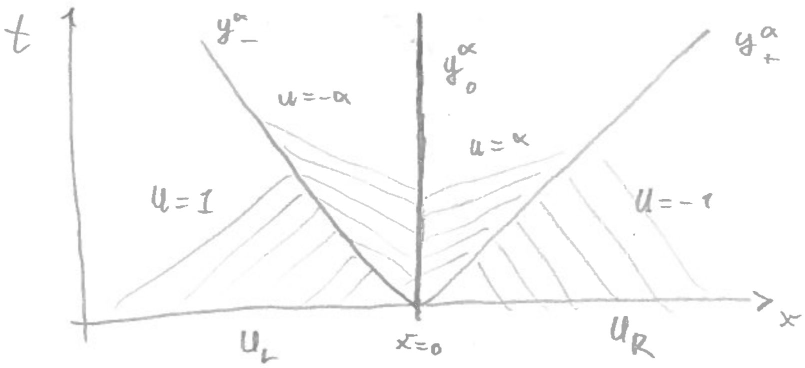
Def Assume (u_L, u_R) satisfies RH-conditions, i.e. $s[u] = [f]$.
We say that (u_L, u_R) satisfies the "Lax (entropy) condition" if

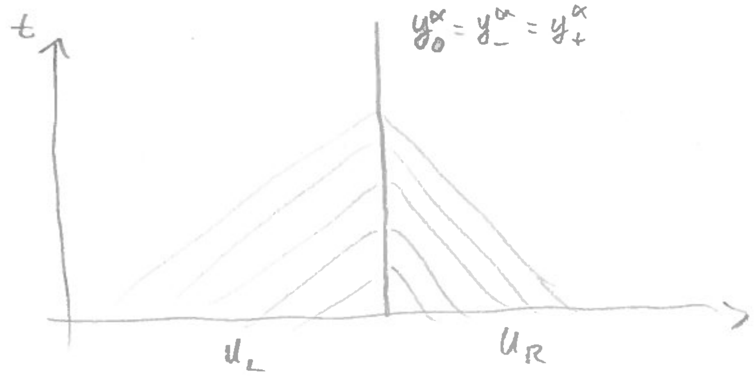
$$\boxed{f'(u_L) > s > f'(u_R).} \qquad (LC)$$

Remk:
We get uniqueness of discontinuous solutions by only allowing for discontinuities that satisfy (LC).
In Example 1, only the solution u_α for $\alpha = 1$ meets (LC)!



Case $\alpha \geq 1$

Discontinuity across y^α_0 violates (LC) $\forall \alpha > 1$, since $f'(u_L) = -\alpha$, $s = 0$, $f'(u_R) = \alpha$ (recall $f(u) := \frac{1}{2}u^2$, $f'(u) = u$).



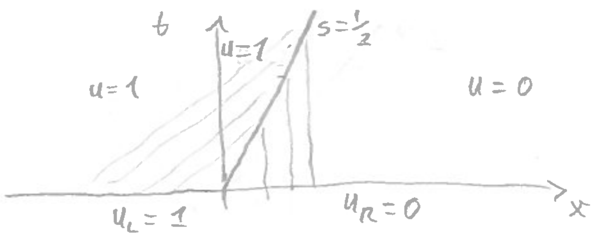
Case $\alpha = 1$

All good!

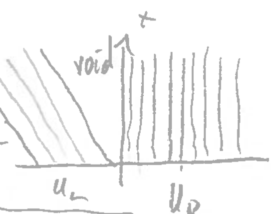
Example 2: • Consider $u_t + uu_x = 0$ with $u_0 = \begin{cases} +1, & x < 0 \\ 0, & x > 0 \end{cases}$

To meet (LC) & RH-condition we solve this with shock surface of speed $s = \frac{1}{2}$, and constant states:

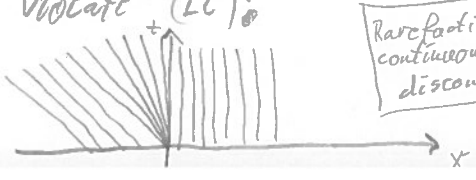
$$u(t,x) := \begin{cases} +1, & x < \frac{1}{2}t \\ 0, & x > 0 \end{cases}$$



• Consider $u_t + uu_x = 0$ with $u_0 = \begin{cases} 0, & x < 0 \\ +1, & x > 0 \end{cases}$.



This results in void region which is not influenced by initial data.
Any discontinuity in void region would violate (LC)!
↳ solve with "Rarefaction Wave":



Rarefaction Wave continuously dissolves discontinuity.

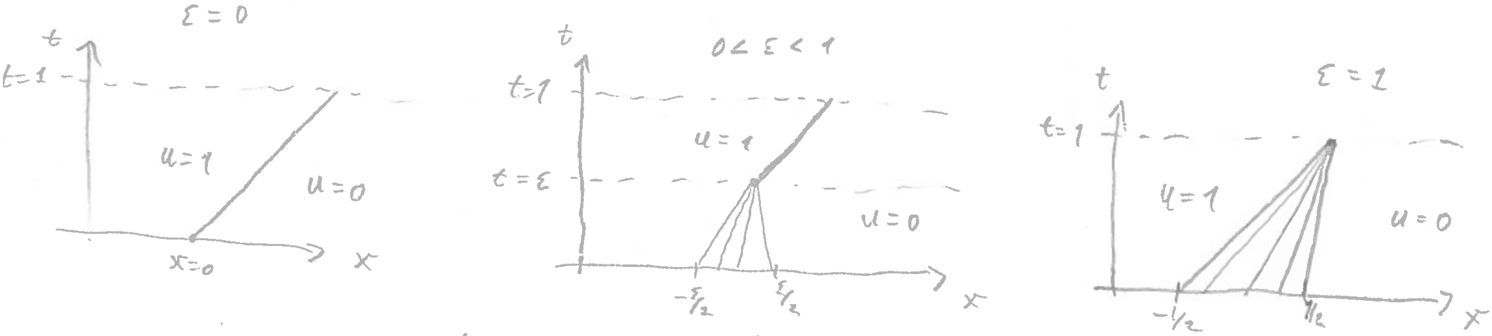
III.5) Irreversibility

Example:

For each $\varepsilon \in [0,1]$, we define solution u_ε of $u_t + uu_x = 0$ by

For $t \le \varepsilon$:
$$u_\varepsilon(t,x) := \begin{cases} 1, & x < t - \varepsilon/2 \\ \dfrac{x - \varepsilon/2}{t - \varepsilon}, & x \in \left(t - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) \\ 0, & x > \varepsilon/2 \end{cases}$$

For $t > \varepsilon$:
$$u_\varepsilon(t,x) := \begin{cases} 1, & x < \frac{t}{2} \\ 0, & x > \frac{t}{2} \end{cases}$$



Each solution u_ε satisfies Lax condition, but still, knowing solution at $t=1$ does not suffice to determine solution for $t<1$.

↳ Irreveribility of discontinuous solutions!

Rmk:

- Lax condition only gives unique solution "forward" in time, "backward" in time Lax condition is automatically violated which results in "backward in time non-uniqueness" (= irreversibility).
- Discontinuous solutions of conservation laws can be approximated by parabolic PDE's (Method of artificial viscosity),

$$u_t + f(u)_x = 0$$
$$\uparrow \varepsilon \searrow 0$$
$$u_t + f(u)_x = \varepsilon B u_{xx}$$

and parabolic PDE's fail to be time-reverible.

IV) Riemann Problems

IV.1) Basic Notions

- Let $u(t,x)$, $f(u) \in \mathbb{R}^N$.
  Consider system of Conservation Laws

$$\boxed{u_t + f(u)_x = 0} \qquad (CL)$$

- Def:
  Let $u_L, u_R \in \mathbb{R}^N$ be constant, (so called "constant states").
  Solving (CL) for u with initial data $u(0,x) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases}$

  is called "Riemann Problem" for (CL) with data (u_L, u_R).

- Observe, (CL) $\Leftrightarrow$ $u_t + \underbrace{df|_u}_{=: A(u)} \cdot u_x = 0$ (CL')
  $A(u)$, $N\times N$-matrix

  Def: Let $M \subset \mathbb{R}^N$, open.
  We say (CL) is "strictly hyperbolic" in M if there exists N mutually distinct eigenvalues $\lambda_1(u), \dots, \lambda_N(u)$ of $A(u)$ $\forall u \in M$.
  Then, we assume WLOG that $\lambda_1(u) < \dots < \lambda_N(u)$.

  Rmk:
  If (CL) is strictly hyperbolic, then (CL) is symmetrizable.

- From now on we assume (CL) is strictly hyperbolic.
- Note, if $A(u) = \text{diag}(\lambda_1(u), \dots, \lambda_N(u))$, then (CL') were a system of N scalar conservation laws $u^k_t + \lambda_k(u)\, u^k_x = 0$, $k \in \{1, \dots, N\}$.
  Then each component u^k is constant along $t \mapsto \gamma_a(t)$, characteristic line, defined by $\begin{cases} \dot\gamma_a(t) = \lambda_k(u(t,\gamma_a(t))) \\ \gamma_a(0) = a \end{cases}$! This motivates for general case:

  Def: We define the k-th "characteristic curve" of (CL), at $a \in \mathbb{R}$, as the solution $\gamma_a(t)$ of ODE $\begin{cases} \dot\gamma_a(t) = \lambda_k(u(t,\gamma_a(t))) \quad [\in \mathbb{R}] \\ \gamma_a(0) = a. \end{cases}$

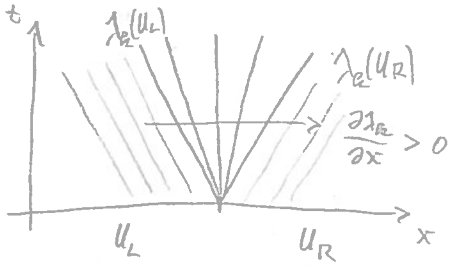
- **Def**

  Let u be a C^1-solution of

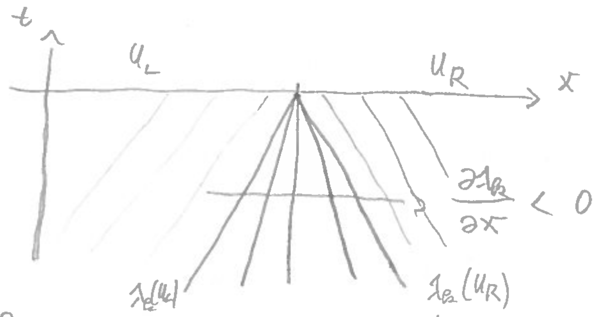
  We call u a "k-simple wave" if u is constant along the k-th characteristic curve.

  We call a k-simple wave u a "k-rarefaction wave", ("k-compression wave")

  if $\frac{\partial}{\partial x}\lambda_k(u(t,x)) > 0$, $\left(\text{if } \frac{\partial}{\partial x}\lambda_k(u(t,x)) < 0\right)$.



k- Rarefaction Wave



k- Compression Wave

[Recall that $\lambda_k(u)$ gives the slope of the k-th characteristic curve. Thus, $\frac{\partial\lambda_k}{\partial x} > 0$ means this slope is increasing as one move from left to right in above picture.

- For (CL) to admit k-rarefaction waves, (CL) must be genuinely nonlinear:

Def: Let $r_k(u)$ be eigenvector of $A(u)$ with $A r_k = \lambda_k r_k$, and $|r_k| = 1$.

We call (λ_k, r_k) the "k-th characteristic field" of (CL).

We call (λ_k, r_k) "genuinely nonlinear" in $\mathcal{M} \subset \mathbb{R}^N$ if

$$\langle r_k(u), \nabla\lambda_k(u)\rangle_{\mathbb{R}^N} > 0 \quad \forall u \in \mathcal{M}.$$

We call (λ_k, r_k) "linearly degenerate" in $\mathcal{M}$ if

$$\langle r_k(u), \nabla\lambda_k(u)\rangle_{\mathbb{R}^N} = 0 \quad \forall u \in \mathcal{M}.$$

Generalizes condition $f''(u) > 0$ for scalar systems.

- Recall that a weak solution of (CL), which is discontinuous across the line $\{x=st\}$, satisfies the Rankine Hugoniot condition

$$\boxed{s[u]=[f(u)]} \qquad \text{(RH)}$$

<u>Def:</u> Assume the k-th characteristic field is linearly degenerate.
We call a (weak) solution of the Riemann problem (u_L, u_R) for (CL) a "contact discontinuity", if it is discontinuous across $\{x=st\}$ and C^1 otherwise, such that

$$\lambda_k(u_L) = s = \lambda_k(u_R).$$

<u>Def:</u> Assume the k-th char. field is genuinely nonlinear.
We call a (weak) solution of the Riemann problem (u_L,u_R) for (CL) which is discontinuous across $\{(t,x=st)\,|\,t\geq 0\}$ and C^1 otherwise an "(admissible) k-shock wave" if s meets "Lax condition":

$$\boxed{\lambda_k(u_L) > s > \lambda_k(u_R)} \qquad \text{(LC)}$$

- <u>Example:</u>
The Euler equations (MA), (MO), (E) can be written in Lagrangian coordinates, replacing the internal energy e by the entropy s by 1^{st} Law of Thermod.

$$\text{ds} \quad \left\{\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ s_t &= 0. \end{aligned}\right\}, \quad \Longleftrightarrow \quad \partial_t \begin{pmatrix} v \\ u \\ s \end{pmatrix} + \overbrace{\begin{bmatrix} 0 & -1 & 0 \\ p_v & 0 & p_s \\ 0 & 0 & 0 \end{bmatrix}}^{=:A} \partial_x \begin{pmatrix} v \\ u \\ s \end{pmatrix} = 0,$$

where $v := \frac{1}{\rho}$ & $p=p(v,s)$, such that $p_v < 0$.
The eigenvalues of A are

$$\underline{\lambda_1 = -\sqrt{-p_v}}, \quad \underline{\lambda_2 = 0}, \quad \underline{\lambda_3 = \sqrt{-p_v}}.$$

$\Rightarrow$ System is strictly hyperbolic.
& If $p_{vv} > 0$, then λ_1 & λ_3 are genuinely nonlinear, while λ_2 is linearly degenerate.

- **Thm** ("Lax's Existence Theorem", 1957)

  Assume (CL) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear. Let $u_L \in \mathbb{R}^N$ be a constant state.

  Then there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^N$ of u_L such that for each $u_R \in \mathcal{U}$ the Riemann problem (u_L, u_R) for (CL) has a solution.

  This solution consists of at most $(N+1)$-constant states separated by admissible shock waves, rarefaction waves and contact discontinuities.

  There is precisely one solution of this form.

- Idea of proof:

  Omit contact discontinuities! (→ All char. fields are gen. nonlinear)

  Step 1: Parameterize state space $\mathcal{U}$ by shock & rarefaction curves!

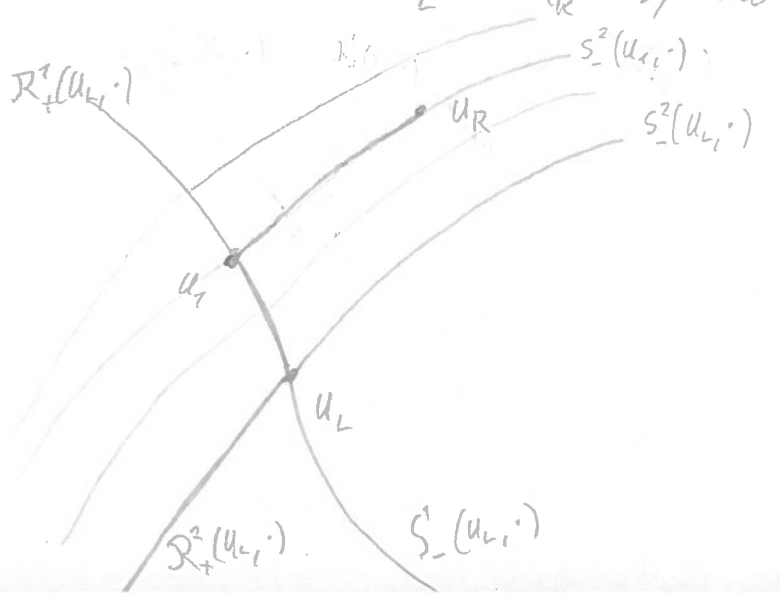
  - Rarefaction curves $\mathcal{R}^1, \dots, \mathcal{R}^N$: $\frac{d\mathcal{R}^k}{d\varepsilon} = r_k(\mathcal{R}^k)$, $\mathcal{R}^k$ k-th rarefaction curve

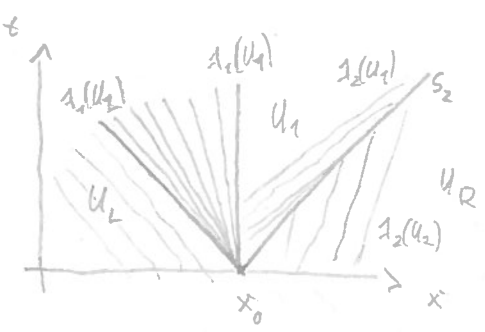
    ↳ $\mathcal{R}^k$ is flow of r_k, k-th eigenvector

  - Shock curves $S^1, \dots, S^N$ are implicit solutions of "Hugoniot locus"

    $$u - u_L - s\big(f(u) - f(u_L)\big) = 0$$

  Step 2: Find the unique intermediate state $u_1, \dots, u_{N-1} \in \mathcal{U}$ which connect u_L to u_R by the parameterization!





- Order curves so that wave speeds increase!
- Use shock curve for negative parameter and rarefaction curve for positive parameter