Weaving 4-Dimensional TQFTs with Ribbon Categories.

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Lisbon Topological Quantum Field Theory Seminar
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Joint work with F. Costantino, N. Geer and B. Haïoun.

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Based on

arXiv:2306.03225 [pdf, other]

Skein (3+1)-TQFTs from non-semisimple ribbon categories

Francesco Costantino, Nathan Geer, Benjamin Haïoun, Bertrand Patureau-Mirand

Comments: 30 pages

Subjects: **Geometric Topology (math.GT)**; Quantum Algebra (math.QA)

arXiv:2302.04493 [pdf, other]

Admissible Skein Modules

Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand

Comments: 15 pages

Subjects: **Geometric Topology (math.GT)**; Quantum Algebra (math.QA)

Definition of a non compact (n+1)-TQFT

A $(n+1)$-cobordism $W : M_1 \to M_2$ between closed oriented n-manifolds M_1 and M_2 is an oriented compact smooth $(n+1)$-manifold W with boundary $\partial W \cong (-M_1) \sqcup M_2$.

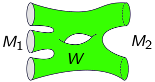
Let $(\mathbf{Cob}_{n+1}, \sqcup)$ (resp. $\mathbf{Cob}^{\text{nc}}_{n+1}$) be the monoidal category whose

- *objects* are closed oriented n-manifolds and
- *morphisms* are equivalence classes of $(n+1)$-cobordisms (resp. where every component has non empty incoming boundary.).

Definition

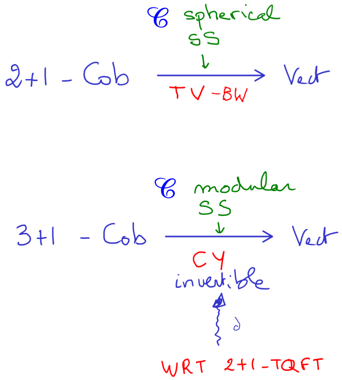
A (non compact) *(n+1)-TQFT* is a symmetric monoidal functor

$$\mathfrak{Z} : (\mathbf{Cob}^{\text{nc}}_{n+1}, \sqcup) \to (\text{Vect}_{\mathbb{K}}, \otimes).$$

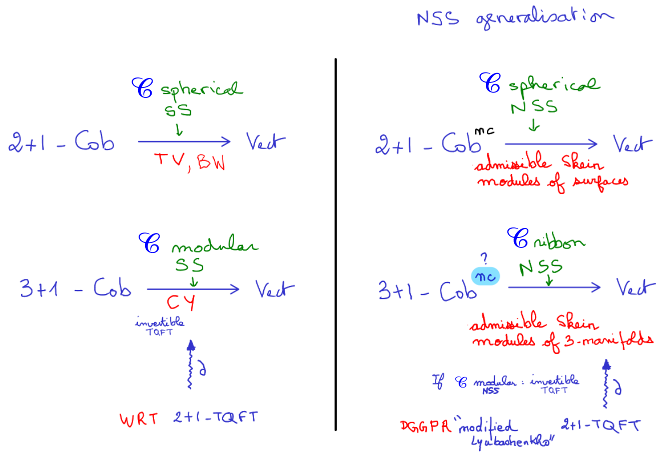


$$\xrightarrow{\mathfrak{Z}} \quad \left(\mathfrak{Z}(M_1) \xrightarrow[\mathfrak{Z}(W)]{\text{linear map}} \mathfrak{Z}(M_2) \right)$$

We learn in the 90's that finite tensor categories $(\mathscr{C}, \circ, \otimes, \mathbb{1})$ which are **semi-simple** can be used to define TQFTs in dimension 3 & 4.



Non-semisimple TQFTs



arXiv:2503.20905 [pdf, ps, other] math.QA math.CT math.GT

Non-semisimple WRT at the boundary of Crane-Yetter

Authors: Benjamin Haïoun

What ribbon categories lead to 3+1-TQFTs ?

A finite unimodular ribbon tensor category or, more generally, a *ribbon chromatic category* $\mathscr{C}$ over a field $\mathbb{K}$:

- ribbon category $((\mathscr{C}, \otimes, \mathbb{1})$ with braiding $\{b_{U,V}\}_{U,V \in \mathscr{C}}$ and pivotal)
- chromatic category
  - "sharp" $\mathbb{K}$-additive (not necessarily abelian) is a
    - $\mathbb{K}$-linear, finite dim Hom-spaces;
    - $\exists U \oplus V$, idempotent complete;
    - $\mathbb{1}$ is simple and absolutely simple;
  - $\mathscr{C}$ has a non zero projective generator G (i.e. $\mathscr{C}$ is finite);
  - $\mathscr{C}$ is unimodular ($P_{\mathbb{1}}^* \simeq P_{\mathbb{1}}$ which implies that $\mathscr{C}$ has a modified trace t on Proj);
  - $\mathscr{C}$ has a chromatic map $\mathsf{c}_{P_{\mathbb{1}}} \in \mathrm{End}_{\mathscr{C}}(G \otimes P_{\mathbb{1}})$ (for example if $\mathscr{C}$ is a finite ribbon tensor category);

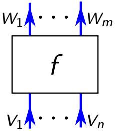
Let $\Delta^0 = \mathrm{ptr}_G^L(b^2 \circ \mathsf{c}_{P_{\mathbb{1}}}) \in \mathrm{End}(P_{\mathbb{1}})$.

$\mathscr{C}$ is *chromatic non-degenerate* if $\Delta^0 \neq 0 \overset{\text{Th}}{\Longrightarrow} \exists$ non-cpct TQFT;

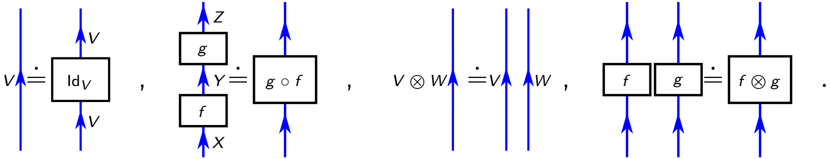
$\mathscr{C}$ is *chromatic compact* if Δ^0 is invertible $\overset{\text{Th}}{\Longrightarrow} \exists$ TQFT.

Ribbon $\mathscr{C}$-graphs

We use Reshetikhin-Turaev ribbon graphs colored by a ribbon category $(\mathscr{C}, \otimes, \mathbb{1})$ where graphs are thickened into bands colored by objects and coupons colored by morphisms:

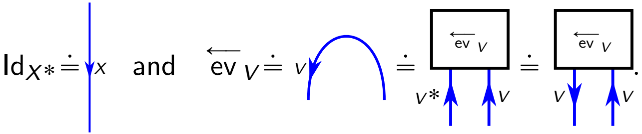


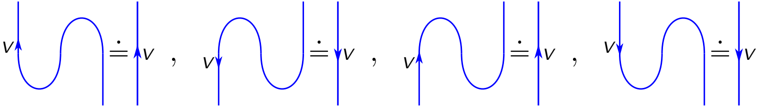
represents a morphism $f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots W_m$.
RT-functor $\mathrm{F} : \mathscr{T}_{\mathscr{C}} \to \mathscr{C}$ associates $\mathscr{C}$-morphisms to ribbon graphs.

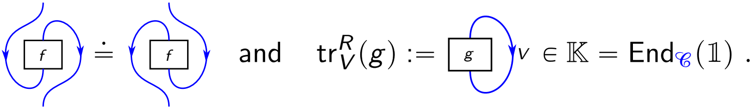


Pivotal graphical calculus

In a pivotal category $\mathscr{C}$ any object V has a dual object V^*, and duality morphisms $\overleftarrow{\mathrm{ev}}_V : V^* \otimes V \to \mathbb{1}$, $\overleftarrow{\mathrm{coev}}_V, \overrightarrow{\mathrm{ev}}_V, \overrightarrow{\mathrm{coev}}_V$

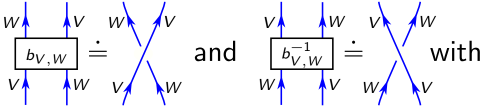


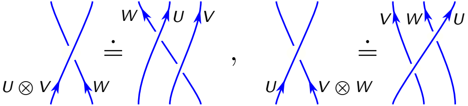




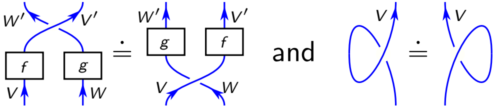
Ribbon graphical calculus

A ribbon category $\mathscr{C}$, is a pivotal category with a braiding: a family of isomorphisms $\{b_{V,W} : V \otimes W \to W \otimes V\}$





and $b_{V,\mathbb{1}} = b_{\mathbb{1},V} = \mathrm{Id}_V$



Admissible Skein module of a 3-manifold M

Let $\mathscr{C}$ be a ribbon category. If M is a closed 3-manifold, the skein module of M is the quotient of the linear span over $\mathbb{K}$ of all finite $\mathscr{C}$-colored closed graph embedded in M modulo the relations

1. Ambient isotopy of the graph in M.
2. Skein relations : Local graphical calculus in a 3-ball $\simeq [0,1]^3$.

An equivalence class of such graph is called a skein.

Example: If $\mathscr{C}$ is the discrete category of an abelian group G, the skein module of M is just the ring of the group $H_1(M, G)$.

Admissible Skein module of a 3-manifold M

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An equivalence class of such graph is called a skein.

Admissible skein module $\mathscr{S}\!k(M)$: Only $\mathscr{C}$-colored graphs with at least an edge colored by a projective object of $\mathscr{C}$ in each component of M. Only allow Proj-skein relations.

$$M \xmapsto{\mathscr{S}\!k} \mathscr{S}\!k(M)$$

Assign to any 3-manifold a finite dimensional vector space ; functorial with diffeomorphisms.
For which categories can it be extended to a 3+1-TQFT ?

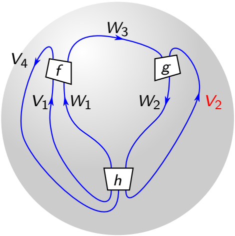
Why using the admissible skein module ?

The idea was behind the definition of modified traces:
In non semi-simple categories, F vanishes on all closed ribbon graphs with some fixed colors $\in \mathscr{C}$.

These vanishing colors form an ideal:

An *ideal* $\mathcal{I}$ is a full subcategory stable by retracts and absorbing for tensor product. There can be a strictly decreasing sequence of ideals:

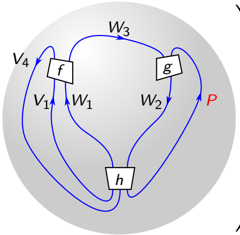
$$\mathscr{C} \supset \mathcal{I}_1 = \ker(\mathrm{tr}_{\mathscr{C}}) \supset \mathcal{I}_2 \supset \cdots \supset \mathcal{I}_n = \mathrm{Proj} \supset \{0\}$$

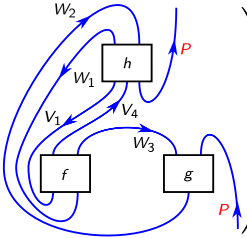


$$\left.\begin{array}{c} \mathbb{1} \\ \uparrow \\ V_2^* \otimes V_2 \in \mathcal{I}_2 \\ \uparrow \\ \mathbb{1} \end{array}\right\} \doteq 0$$

Why using the admissible skein module ?

For example, if $\mathscr{J}_2 = \mathrm{Proj}$,

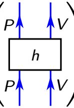
F' (

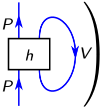
) $:= \mathrm{t}_P($F (

))

$$\{\mathrm{t}_P : \mathrm{End}_{\mathscr{C}}(P) \to \mathbb{K}\}_{P \in \mathrm{Proj}}$$

F' invariant by Proj-skein relation $\iff$ t is an m-trace on Proj:

$\mathrm{t}(f \circ g) = \mathrm{t}(g \circ f)$ and t$\circ$F (

) = t$\circ$F (

),

Existence of m-trace

Theorem

There exists an unique m-trace on Proj *iff* $\mathscr{C}$ *is absolutely unimodular iff*

$$\exists \quad \mathbb{1} \xrightarrow{\eta} P_{\mathbb{1}} \quad \textit{and} \quad P_{\mathbb{1}} \xrightarrow{\varepsilon} \mathbb{1}$$

which generate their Hom*-spaces.*

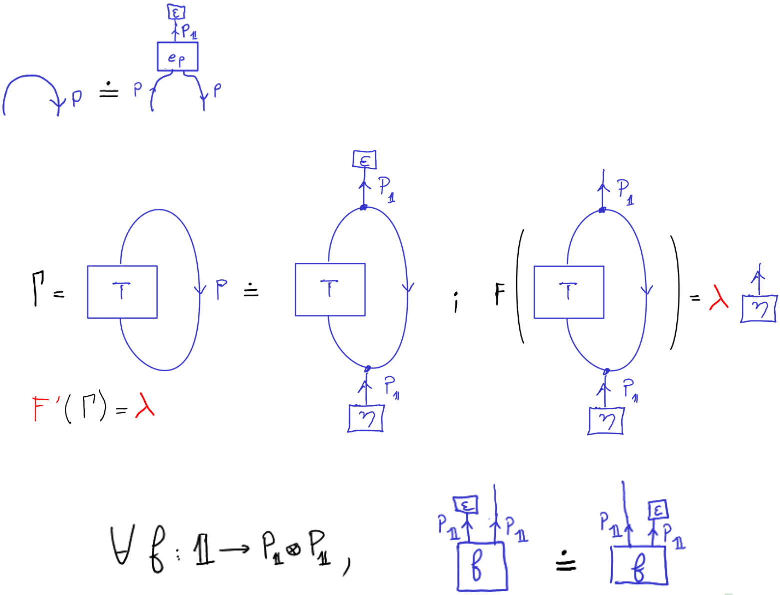
Here $P_{\mathbb{1}}$ is the projective cover of $\mathbb{1}$ and the proof uses the Fitting Lemma which implies that $\mathrm{End}_{\mathscr{C}}(P_{\mathbb{1}})$ is a local ring:

$$\mathrm{End}_{\mathscr{C}}(P_{\mathbb{1}}) = \mathbb{K}\mathrm{Id} \oplus J$$

where J is a nilpotent ideal.

$$(P_{\mathbb{1}} \xrightarrow{f} P_{\mathbb{1}}) = \lambda \mathrm{Id} + n \implies \varepsilon \circ f = \lambda \varepsilon \quad \text{and} \quad f \circ \eta = \lambda \eta.$$

Definition of F'

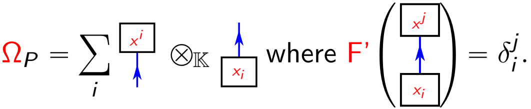


Non degeneracy of the m-trace

Theorem

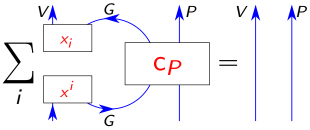
The m-trace t *on* Proj *is non degenerate: The pairing* $\mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, P) \times \mathrm{Hom}_{\mathscr{C}}(P, \mathbb{1}) \to \mathbb{K}$ *given by* $(f, g) \mapsto \mathsf{t}_P(f \circ g)$ *is non degenerate.*

The copairing, given by any dual bases, is then a well defined element $\Omega_P = \sum_i x^i \otimes_{\mathbb{K}} x_i \in \mathrm{Hom}_{\mathscr{C}}(P, \mathbb{1}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, P)$.



The chromatic map as a "non semisimple Kirby color"

There is no non-semisimple Kirby color but we have an analogous notion introduced in [CGPT18]: A chromatic map for a projective generator G, based on a projective object P, is a map $\mathsf{c}_P \in \mathrm{End}_{\mathscr{C}}(G \otimes P)$ such that $\forall V \in \mathscr{C}$,



where $\{x_i\}_i$ and $\{x^i\}_i$ are dual bases with respect to the m-trace.

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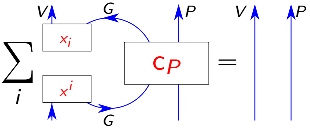

where $\{x_i\}_i$ and $\{x^i\}_i$ are dual bases with respect to the m-trace.

Theorem:[CGPV] Finite tensor category have chromatic maps.

arXiv:2305.14626 [pdf, ps, other] math.QA

Chromatic maps for finite tensor categories

Authors: Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand, Alexis Virelizier

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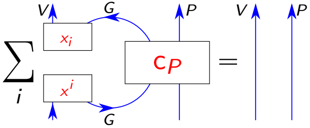
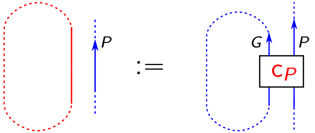


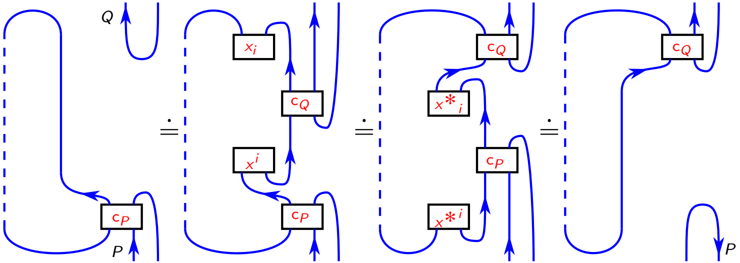
where $\{x_i\}_i$ and $\{x^i\}_i$ are dual bases with respect to the m-trace.
Theorem:[CGPV] Finite tensor category have chromatic maps.

Chromatic maps allow to define skein elements of $\mathscr{S}\!k(M)$ with ribbon circles colored by a "virtual" object (red). Red circles have the slidding property.

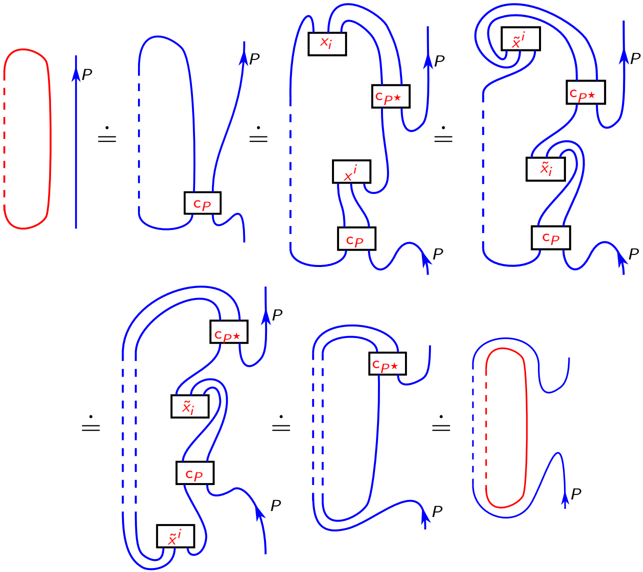
Extension of $\mathcal{S}\!k(M)$ to bichrome graphs with red circles:



Independent of the choices made to turn the red circle blue:



Sliding property for the chromatic map



Summary for ribbon chromatic category

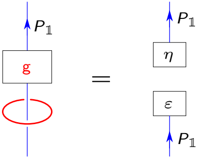
A ribbon chromatic category $\mathscr{C}$ has

1. Distinguished objects: the unit $\mathbb{1}$, its projective cover $\mathbb{1} \xrightarrow{\eta} P_{\mathbb{1}} \xrightarrow{\varepsilon} \mathbb{1}$, a projective generator G.
2. An m-trace $\{\mathrm{t}_P : \mathrm{End}_{\mathscr{C}}(P) \to \mathbb{K}\}_{P \in \mathrm{Proj}}$ giving a renormalized invariant $\mathrm{F}' : \mathscr{S}\!k(S^3) \to \mathbb{K}$.
3. Copairings in $\mathrm{Hom}_{\mathscr{C}}(P, \mathbb{1}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, P)$:
$$\left\{\Omega_P = \sum_i x^i \otimes_{\mathbb{K}} x_i\right\}_{P \in \mathrm{Proj}}.$$
4. Chromatic maps $\{\mathrm{c}_P \in \mathrm{End}_{\mathscr{C}}(G \otimes P)\}_{P \in \mathrm{Proj}}$.

Only one algebraic tool is missing in this list to extend $\mathscr{S}\!k$ to a TQFT following ideas of Walker-Reutter and using the presentation of $\mathbf{Cob}_{3+1}$ by Juhász.

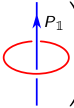
Gluing map

A gluing map $\mathsf{g} \in \mathrm{End}_{\mathscr{C}}(P_{\mathbb{1}})$ is a morphism of $\mathscr{C}$ that satisfies



i.e. $\mathsf{g} \circ \Delta^0 = \eta \circ \varepsilon$

where $\Delta^0 = \mathsf{F}\left(\right) = \mathrm{ptr}^L_G(b^2 \circ \mathsf{c}_{P_{\mathbb{1}}}) \in \mathrm{End}(P_{\mathbb{1}})$.



There is no gluing map if $\Delta^0 = 0$!

Existence of gluing map

Theorem

If $\Delta^0 \neq 0$, there exists a gluing map.

The proof uses the Frobenius structure of $\text{End}_{\mathscr{C}}(P_{\mathbb{1}}) = \mathbb{K}\,\text{Id} \oplus J$ given by the m-trace:

$$\text{End}_{\mathscr{C}}(P_{\mathbb{1}}) \times \text{End}_{\mathscr{C}}(P_{\mathbb{1}}) \to \mathbb{K} \quad (f, g) \mapsto \mathsf{t}_{P_{\mathbb{1}}}(f \circ g).$$

The morphism $\eta \circ \varepsilon$ is orthogonal to J thus $J^{\perp} = \mathbb{K}\eta \circ \varepsilon$.

If $(P_{\mathbb{1}} \xrightarrow{f} P_{\mathbb{1}}) \neq 0$, choose $g \in J^k$ such that $g \circ f \neq 0$ with k maximal.
Then $\forall n \in J$, $n \circ g \circ f = 0$ so $g \circ f \in J^{\perp} = \mathbb{K}\eta \circ \varepsilon$.
Then λg is a gluing map.

Summary for ribbon chromatic non-degenerate category

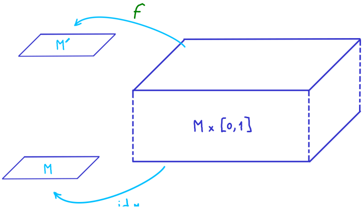
A ribbon chromatic non-degenerate category $\mathscr{C}$ has

1. An m-trace $\{\mathrm{t}_P : \mathrm{End}_{\mathscr{C}}(P) \to \mathbb{K}\}_{P \in \mathrm{Proj}}$ giving a renormalized invariant $\mathrm{F}' : \mathscr{S}\!k(S^3) \to \mathbb{K}$.
2. Copairings in $\mathrm{Hom}_{\mathscr{C}}(P, \mathbb{1}) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, P)$:
$$\left\{\Omega_P = \sum_i x^i \otimes_{\mathbb{K}} x_i\right\}_{P \in \mathrm{Proj}}.$$
3. Chromatic maps $\{\mathrm{c}_P \in \mathrm{End}_{\mathscr{C}}(G \otimes P)\}_{P \in \mathrm{Proj}}$.
4. A gluing map $\mathrm{g} : P_{\mathbb{1}} \to P_{\mathbb{1}}$.

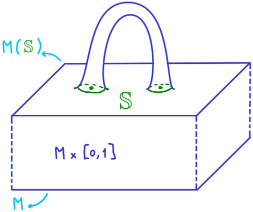
This is all you need to extend $\mathscr{S}\!k : \mathscr{M}an_3 \to \mathrm{Vect}$ to a non compact $3+1$-TQFT
$$\mathscr{S}\!k : \mathbf{Cob}_{3+1} \to \mathrm{Vect}$$

Juhász's presentation of the smooth (n+1)-cobordism category

- Generators:
  1. mapping cylinders indexed by **diffeomorphisms** f of n-manifold,
  2. k-handles on top of trivial cylinder indexed by their **framed attaching sphere** $\mathbb{S} \simeq S^{k-1} \times B^{n-k+1}$.



and



$\{W(f)\}_{f:M \xrightarrow{\sim} M'}$ $\{W(\mathbb{S})\}_{\mathbb{S} \subset M}$

Juhász's presentation of **Cob**

Juhász's presentation of the smooth (n+1)-cobordism category

- Generators:
  1. mapping cylinders indexed by **diffeomorphisms** f of n-manifold,
  2. k-handles on top of trivial cylinder indexed by their **framed attaching sphere** $\mathbb{S} \simeq S^{k-1} \times B^{n-k+1}$.
- Relations:
  - R1 composition of diffeo ; isotopies have trivial mapping cylinders
  - R2 conjugation of attaching sphere by diffeo
  - R3 handles with disjoint attachment commute
  - R4 (k)–$(k+1)$ handle cancellation when $\text{belt}_k \pitchfork S^k = \{*\}$
  - R5 handles do not depend of the orientation of the attaching sphere

Let $\mathcal{F}$ be the free category whose objects are closed oriented n-manifolds and morphisms generated by generators.

Theorem (Juhász, 2018)

$$\mathcal{F}/R \cong \textbf{Cob}$$

Juhász's presentation of **Cob**

Juhász's presentation of the smooth (n+1)-cobordism category

- Generators:
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Let $\mathscr{F}^{\mathrm{nc}}$ be the subcategory of $\mathscr{F}$ with no 0-handle.

Corollary

$$\mathscr{F}^{\mathrm{nc}}/R^{\mathrm{nc}} \cong \mathbf{Cob}^{\mathrm{nc}}$$

$\mathscr{C}$ is a ribbon chromatic non-degenerate category.

$$\text{3-manifold } M \longmapsto \mathcal{S}k(M) \text{ admissible skein module}$$

is functorial from the category of diffeomorphisms between 3-manifolds.

We need to associate maps $\chi_{\mathbb{S}} = \mathcal{S}k(W(\mathbb{S}))$

$$\mathcal{S}k(M) \xrightarrow{\chi_{\mathbb{S}}} \mathcal{S}k(M(\mathbb{S}))$$

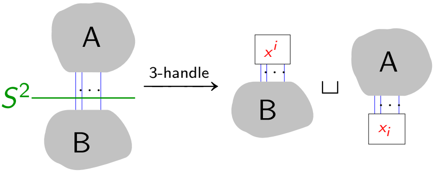
$$[M, \Gamma] \overset{\chi_{\mathbb{S}}}{\longmapsto} [M(\mathbb{S}), \Gamma']$$

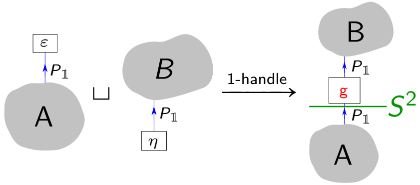
to any 4,3,2,1-handle and check the relations.
Here if $W(\mathbb{S})$ comes from gluing a $k+1$-handle, then $M(\mathbb{S})$ is obtained from M by index $k+1$ surgery.

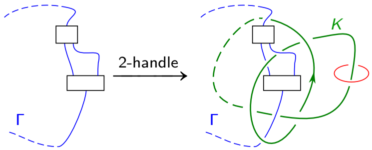
Summary for $\mathscr{Sk}$(index k handles)

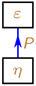
$$[M,\Gamma_1]\sqcup[S^3,\Gamma_2] \xrightarrow{\text{4-handle}} F'(\Gamma_2).[M,\Gamma_1]$$

A and B might be
or not be connected.





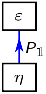


$[M,\Gamma] \xrightarrow{\text{0-handle}} [M,\Gamma]\sqcup[S^3,\zeta$  $]$

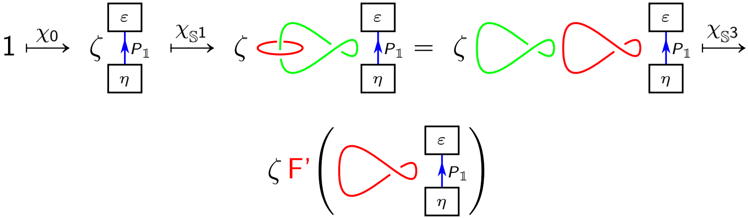
only for $\mathscr{C}$ chromatic compact

i.e. if $\mathsf{g} = \zeta^{-1}\mathrm{Id}$ is invertible

Example: $\mathbb{C}P^2$

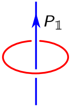
The 4-manifold $\mathbb{C}P^2$ is obtained by attaching successively exactly one index 0, 2 and 4-handle. The index 0 handle introduces ζ 

in S^3. The index 2 handle is attached along a 1-framed unknot. The surgered manifold is still S^3 and the meridian is a (-1)-framed unknot. The index 4 handle will evaluate this skein in S^3 using F'.



A ribbon chromatic category $\mathscr{C}$ is

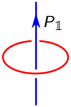
1. **chromatic non degenerate** if



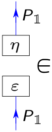
$\neq 0 \in \mathrm{End}_{\mathscr{C}}(P_{\mathbb{1}})$

$\Longrightarrow$ $\mathscr{S}\!k$ extends to a non-compact TQFT.

2. **chromatic compact** if



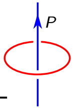
$= \zeta$



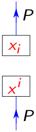
$\in \mathrm{End}_{\mathscr{C}}(P_{\mathbb{1}})$

$\Longrightarrow$ $\mathscr{S}\!k$ extends to a TQFT.

3. **factorizable** if $\forall P \in \mathrm{Proj}$,



$= \zeta \sum_i$



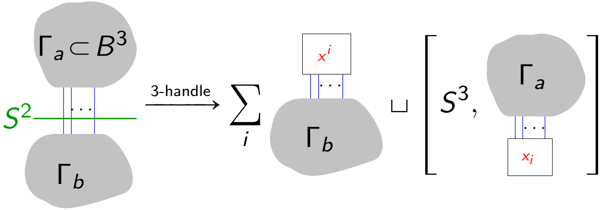
$\in \mathrm{End}_{\mathscr{C}}(P)$

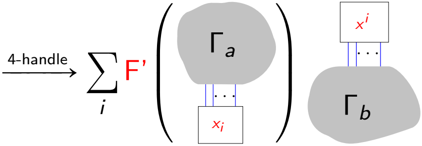
$\Longrightarrow$ $\mathscr{S}\!k$ extends to an invertible TQFT.

$\mathscr{C}$ factorizable $\Longrightarrow$ chromatic compact $\Longrightarrow$ chromatic non degenerate

3-4-handle cancellation

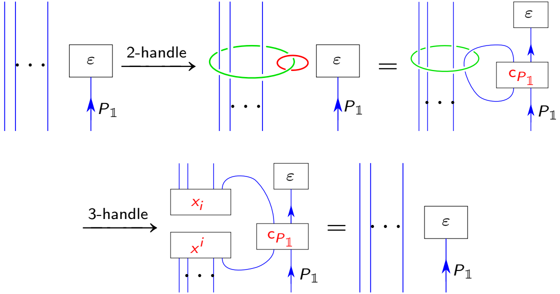
3-4-handle cancellation happens when 2-surgery (cutting) create a 3-sphere S^3 which is then killed by a 4-handle.





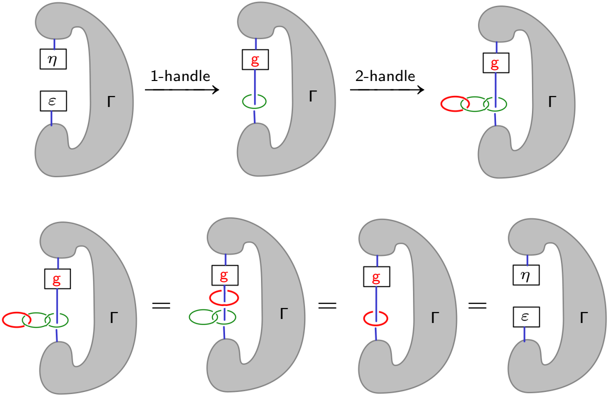
2-3-handle cancellation

2-3-handle cancellation happen when we do surgery along an unknot, creating a 2-sphere that is then cut by a 2-surgery.



1-2-handle cancellation

1-2-handle cancellation happens when we self connect a component of M_1, creating a $S^2 \times S^1$ factor which is then killed by $\mathbb{S}^1$-surgery:

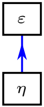


0-handle for chromatic compact $\mathscr{C}$

A gluing map is invertible iff $\mathscr{C}$ is chromatic compact.
Then $\mathrm{g} = \zeta^{-1}\,\mathrm{Id}_{P_{\mathbb{1}}}$ and the the map

$$\mathcal{S}k(M) \xrightarrow{\text{0-handle}} \mathcal{S}k(M \sqcup S^3)$$

is adding a component S^3 which contains the skein

$\Gamma_0 = \zeta$ 

And the 0-1 handle cancellation is a consequence of $\varepsilon \circ \mathrm{g} = \zeta^{-1}\varepsilon$.

What about exotic 4-manifolds ?

There are no example but no obstruction to the possibility of detecting exotic pairs of 4-manifolds.

$$\mathscr{Sk}(W_1 \sharp W_2) \approx \mathscr{Sk}(W_1)\mathscr{Sk}(W_2)$$

So we are looking for a chromatic non degenerate category $\mathscr{C}$ for which

$$\mathscr{Sk}(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}) = \mathscr{Sk}(S^2 \times S^2) = 0$$

For $\mathscr{C} = H$-mod (H a ribbon Hopf algebra) it would mean that

$$\lambda(\theta)\lambda(\theta^{-1}) = \lambda \otimes \lambda(R_{21}R_{12}) = 0$$

Thank you for your attention!