Smoluchowski coagulation equation with a flux of dust particles

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Smoluchowski's coagulation equation 1917

$f_t(x)$ density of clusters of size $x > 0$ at time $t \geq 0$

$$\partial_t f_t(x) = \mathbb{K}[f](x,t)$$

with

$$\mathbb{K}[f](x,t) := \frac{1}{2}\int_0^x K(x-y,y) f_t(x-y) f_t(y) dy - \int_0^\infty K(x,y) f_t(x) f_t(y) dy.$$

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- mass-conserving solutions [Banasiak-Lamb-Laurençot 2019]

$$M_1(t) = M_1(0), \quad \text{with } M_1(t) := \int_0^\infty x f_t(x)$$

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Applications: coagulation in open systems (input of dust), formation of soot, aerosol growth [Friedlander 2000]

Continuity equation for the mass variable

$xf_t(x)$ mass variable satisfies the continuity equation (for sufficiently regular f)

$$\partial_t(xf_t(x)) + \partial_x J_{f_t}(x) = 0$$

with the mass flux defined by

$$J_{f_t}(x) = \int_0^x \int_{x-y}^{\infty} y\, K(y,z)\, f_t(y)\, f_t(z) \mathrm{d}z \mathrm{d}y$$

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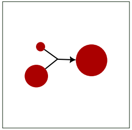
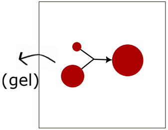
- mass-conserving solutions

$$J_{f_t}(x) \to 0, \quad \text{as} \quad x \to 0 \quad \text{and} \quad x \to \infty$$



- gelling solutions (with mass flux leaving at infinity)

$$J_{f_t}(x) \to 1, \text{ as } x \to \infty$$



- flux solutions (with a constant mass flux from zero)

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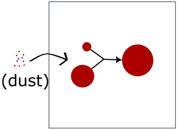


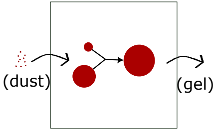
Long time behaviour for flux solutions 1) similar sizes

Class of kernels: $K(x,y) \approx x^{\gamma+\lambda}y^{-\lambda} + x^{-\lambda}y^{\gamma+\lambda}$

1) Region where coagulation between similar sizes dominates: $|\gamma + 2\lambda| < 1$

- stationary solutions: constant flux solutions
  [F., Lukkarinen, Nota, Velázquez 2024]

$$J_f(x) = 1$$



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but this solution is not always unique.

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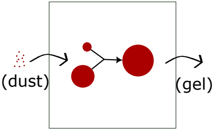

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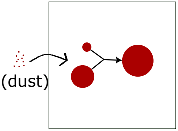
- self-similar solution for homogeneous kernels
  $K(cx, cy) = c^{\gamma}K(x,y)$ with zero initial data $(\gamma < 1)$
  [F., Franco, Velázquez 2022]



$$f_t(x) = \frac{t}{L(t)^2}\Phi\left(\frac{x}{L(t)}\right), \quad L(t) = t^{\frac{2}{1-\gamma}}$$

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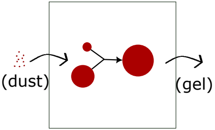

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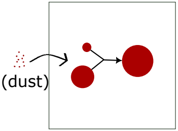
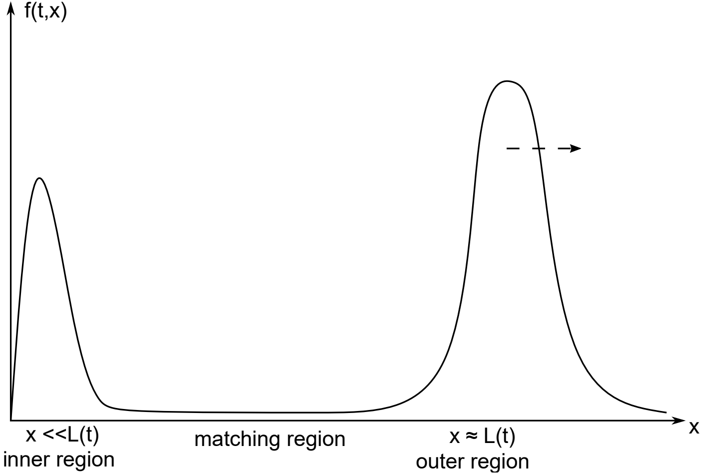


$$f_t(x) = \frac{t}{L(t)^2} \Phi\left(\frac{x}{L(t)}\right), \quad L(t) = t^{\frac{2}{1-\gamma}}$$

Expected long time behaviour: convergence towards a constant flux solution in a self-similar manner. [Davies, King, Wattis 1999]

Long time behaviour for flux solutions 1) similar sizes



Long time behaviour for flux solutions 2) different sizes

Class of kernels: $K(x,y) \approx x^{\gamma+\lambda} y^{-\lambda} + x^{-\lambda} y^{\gamma+\lambda}$

2) Coagulation between particles of different sizes dominates: $|\gamma + 2\lambda| \geq 1$

- No stationary solution exists [F., Lukkarinen, Nota, Velázquez 2021]
- No flux solution is expected to exist

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Main goals:

- To construct a flux solution for general initial data for $|\gamma + 2\lambda| < 1$ and $\gamma < 1$.
- To show non-existence of flux solutions if $|\gamma + 2\lambda| > 1$. (under construction)

Coagulation kernels

We assume that $K \in C(\mathbb{R}_*^2)$ satisfies

$$K(x,y) \geq 0, \quad K(x,y) = K(y,x)$$

$$c_1 \left(x^{\gamma+\lambda} y^{-\lambda} + y^{\gamma+\lambda} x^{-\lambda} \right) \leq K\left(x,y\right) \leq c_2 \left(x^{\gamma+\lambda} y^{-\lambda} + y^{\gamma+\lambda} x^{-\lambda} \right)$$

$$\gamma, \lambda \in \mathbb{R}, \quad c_1, c_2 > 0.$$

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- $\gamma < 1$, (and $\gamma + \lambda, -\lambda < 1$) no gelation, hence
$$M_1(t) = M_1(0) + t$$
- $|\gamma + 2\lambda| < 1$ ensures existence of a constant flux solution, $J_f(x) = 1$
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Motivation:

- *nm* scale: **free molecular kernel** ($\lambda = 1/2$, $\gamma = 1/6$) $\rightarrow$ non-existence
- μm scale: **diffusion kernel** ($\lambda = 1/3$, $\gamma = 0$) $\rightarrow$ existence

Definition (Flux solution, weak formulation)

A time-dependent measure $f \in C([0,T], \mathcal{M}_+(\mathbb{R}_*))$ is a weak flux solution with initial data $f_0 \in \mathcal{M}_+(\mathbb{R}_*)$ such that $xf_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*)$, in case

(i) $xf \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$

(ii) for almost every $(t,z) \in [0,T] \times \mathbb{R}_*$

$$\int_{(0,z]} xf_t(\mathrm{d}x) - \int_{(0,z]} xf_0(\mathrm{d}x) = -\int_0^t J_{f_s}(z)\mathrm{d}s + t \tag{1}$$

where

$$\int_0^t J_{f_s}(z)\mathrm{d}s := \int_0^t \iint_{\Omega_z} xK(x,y) f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s \tag{2}$$

is finite for all $t \in [0,T]$ and all $z \in \mathbb{R}_*$, with
$\Omega_z := \{(x,y) \in \mathbb{R}_*^2 \colon 0 < x \leq z, z - x < y\}$.

Properties of flux solutions

Proposition

Let $\gamma < 1$. Then f satisfies the weak coagulation equation

$$\int_{(0,\infty)} x\varphi(t,x)f_t(dx) = \int_{(0,\infty)} x\varphi(0,x)f_0(dx) + \int_0^t \int_{(0,\infty)} x\partial_s\varphi(s,x)f_s(dx)ds$$

$$+ \frac{1}{2}\int_0^t \int_{(0,\infty)} \int_{(0,\infty)} K(x,y)[(x+y)\varphi(s,x+y) - x\varphi(s,x) - y\varphi(s,y)]f_s(dx)f_s(dy)d$$

for every $\varphi \in C_c^1([0,T] \times \mathbb{R}_*)$ and almost every $t \in [0,T]$, together with the flux boundary condition (in some weak sense),

$$\int_0^t J_{f_s}(z)\mathrm{d}s \to t, \text{ as } z \to 0, \quad \text{a.e. } t \in [0,T].$$

Properties of flux solutions

Proposition (Mass is linearly increasing)

Let $\gamma < 1$ and $|\gamma + 2\lambda| < 1$. Then,

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Fix $\varepsilon > 0$ arbitrarily. Since $|\gamma + 2\lambda| < 1$, there is a small positive δ such that

$$J^1_{f_t}(z;\delta) + J^3_{f_t}(z;\delta) \leq \varepsilon C_T.$$

On the other hand, using the upper bound of the kernel, it holds

$$J^2_{f_t}(z;\delta) \leq C \int_{[\frac{\delta}{1+\delta}z,\infty)} x^\gamma f_t(dx) \int_{[\frac{\delta}{1+\delta}z,\infty)} x f_t(dx)$$

Since $M_1(f_t) < \infty$, for all $t \in [0, T]$, and $\gamma < 1$, there is a large enough z_*, depending on ε and δ, such that, for all $z > z_*$,

$$J^2_{f_t}(z;\delta) \leq \varepsilon C_T.$$

Properties of flux solutions

→ f behaves like a constant flux solution near zero

- Upper bound

$$f_t(\mathrm{d}x) \lesssim \frac{1}{x^{\frac{\gamma+3}{2}}} C_t(t + M_1(f_0)), \quad x > 0$$

- Asymptotic lower bound

  For each t there is a constant $\delta > 0$ and a constant b, satisfying $0 < b < 1$, such that,

$$f_t(x) \gtrsim \frac{1}{x^{\frac{\gamma+3}{2}}} C_{t,b}, \quad x \in \left(0, \frac{\delta}{\sqrt{b}}\right)$$

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→ no dust in the system

$$\int_0^t \int_{(0,x_0]} x f_s(\mathrm{d}x)\mathrm{d}s \leq C_T x_0^{\frac{1-\gamma}{2}}.$$

Existence of flux solutions

Theorem

Assume that $|\gamma + 2\lambda| < 1$ *and* $\gamma < 1$. *Given an initial data* $f_0 \in \mathcal{M}_+(\mathbb{R}_*)$ *such that the mass measure satisfies* $xf_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*)$, *there exists a weak flux solution in the sense of the Definition.*

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Proposition (Coagulation equation with constant-in-time source term)

Assume that $-\lambda, \gamma + \lambda < 1$. Let $f_0 \in \mathcal{M}_+(\mathbb{R}_*)$ be the initial data, with $\mathrm{spt}(f_0) \subset [a, +\infty)$ for some $a > 0$ and $xf_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*)$. Assume that $\eta \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ is a source term with $\mathrm{spt}(\eta) \subset [a, +\infty)$. Then, for every $T > 0$, there exists a weak solution $f \in C([0, T], \mathcal{M}_+(\mathbb{R}_*))$ to

$$\partial_t f_t = \frac{1}{2}\int_0^x K(x-y, y) f_t(x-y) f_t(y) \mathrm{d}y + \int_0^\infty K(x, y) f_t(x) f_t(y) \mathrm{d}y + \eta(x).$$

[Escobedo, Mishler 2006] time-dependent source, homogeneous kernels with $\gamma \in [0, 1)$

Remark: Interestingly, solutions with source also exist for $|\gamma + 2\lambda| \geq 1$.

[Cristian, F., Franco, Nota, Lukkarinen, Velázquez 2023]

Construction of a flux solution

- For each $\varepsilon \in (0,1)$, let f^{ε} be a solution to the coagulation equation with source $\eta_{\varepsilon} = \frac{1}{\varepsilon}\delta_{\varepsilon}$ and initial data $f_0|_{[\varepsilon,+\infty)}$
- For each $M \in \mathbb{N}$, consider the family of the solutions restricted to the closed interval $I_M = [2^{-M}, 2^M]$.

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Construction of a diagonal sequence

- $M = 1$, by compactness we find a limit point F^1 and a sequence $(\varepsilon_i)_{i=1}^\infty$ such that $xf^{\varepsilon_i}|_{I_1} \to F^1$.
- $M = 2$, by compactness we find a limit point F^2 and a subsequence $(\varepsilon_{i_k})_{k=1}^\infty$ such that $xf^{\varepsilon_{i_k}}|_{I_2} \to F^2$. Moreover, $F^2|_{I_1} = F^1$.
- ...

Candidate solution as the limit of a diagonal subsequence

- Take a diagonal subsequence $(\varepsilon(i))_{i=1}^{\infty}$ and a limiting function F_t, defined pointwise in time by

$$\langle \varphi, F_t \rangle = \lim_{i \to \infty} \left\langle \varphi, x f^{\varepsilon(i)}|_{I_i} \right\rangle, \quad \varphi \in C_c(\mathbb{R}_*)$$

- $t \mapsto F_t$ is continuous
- canditate solution: $f \in C([0,T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, such that $xf = F$.

- Final step: Show that f verifies the flux equation in the sense of the Definition.

Long time behaviour for the constant kernel

Theorem

If the coagulation kernel is constant, $K(x, y) \equiv 2$, there exists a unique solution f_t to the flux equation with the initial data $f_0 = 0$. This solution converges weakly as a measure on $\mathbb{R}_*$ to the stationary solution of the flux equation, i.e.,

$$f_t(\mathrm{d}x) \to \frac{1}{\sqrt{2\pi}} x^{-\frac{3}{2}} \mathrm{d}x, \quad t \to \infty.$$

The proof relies on the use of the Bernstein transform $B_{f_t}(\lambda) = \int_{\mathbb{R}_*} (1 - e^{-\lambda x}) f_t(dx)$.

Non-existence (under construction)

Theorem

If $\gamma + 2\lambda > 1$ *then there are no flux solutions in the sense of the definition satisfying the pointwise in time upper bound*

$$\frac{1}{R}\int_{[R/2,R]} f_t(\mathrm{d}x) \le \frac{1}{R^{\frac{\gamma+3}{2}}} C_T, \quad R > 0 \tag{3}$$

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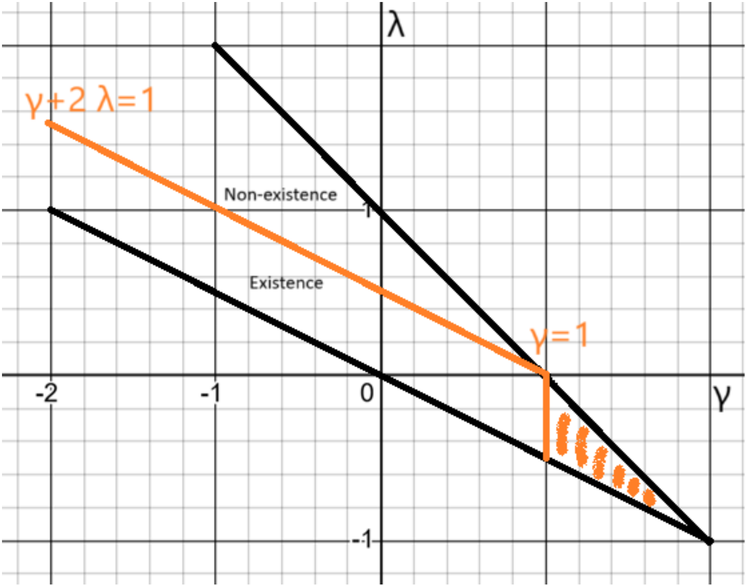
The idea is to use the boundedness of the flux to obtain a bound for the moment $1-\lambda$ near the origin. Then the upper estimate (3) allow to conclude that (in some weak sense)

$$\int_0^t \int_{(0,z]} \int_{(z-x,\infty)} xK(x,y) f_s(dy) f_s(dx) ds \leq C_T z^{\frac{\gamma+2\lambda-1}{2}}.$$

Therefore, taking $z \to 0$ and using $\gamma + 2\lambda > 1$, yields $\int_0^t J ds \to 0$ as $z \to 0$, which is a contradiction.


λ
γ+2 λ=1
Non-existence
1
Existence
γ=1
-2
-1
0
γ
-1

Coagulation equation with a source term $\gamma + 2\lambda > 1$

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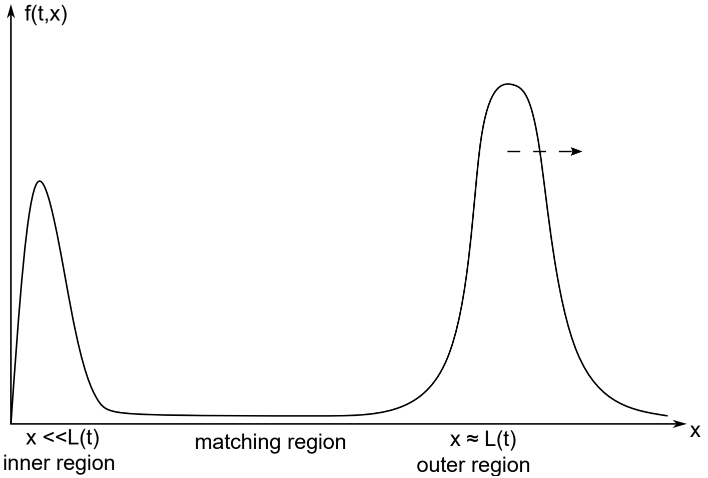
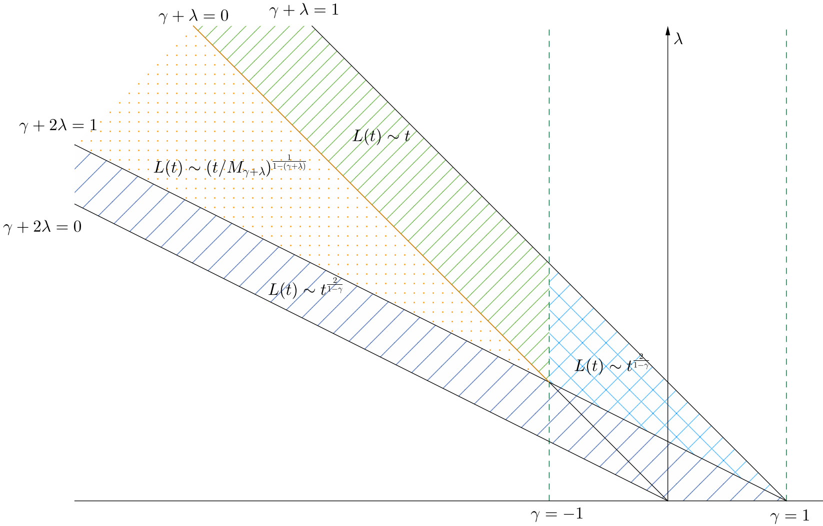


Anomalous self-similarity



[F., Franco, Velázquez 2022], [Cristian, F., Franco, Velázquez 2023], [F., Franco, Nota, Lukkarinen, Velázquez 2023]

Thank you for your attention!