Lagrangian mean curvature flow and the Gibbons-Hawking ansatz

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  - A (unique) sLag exists in a Hamiltonian isotopy class of Lagrangians iff a stability condition holds (Thomas).
  - A Lagrangian can be decomposed into a sequence of sLags $\sim$ Harder–Narasimhan filtration (Thomas–Yau, Douglas, Bridgeland, Joyce).

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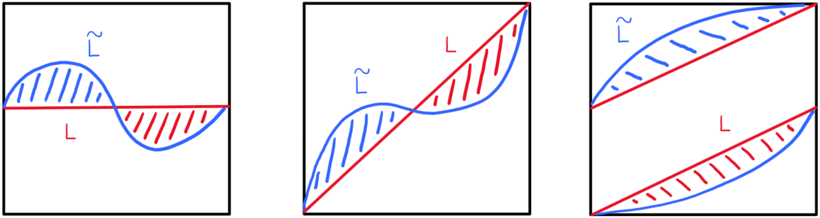

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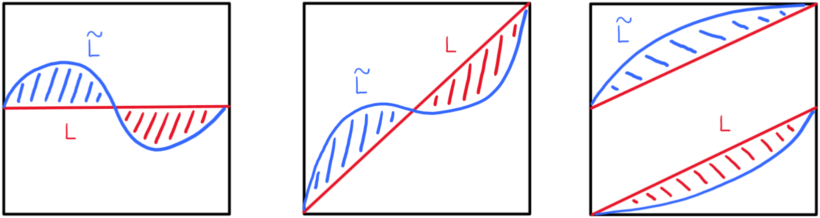

- Straight lines are the "best" representatives.

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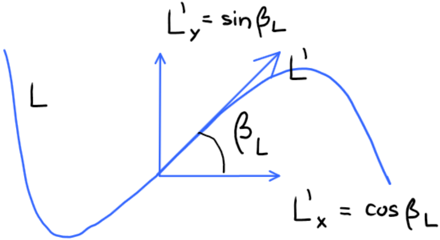

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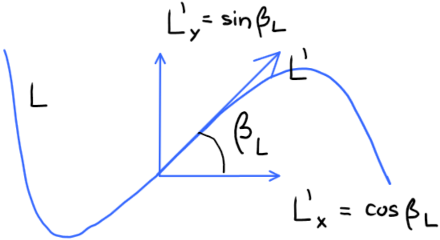

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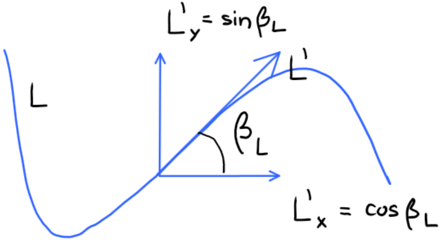

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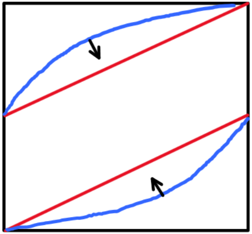

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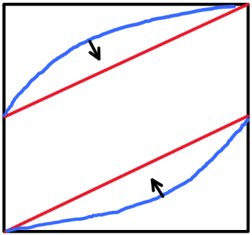

In the direction of its curvature $= L'' \rightsquigarrow$ Curve shortening flow $\{L_t\}_{t\geq 0}$:
$$\partial_t L_t = L_t''$$

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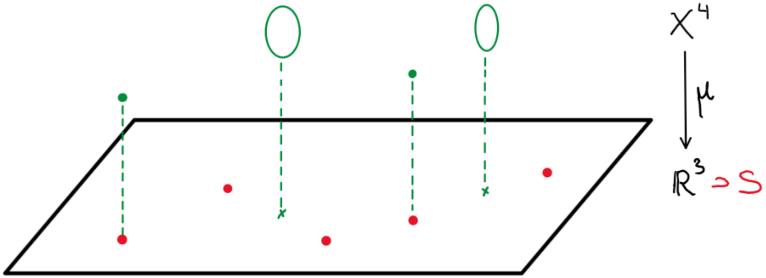

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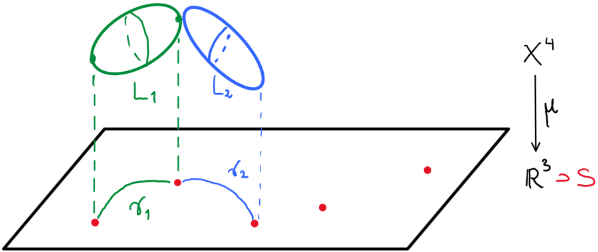

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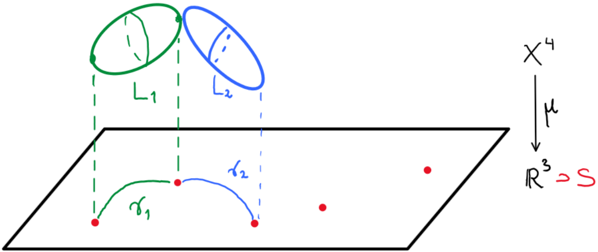

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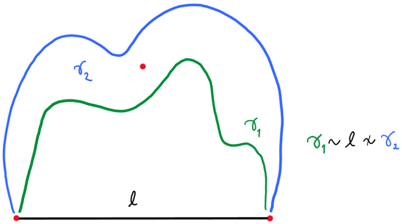

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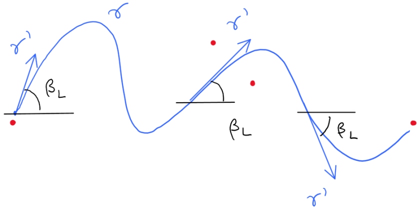

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$\implies$ L is a sLag iff γ is a straight line.

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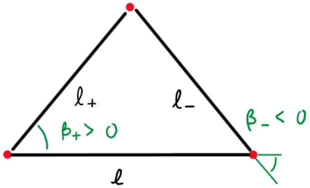

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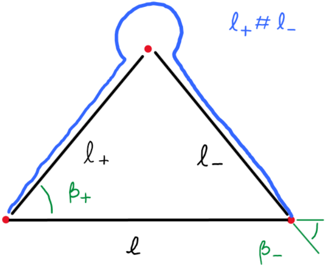

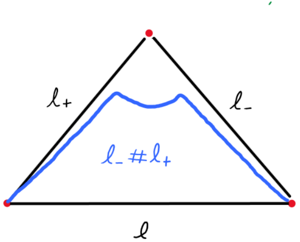

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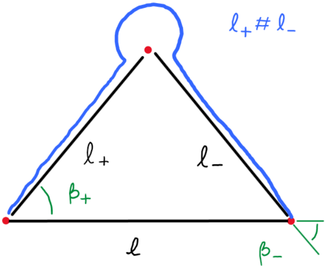

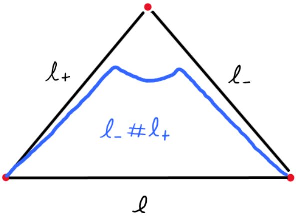

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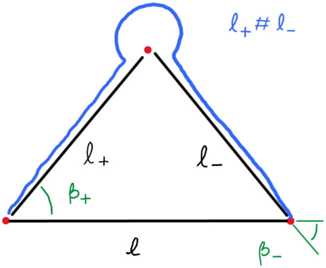

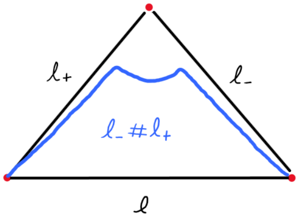

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  $\mu^{-1}(\ell)$ is the unique circle-invariant sLag in $[L]$.

Tackling the Conjectures

Theorem (Jason Lotay, –)

$L \subseteq X$ a circle-invariant Lagrangian sphere, then:

$$L \sim_{\mathrm{U}(1)} \textit{sLag} \quad \Leftrightarrow \quad L \textit{ is } \mathrm{U}(1)\textit{-stable}.$$

In this situation, such a sLag is unique.

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Conclusion: We have proved circle-invariant versions of:

- The Thomas conjecture.
- The Thomas–Yau conjecture.

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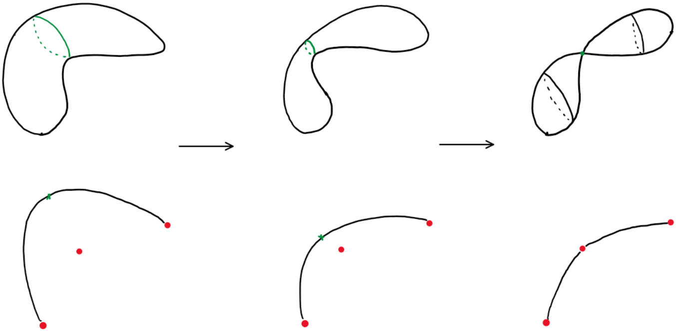

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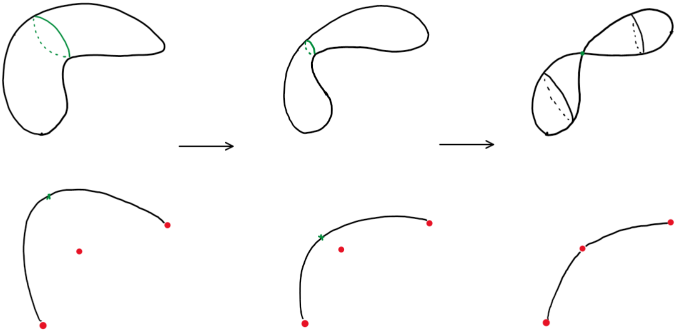

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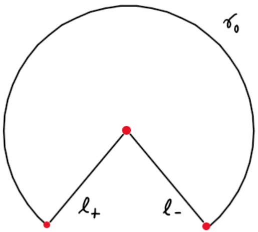
- Consider, for example, the convex curve $\gamma_0 \sim \ell_+ \# \ell_-$.



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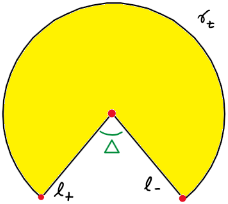

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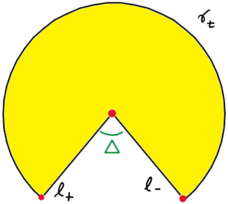

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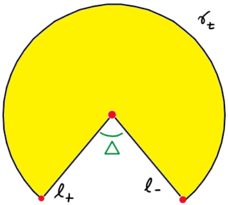

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$\implies$ there is a finite time singularity as otherwise $\mathrm{Area}(D_t)$ would become zero in finite time.

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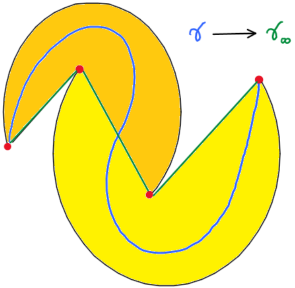

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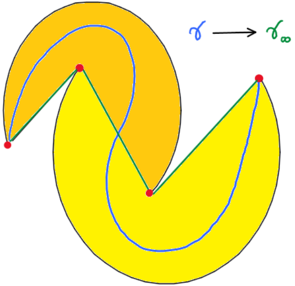
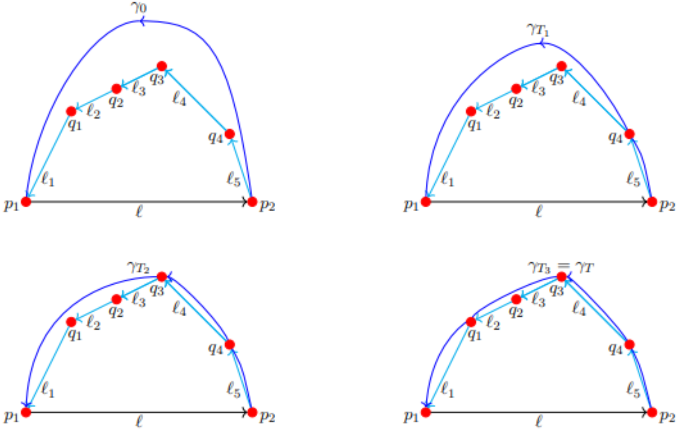


3. Restart the flow (for a piecewise smooth curve) after a singularity.

Flow through singularities and long-time convergence

Theorem (Jason Lotay, –)



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Let L_0 be a circle-invariant, almost calibrated Lagrangian in X. There is a continuous family $\{L_t\}_{t\in[0,+\infty)}$ so that the following holds.

(a) $\exists\, 0 < T_1 \leq \ldots \leq T_l < \infty$ *such that the family* $\{L_t\}_{t\in[0,\infty)\setminus\{T_1,\ldots,T_l\}}$ *satisfies MCF.*

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- Proves a circle invariant version of one of Joyce's conjectures.

Obrigado!

Structure of finite time "neck pinch" singularities

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Let L_0 be embedded, almost calibrated, circle-invariant and $\{L_t\}_{t\in[0,T)}$ with the smooth LMCF starting at L_0 developing a finite time singularity at $p \in X$ when $t \to T < \infty$.

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