \{\pm 1\}$

	$\mathbb{Z}_2$ can be gauged $\omega(\underline{1}, \underline{1}, \underline{1}) = 1$	$\mathbb{Z}_2$ cannot be gauged $\omega(\underline{1}, \underline{1}, \underline{1}) = -1$
$d(\underline{1}) = 1$	$\mathrm{Vect}_{\mathbb{Z}_2}$	$\mathrm{Vect}^{\omega}_{\mathbb{Z}_2}$
$d(\underline{1}) = -1$	SVect	SVect^{ω}

... pivotal fusion cat.

E.g. $G = \mathbb{Z}_2$ $\quad H^3(\mathbb{Z}_2, \mathbb{C}^\times) \cong \mathbb{Z}_2$

$$\left.\begin{array}{l} H^3(\mathbb{Z}_2, \mathbb{C}^\times) \cong \mathbb{Z}_2 \\ \{d : \mathbb{Z}_2 \to \mathbb{C}^\times\} \cong \mathbb{Z}_2 \end{array}\right\} \text{4 choices}$$

Write multiplicatively: $\mathbb{Z}_2 \cong \{\pm 1\}$

	$\mathbb{Z}_2$ can be gauged $\omega(\underline{1}, \underline{1}, \underline{1}) = 1$	$\mathbb{Z}_2$ cannot be gauged $\omega(\underline{1}, \underline{1}, \underline{1}) = -1$
$d(\underline{1}) = 1$	$\mathrm{Vect}_{\mathbb{Z}_2}$ gauging well-defined on oriented surfaces	$\mathrm{Vect}^{\omega}_{\mathbb{Z}_2}$
$d(\underline{1}) = -1$	SVect gauging needs a spin structure	SVect^{ω}

... pivotal fusion cat.

E.g.: simplest non-group case : Fib

- simple obj $\mathbb{1}$, φ
- tensor $\varphi \otimes \varphi \simeq \mathbb{1} \oplus \varphi$
- associator : 2 choices
- pivotal : 1 choice

Pivotal fusion cat. classified for :

$\#(\text{simple obj}) =$

... pivotal fusion cat.

E.g.: simplest non-group case: Fib

- simple obj $\mathbb{1}$, φ
- tensor $\varphi \otimes \varphi \cong \mathbb{1} \oplus \varphi$
- associator : 2 choices
- pivotal : 1 choice

Pivotal fusion cat. classified for:

Ostrik '02, '13

$\#(\text{simple obj}) = 1, 2, 3$

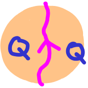
Fusion category symmetry restricts 2d QFT

pivotal fusion cat.

$$\mathcal{F} \xrightarrow{\rho} \mathcal{E}(Q)$$

pivotal mon. functor

topol. defects



1) 2d TFT

Suppose
- Q is a 2dTFT (sym. mon. fun. $\mathrm{Bord}_{2,1} \to \mathrm{Vect}$)
- has unique vacuum (state space on S^1 is 1-dim.)

Then $\mathcal{F}$ admits a fibre functor

($\Leftrightarrow$ $\mathcal{F} \cong \mathrm{Rep}\, H$ for a ssi Hopf alg. H)

Thorngren, Wang '19

... fusion cat. sym. restricts 2dQFT

$$\mathcal{F} \xrightarrow{\rho} \mathcal{E}(Q)$$

2) 2d CFT

What is the unitary CFT (with unique vacuum) of smallest central change such that ρ exists and is full ?

$$\mathcal{F} = \mathrm{Vect}_{\mathbb{Z}_2} \quad : \quad c = 1/2 \quad (\text{Ising CFT})$$

$$\mathcal{F} = \mathrm{Vect}^{\omega}_{\mathbb{Z}_2} \quad : \quad c = 1 \quad (\text{e.g. } su(2)_1 \text{ WZW})$$

... fusion cat. sym. restricts 2dQFT

$$\mathcal{F} \xrightarrow{\rho} \mathcal{E}(Q)$$

3) Renormalisation group flows :

Chang, Lin, Shao, Wang, Yin '18, Komargodski, Ohmori, Roumpedakis, Seifnashri '20, Bhardwaj, Bottini, Pajer, Schäfer-Nameki '23, Cordova, García-Sepúlveda, Holfester '24

combine 1 & 2

RG flow

2d CFT in the UV $\longrightarrow$ massive theory in the IR

assume $\mathcal{F}$ preserved along entire flow

ground states form a 2dTFT

If $\mathcal{F}$ does not allow for a fibre functor, a massive IR theory must have more than one ground state.

(more math : possible gnd state degeneracies $\leftrightarrow$ $\mathcal{F}$-module categories)

Generalised orbifolds

Fröhlich, Fuchs, Schweigert, IR '09
Carqueville, IR '12
Carqueville, Schaumann, IR '17

a.k.a. gauging non-invertible top. symmetries

a.k.a. internal state sum models

Idea:

Fill world volume of QFT with a defect foam, impose conditions to ensure independence of the choice of foam

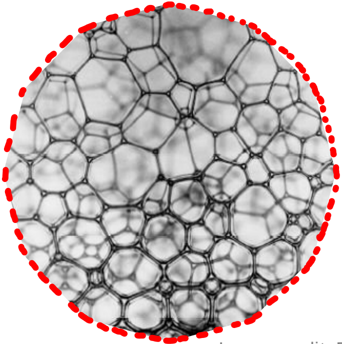


Image credit: ESA

... gauging non-inv. top. sym.

Gauging produces

- oriented theory : orbifold data — Carqueville, Schaumann, IR '17
- framed theory : higher idempotents/condensations — Douglas, Reutter '18; Gaiotto, Johnson-Freyd '19

For d-dim theory:

Data: defect labels $\mathcal{A} = (\mathcal{A}_{d-1}, \mathcal{A}_{d-2}, \dots, \mathcal{A}_1, \mathcal{A}_0^{\pm})$

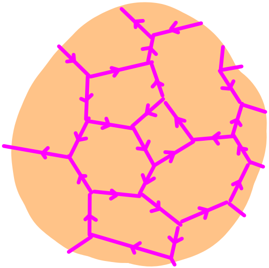
Cond.: dual to oriented d-dim Pachner moves

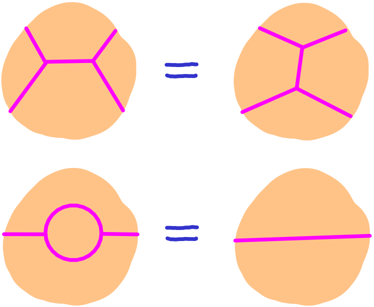
... gauging non-inv. top. sym.

1d

Can change # of A's : $A^2 = A$ idempotent

2d





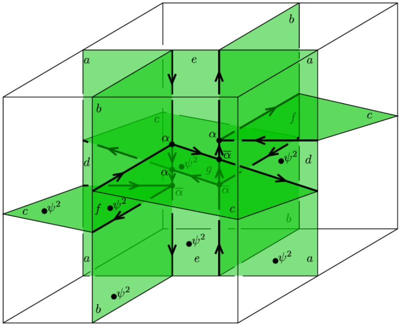
$\rightsquigarrow$ symmetric Δ-separable Frobenius algebra

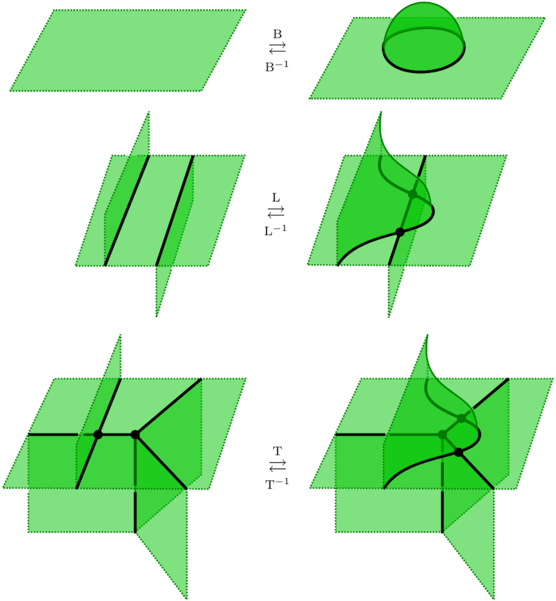
(for all orientations)

... gauging non-inv. top. sym.

Carqueville, Schaumann, IR '17

3 d





pictures from: Mulevicius, PhD Thesis, '21

... gauging non-inv. top. sym.

Carqueville, Schaumann, IR '17

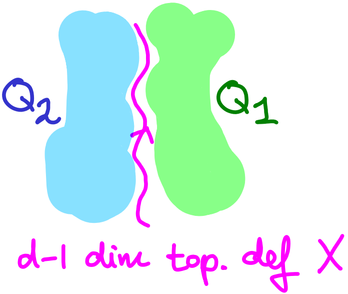
Thm.: Gauging an oriented d-dim TFT by $\mathcal{A}$ is again an oriented d-dim TFT.

In an oriented d-dim QFT where one can describe topol. defects, the same data & cond. $\mathcal{A}$ can be gauged to give a new d-dim QFT.

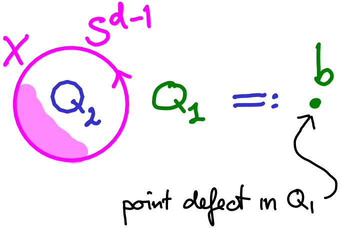
Invertible bubbles and gauging

Carqueville, IR '12

Given: topol. defect separating d-dim QFTs Q_1 and Q_2



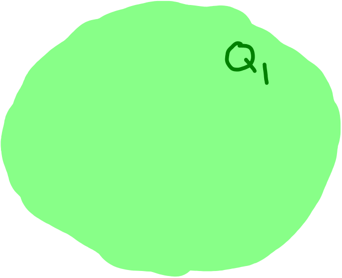
such that



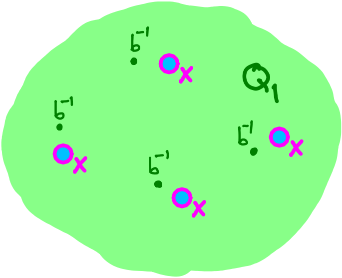
is invertible as a point defect in Q_1.

Then: Q_1 is obtained from Q_2 by gauging a top. sym.

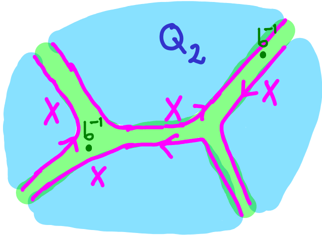
... invertible bubbles



=

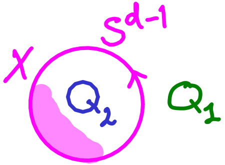


=

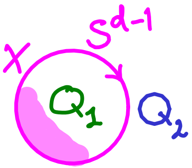


... invertible bubbles

Gauging has an inverse gauging if



and



invertible

e.g.:
- In 1d only true if X invertible
- In 2d ssi case & simple X : always true
- In 3d ...?

Get equivalence relation on d-dim. QFTs.

Example : rational 2d CFTs

V : rational vertex operator algebra

$\hookrightarrow$
- all V-modules are semisimple
- finite # of simple V-modules
- in fact : $\mathrm{Rep}\,V$ modular fusion cat.

Fröhlich, Fuchs, Schweigert, IR '09

Thm.:

Any two oriented 2d CFTs s.th. space of fields contains $V_{hol} \otimes V_{antihol}$ (with unique vac. and non-deg. 2pt corr.) can be obtained from each other by gauging a topol. sym.

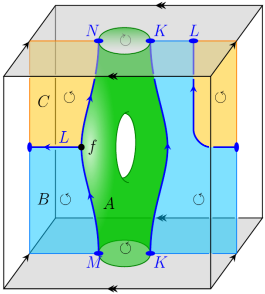
⚠ Wrong for just group orbifolds!

Example: 3d TFT of surgery type

modular fusion category $\mathcal{C}$

$\rightsquigarrow$ 3d TFT $Z_{\mathcal{C}}$ with

- line defects $\longleftrightarrow$ objects in $\mathcal{C}$
- surface defects $\longleftrightarrow$ $\mathcal{C}$-module cat (with trace)



Kapustin, Saulina '10
Fuchs, Schweigert, Valentino '12
Carqueville, Schaumann, IR '17

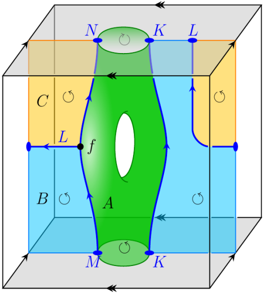
Example: 3d TFT of surgery type

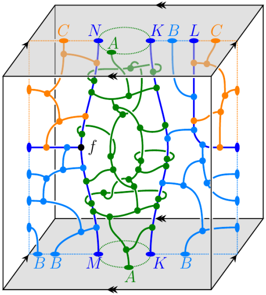
modular fusion category $\mathcal{C}$

$\rightsquigarrow$ 3d TFT $Z_{\mathcal{C}}$ with

- line defects $\longleftrightarrow$ objects in $\mathcal{C}$
- surface defects $\longleftrightarrow$ ~~$\mathcal{C}$-module cat (with trace)~~

  $\to$ sym. sep. Frob. alg.



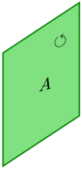


Kapustin, Saulina '10
Fuchs, Schweigert, Valentino '12
Carqueville, Schaumann, IR '17

... 3d TFT

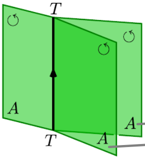
Orbifold data

geom | **alg**

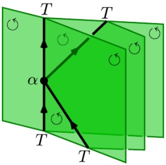


sym. Δ-sep. Frob. alg $A \in \mathcal{C}$

(typically not simple)



A - $A \otimes A$ - bimodule $T \in \mathcal{C}$



morphism $\alpha : T \otimes T \to T \otimes T$
compatible with A-actions

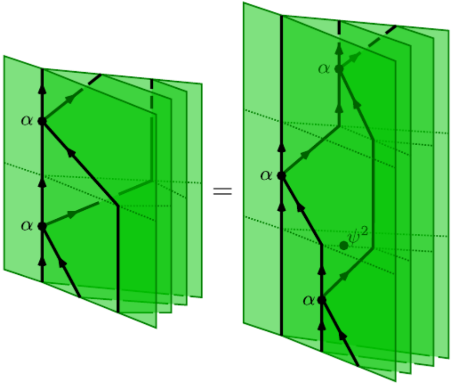
and $\quad \bar{\alpha}$: inverse to α (up to factors)

$\psi : A \longrightarrow A$ bimodule isom. $\quad \phi \in \mathbb{k}^{*}$

... 3d TFT

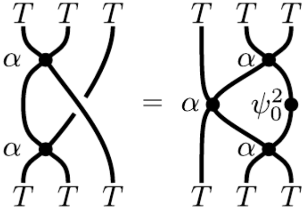
Conditions on orbifold data

geom

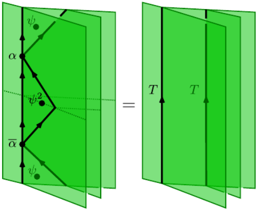


(O1)

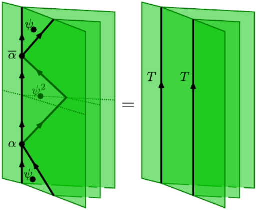
alg



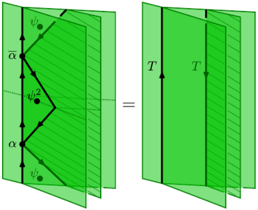
(O1)



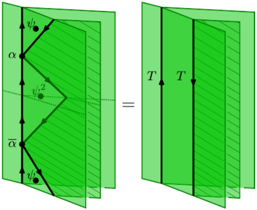
(O2)



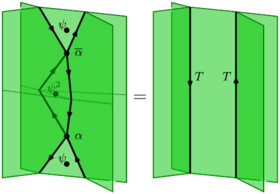
(O3)



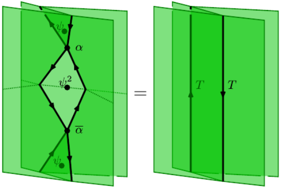
(O4)



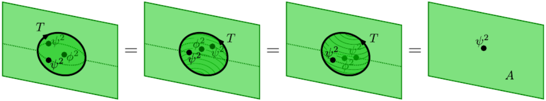
(O5)



(O6)



(O7)



(O8)

... 3d TFT

Carqueville, Mulevicius, Schaumann, Scherl, IR '21
Mulevicius, '22

Theorem:

Let $\mathcal{C}, \mathcal{D}$ be modular fusion categories.

The following are equivalent

1) There is an orbifold datum $\mathcal{A}$ in $\mathcal{C}$ s.th.

$$Z_{\mathcal{C}}/\mathcal{A} \simeq Z_{\mathcal{D}}$$

2) $\mathcal{C}$ and $\mathcal{D}$ are Witt-equivalent.

$\mathcal{C} \boxtimes \overline{\mathcal{D}} \simeq Z(\mathcal{F})$ for some spherical fusion cat. $\mathcal{F}$

The E_6-example

Mulevicius, IR '20
Mulevicius, '22

11 simple obj $\qquad\qquad$ 3 simple obj

condensation

$$\mathcal{C}(\mathfrak{sl}_2, 10) \xleftarrow[\text{gen. orb. by some } \mathcal{A} \text{ in } \mathcal{C}(\mathfrak{sp}_4,1)]{} \underbrace{\mathcal{C}(\mathfrak{sp}_4, 1)}_{\text{of Ising type}} \qquad \mathcal{C}(\mathfrak{sl}_2, 10) \xrightarrow[A_{E_6} = \underline{0} \oplus \underline{6}]{\text{condensation}} \mathcal{C}(\mathfrak{sp}_4, 1)$$

Here

$$\mathcal{A} = (A = \mathbb{1} \oplus \mathbb{1} \;, \; \dots)$$

... the E_6-example

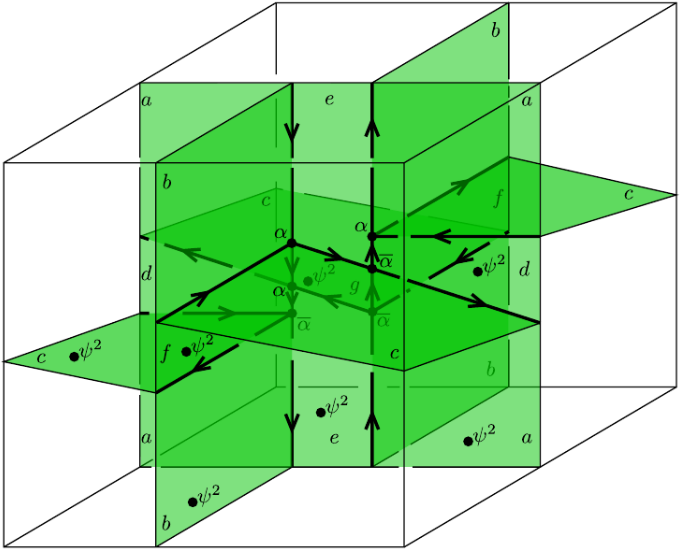
Aside: Compute #(simples) via orbifold:

$$\#(\text{simples in } \mathcal{C}/\!_{A}) = Z_{\mathcal{C}/A}(T^3)$$

$$= Z_{\mathcal{C}}(T^3 + \text{foam})$$

$$= \ldots$$

Foam:



a
b
e
a
b
c
f
c
α
α
$\overline{\alpha}$
d
ψ^2
g
α
$\overline{\alpha}$
$\overline{\alpha}$
d
c
f
ψ^2
c
b
a
e
a
b

Formula:

$$
\begin{aligned}
&Z_{\mathcal{C}}^{\mathrm{orb},\mathbb{A}}(T^3) = \\
&\quad \phi^2 \cdot \sum_{\substack{a,b,c,d,e,f,g \in B \\ r,s,t,u,v,w \in I}} \frac{\psi_a^2 \psi_c^2 \psi_e^2 \psi_f^2}{\psi_b^2 \psi_d^2 \psi_g^2} \, f_{afc,\ bg}^{e,\ r} \, f_{caf,\ gd}^{e,\ s} \, f_{fca,\ db}^{e,\ u} \, g_{acf,\ db}^{e,\ t} \, g_{fac,\ gd}^{e,\ v} \, g_{cfa,\ bg}^{e,\ w} \\
&\qquad \cdot \sum_{x,y,z,k,l,m \in I} L_{eba,\ ur}^{bcf,\ wt}\Big|_x \, L_{efd,\ vt}^{dca,\ us}\Big|_y \, L_{ecg,\ wv}^{gaf,\ sr}\Big|_k \\
&\qquad\qquad \cdot \frac{\dim z}{\dim l} \, \frac{\dim v}{\dim k} \, F_{k\,u}^{(r\,z\,y)v} \, F_{l\,w}^{(s\,k\,x)u} \, G_{r\,m}^{(t\,x\,z)u} \, G_{t\,l}^{(s\,y\,m)u} \\
&\qquad\qquad \cdot T_{xyz,\ klm} \ ,
\end{aligned}
\tag{3.12}
$$

A

C

where

$$
L_{eda,\ ur}^{bcf,\ wt}\Big|_x = N_{{}_e t_{da} \ {}_b t_{cf}}^{w} \, N_{{}_e t_{ad} \ {}_b t_{cf}}^{t} \, \frac{\dim x}{\dim {}_b t_{fc}} \, F_{{}_b t_{fc} \ t}^{({}_e t_{ad} \ {}_b t_{cf} \ x)r} \, G_{w \ {}_b t_{fc}}^{({}_e t_{da} \ {}_b t_{cf} \ x)u} \ ,
$$

$$
T_{xyz,\ klm} = \sum_{p,j \in I} \frac{\dim j}{\dim p} \, \frac{\theta_j}{\theta_p \theta_x} \, R^{-(z\,x)m} \, G_{p\,k}^{(p\,z\,y)p} \, G_{p\,l}^{(p\,k\,x)j} \, F_{l\,p}^{(p\,y\,m)j} \, F_{m\,p}^{(p\,z\,x)j} ,
$$

(Naively, in E_6-ex., this sum has ~600 Mio. terms)

... the E_6-example

Aside: Compute #(simples) via orbifold:

$$\#(\text{simples in } \mathcal{C}/\!\!\mathcal{A}) = Z_{\mathcal{C}/\!\!\mathcal{A}}(T^3)$$

$$= Z_{\mathcal{C}}(T^3 + \text{foam})$$

$$= \ldots$$

$$\underset{E_6\text{-ex.}}{=} 11$$

Summary

Topological defects contain important information about QFT

In 2d :
- finite group $\rightsquigarrow$ pivotal fusion cat. $\mathcal{F}$
- constraints on QFT from presence of $\mathcal{F}$

Generalised orbifolds as an equivalence relation
("bubble condition")

- 2d RCFT
- 3d TFT

Outlook

- non-semisimple symmetries & their gauging
  → Aaron Hofer
- other geometric structures (spin , spin_c)
  → Jannik Grőne
- surface defects labelled by Lie group el.
  → Azat Gainutdinov, Bangxin Wang
- non-topological defects and integrability in 2d
  → Federico Ambrosino , Gérard Watts
- skein theoretic description of defect TFT
  → Patrick Kinnear