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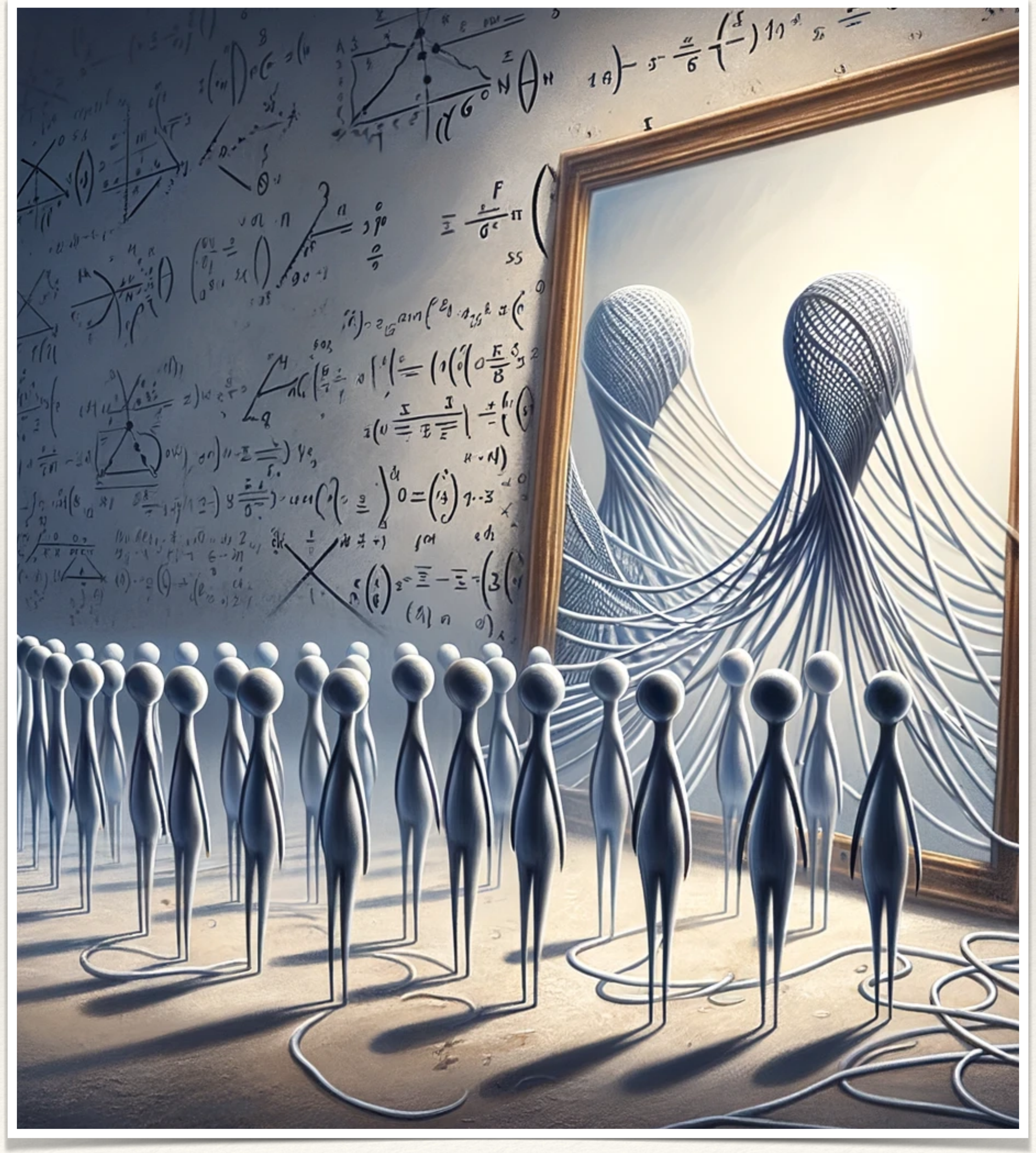
The stringor bundle and the spinor bundle on loop space

joint with Peter Kristel
and Konrad Waldorf

SFB 1085 Higher invariants
University of Regensburg



Index theory

Let M be a compact spin manifold.

$$\mathbb{Z} \quad \ni \quad \mathrm{ind}(D)$$

$$\dim\ker(D) - \dim\ker(D^*)$$

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$\dim\ker(D) - \dim\ker(D^*)$

Atiyah-Singer
index theorem

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… with an S^1-action

$$\mathbb{Z}[q] \quad \ni \quad \mathrm{ind}_{S^1}(D)$$

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dim ker(D) − dim ker(D^*)

Atiyah-Singer index theorem

… with an S^1-action

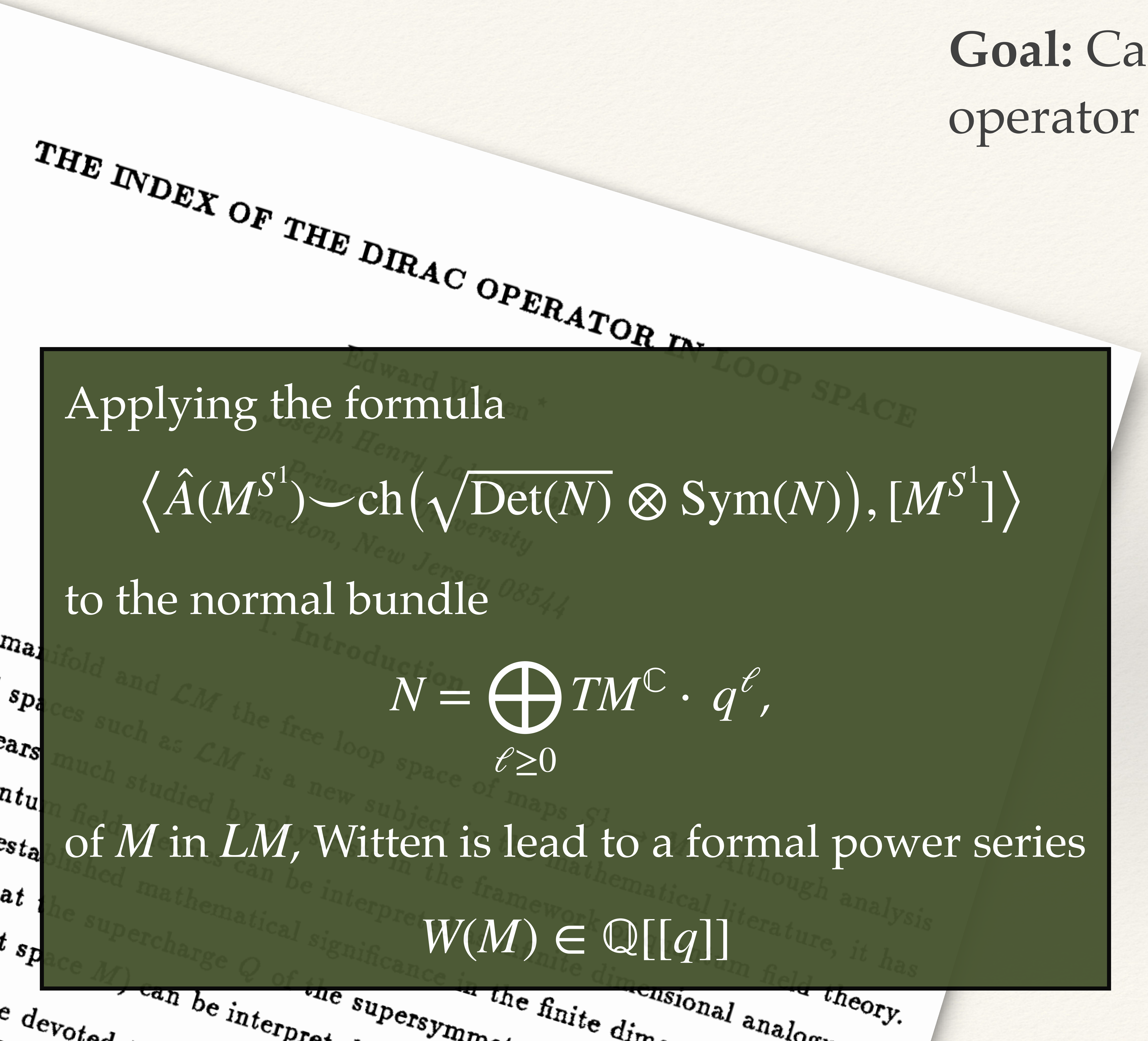
normal bundle of M^{S^1} in M

$$\mathbb{Z}[q] \quad \ni \quad \operatorname{ind}_{S^1}(D) \quad = \quad \left\langle \hat{A}(M^{S^1}) \smile \operatorname{ch}\left(\sqrt{\operatorname{Det}(N)} \otimes \operatorname{Sym}(N)\right), [M^{S^1}] \right\rangle$$

Atiyah-Segal fixed point formula

Goal: Calculate the index of the Dirac operator on $LM = C^\infty(S^1, M)$.

Loop rotation turns LM into an S^1-manifold, with fixed point set $LM^{S^1} = \{\text{constant loops}\} \cong M$



THE INDEX OF THE DIRAC OPERATOR IN LOOP SPACE

Edward Witten *

Joseph Henry Laboratories

Princeton University

Princeton, New Jersey 08544

1. Introduction

Applying the formula

$$\left\langle \hat{A}(M^{S^1}) \smile \mathrm{ch}\left(\sqrt{\mathrm{Det}(N)} \otimes \mathrm{Sym}(N)\right), [M^{S^1}] \right\rangle$$

to the normal bundle

$$N = \bigoplus_{\ell \geq 0} TM^{\mathbb{C}} \cdot q^{\ell},$$

of M in LM, Witten is lead to a formal power series

$$W(M) \in \mathbb{Q}[[q]]$$

Wittens idea: Just formally apply the fixed point formula to LM.

Witten genus and the string condition

$$\begin{tikzcd}
 & & \mathrm{tmf}_* \arrow[d] & \text{if M is string [Ando-Hopkins-Rezk]} & \\
 & & \mathrm{MF}^*_{\mathbb{Z}} \arrow[dl] \arrow[dr] & & \\
\text{if $p_1(M)$ vanishes rationally [Zagier]} & \mathrm{MF}^* \arrow[dr, "\text{Fourier series}"] & & \mathbb{Z}[[q]] \arrow[dl] & \text{if M is spin} \\
W(M) \arrow[rr] & & \mathbb{Q}[[q]] & \text{if M is oriented} &
\end{tikzcd}$$

$\frac{1}{2}p_1(M) = 0$ integrally

Each coefficient is the index of a twisted Dirac operator on M

What should this be used for?

Conjecture. [Stolz '95] If a compact string manifold M admits a metric of positive Ricci curvature, then $W(M) = 0$.

Hope: The loop space Dirac operator satisfies a „Lichnerowicz formula“ involving the Ricci curvature of M

Hope: $2|1$-dimensional extended ETF's are cocycles for tmf.

Has been applied to string theory. [Tachikawa-Yamashita '24]

Conjecture. [Stolz-Teichner '05] The Witten genus is the partion function of a $2|1$-dimensional extended euclidean field theory with target M.

Problem: There is no loop space Dirac operator

This works on any spin manifold, hence cannot be used to tackle the Stolz conjecture

Taubes ['89] constructed a Dirac operator on ``formal loop space” that has the correct index.

Hope: Using the string structure, one may construct an extension of Taubes' Dirac operator to the full loop space (with the same index) which satisfies some Lichnerowicz type formula using the Ricci curvature.

spinor bundle

Problem: There is no loop space ~~Dirac operator~~

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Taubes ['89] constructed a Dirac operator on ``formal loop space'' that has the correct index.

Hope: Using the string structure, one may construct an extension of Taubes' Dirac operator to the full loop space (with the same index) which satisfies some Lichnerowicz type formula using the Ricci curvature.

In this talk: The loop space spinor bundle.

Spin structures in finite dimensions

Definition. A spin structure on an oriented manifold M is a reduction of its structure group $\mathrm{SO}(d)$ to the spin group $\mathrm{Spin}(d)$.

There is an irreducible graded representation S of $\mathrm{Spin}(d)$ on which the Clifford algebra $\mathrm{Cl}(\mathbb{R}^d)$ acts compatibly with the action of $\mathrm{SO}(d)$ on $\mathrm{Cl}(\mathbb{R}^d)$, the *spinor representation*.

Given a spin structure on M, we may form the spinor bundle

$$\mathcal{S} = \mathrm{Spin}(M) \times_{\mathrm{Spin}(d)} S,$$

which is a bundle of graded left modules for $\mathrm{Cl}(TM)$.

The means that automorphisms of the Clifford algebra are implemented on S, $\tilde{g}c(v)\tilde{g}^*\Psi = c(gv)\Psi$.

Spin structures in finite dimensions

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which is a bundle of graded left modules for $\mathrm{Cl}(TM)$.

The Dirac operator is then given by Clifford multiplication:

$$D = \sum_{j=1}^{d} c(e_j) \nabla_{e_j}$$

The loop space spinor bundle

LM is a Fréchet-manifold with tangent bundle $TLM = LTM$.

The structure group of the tangent bundle of LM is canonically reduced to $L\mathrm{O}(d)$, in the sense that

$$TLM \cong L\mathrm{O}(M) \times_{L\mathrm{O}(d)} L\mathbb{R}^d$$

If M is oriented or spin, it is further reduced to $L\mathrm{SO}(d)$, resp. $L\mathrm{Spin}(d)$.

Definition. [Killingback '87] A *loop space spin structure* for a spin manifold M is a lift of structure groups for the principal $L\mathrm{Spin}(d)$-bundle $L\mathrm{Spin}(M)$ to the basic central extension $\widetilde{L\mathrm{Spin}(d)}$.

Fact. [Pressley-Segal '86] If $d \geq 5$, there exists a unitary graded representation S of $\widetilde{L\mathrm{Spin}(d)}$ on which the Clifford algebra $\mathrm{Cl}(L\mathbb{R}^d)$ acts compatibly, called the *spinor representation*.

The loop space spinor bundle

So given a loop space spin structure P, we may form a loop space spinor bundle

$$\mathcal{S} = P \times_{\widetilde{L\mathrm{Spin}(d)}} S,$$

a Hilbert space bundle over LM which is a bundle of graded irreducible modules for the Clifford algebra bundle

$$\mathrm{Cl}(TLM) = L\mathrm{O}(M) \times_{L\mathrm{O}(d)} \mathrm{Cl}(L\mathbb{R}^d).$$

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So what's the problem?

Finding a suitable space of sections for the Dirac operator to act on in order to establish Fredholm properties is a difficult analytic problem.

But there is a more fundamental problem:

Theorem. [McLaughlin '87] A loop space spin structure on a spin manifold M exists if and only if

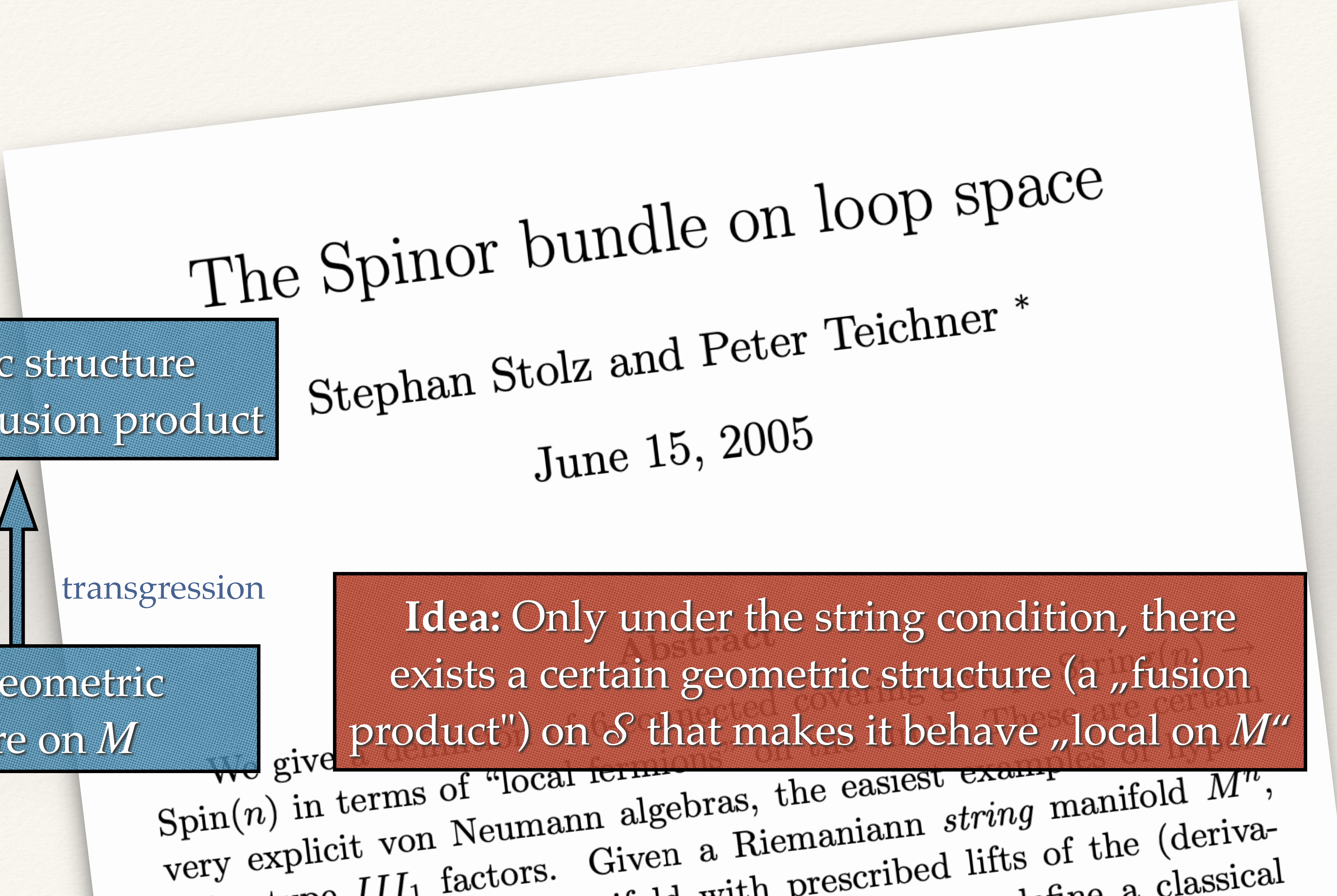
$$\tau\left(\tfrac{1}{2}p_1(M)\right) = 0.$$

This is weaker than the string condition!

For such manifolds, the Stolz conjecture is probably not true, so we must be missing some geometric input.

$$\begin{tikzcd}
H^k(M) \arrow[d, "\mathrm{ev}^*"] \arrow[dd, bend right=60, dashed, "\tau"', "\text{transgression}"' near start] \\
H^k(LM \times S^1) \arrow[d, "/[S^1]"] \\
H^{k-1}(LM)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Geometric structure on } LM \text{ with fusion product} \arrow[d, shift right, "\text{degression}"'] \\
\text{Higher geometric structure on } M \arrow[u, shift right, "\text{transgression}"']
\end{tikzcd}$$



The Spinor bundle on loop space

Stephan Stolz and Peter Teichner *

June 15, 2005

Abstract

We give …

Spin(n) in terms of "local ferm…

very explicit von Neumann algebras, the easiest examp…

type III_1 factors. Given a Riemaniann *string* manifold M^n,

… with prescribed lifts of the (deriva-

… define a classical

Idea: Only under the string condition, there exists a certain geometric structure (a „fusion product") on $\mathcal{S}$ that makes it behave „local on *M*"

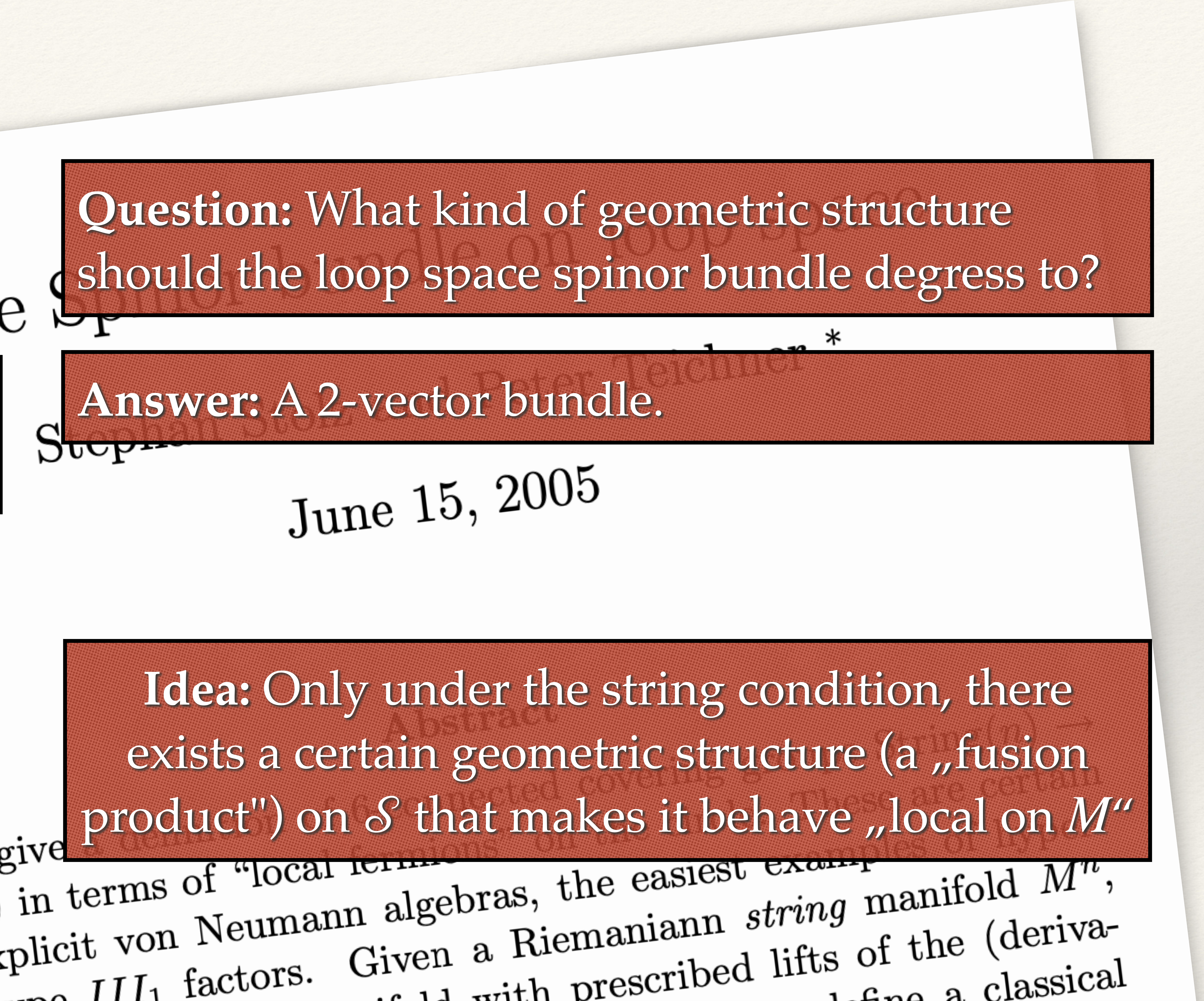
Transgression of bundle gerbes

[Waldorf ’10]

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{bundle gerbes with}\\ \text{connection on } M\end{array}\right\} \arrow[r, "\text{transgression}"] & \left\{\begin{array}{c}\text{complex line bundles with}\\ \text{superficial connection and}\\ \text{fusion product on } LM\end{array}\right\} \\
\mathscr{G} \arrow[r, mapsto] & \mathscr{L}
\end{tikzcd}$$

Comes with the additional structure of a fusion product.

$$\mathscr{L}_\gamma = \left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of trivializations of } \gamma^*\mathscr{G}\end{array}\right\}$$



The Spinor Bundle on Loop Space

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We give

pin(n) in terms of “local fermions” … the easiest examp

very explicit von Neumann algebras, the easiest examp … string manifold M^n,

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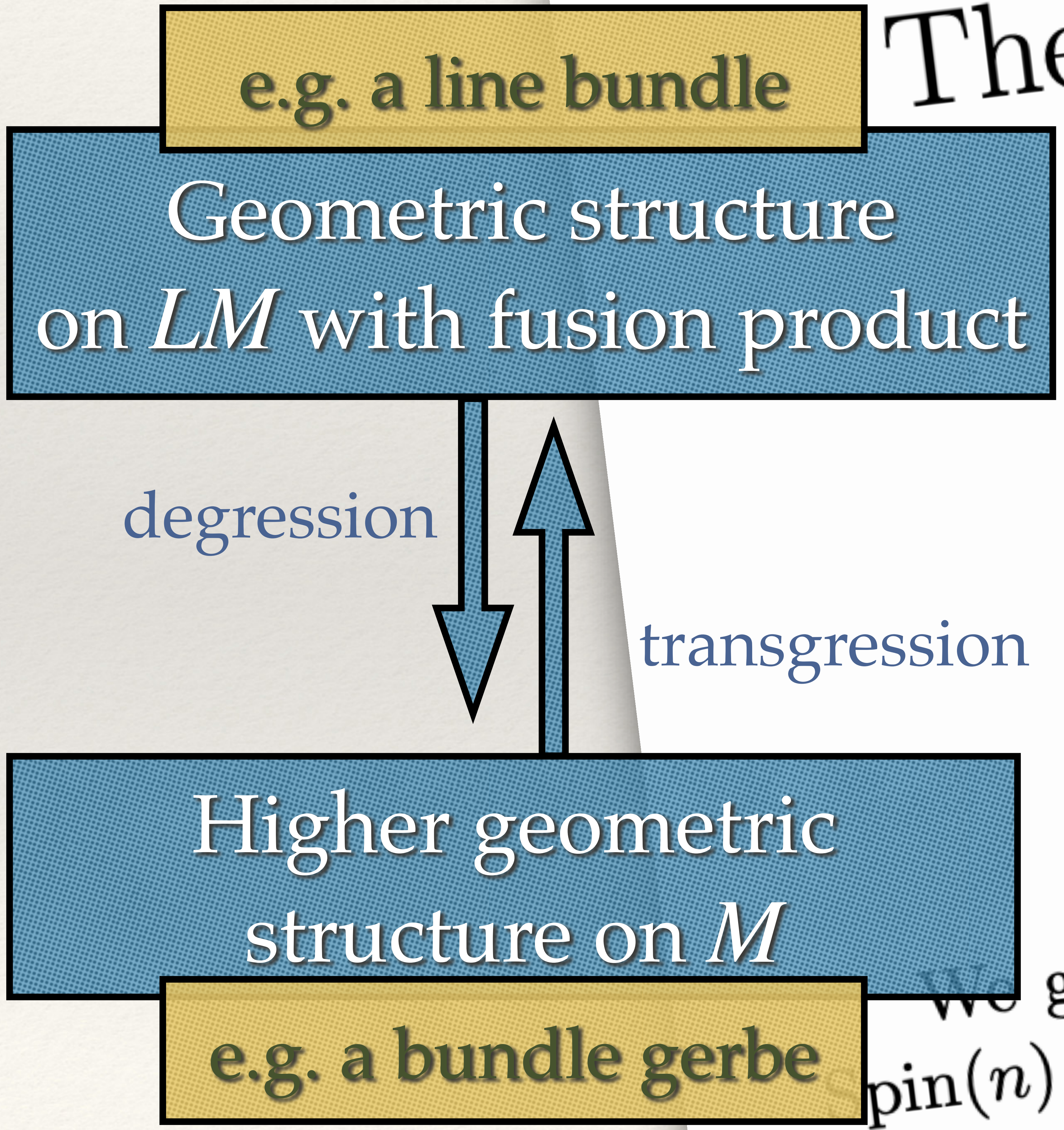
Question: What kind of geometric structure should the loop space spinor bundle degress to?

Answer: A 2-vector bundle.



e.g. a line bundle

Geometric structure on *LM* with fusion product

degression

transgression

Higher geometric structure on *M*

e.g. a bundle gerbe

Idea: Only under the string condition, there exists a certain geometric structure (a „fusion product") on $\mathcal{S}$ that makes it behave „local on *M*“

2-vector bundles

Trivial algebra bundle

bimodule isomorphism

~~Bundle gerbe product~~

$\mathscr{L}_{23} \otimes \mathscr{L}_{12} \longrightarrow \mathscr{L}_{13}$

$$\begin{tikzcd}
\underline{\mathbb{C}} \arrow[d] & \mathscr{L} \arrow[d] & \mu \arrow[d, dotted, dash] & \text{associativity of } \mu \arrow[d, dotted, dash] \\
Y \arrow[d] & Y^{[2]} \arrow[l, shift left=1] \arrow[l, shift right=1] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & Y^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \\
M & & &
\end{tikzcd}$$

2-vector bundles

algebra bundle

$\mathscr{A}_2$-$\mathscr{A}_1$-bimodule bundle

bimodule isomorphism $\mathscr{M}_{23} \otimes_{\mathscr{A}_2} \mathscr{M}_{12} \longrightarrow \mathscr{M}_{13}$

$$\begin{tikzcd}
\mathscr{A} \arrow[d] & \mathscr{M} \arrow[d] & \mu \arrow[d, dotted, no head] & \text{associativity of } \mu \arrow[d, dotted, no head] \\
Y \arrow[d] & Y^{[2]} \arrow[l, shift left] \arrow[l, shift right] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & Y^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \\
M & & &
\end{tikzcd}$$

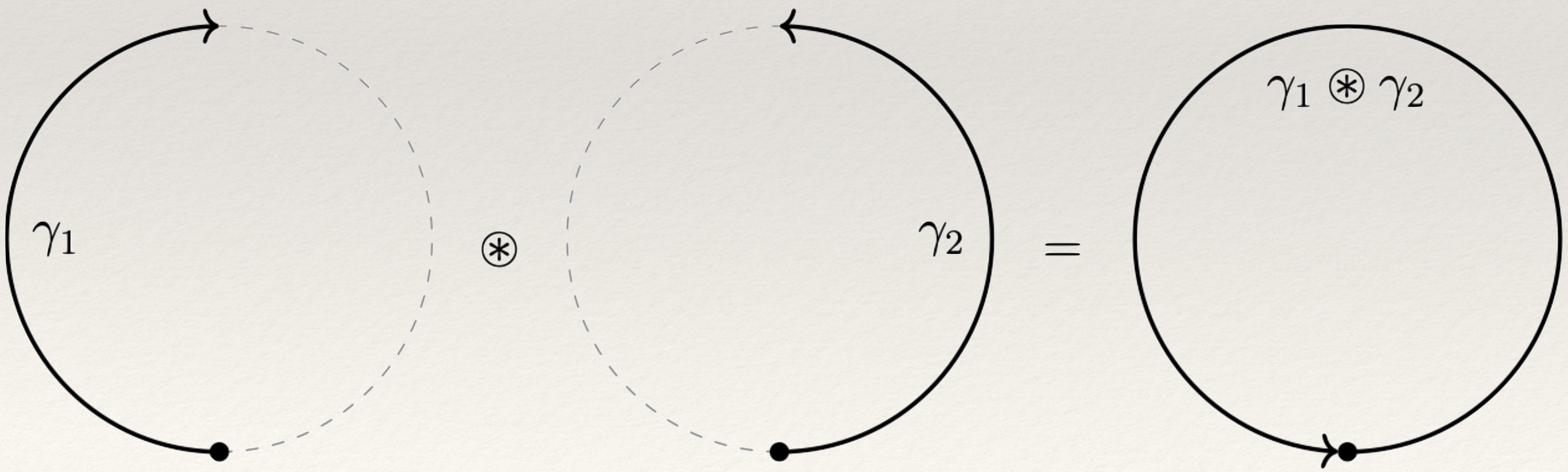
Notice: Both bundle gerbes and algebra bundles are examples of 2-vector bundles.

The stringor bundle

a bundle of graded $\mathrm{Cl}(TPM)_2$-$\mathrm{Cl}(TPM)_1$-bimodules

$$\begin{tikzcd}
\iota^*\mathcal{S} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{S} \arrow[d] \\
PM^{[2]} \arrow[r, "\iota"] & LM
\end{tikzcd}$$

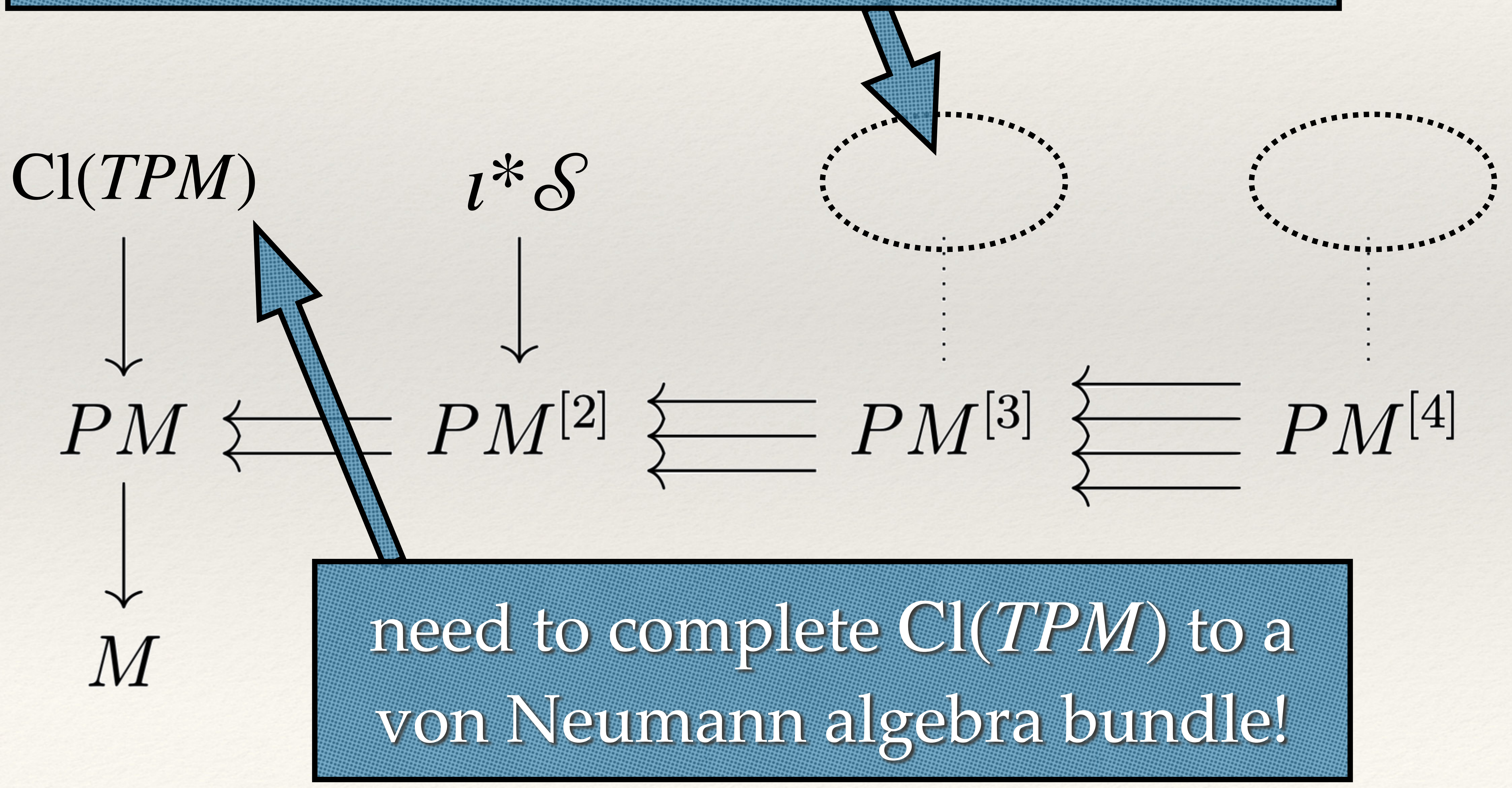
a bundle of graded $\mathrm{Cl}(TLM)$-modules



$\Longrightarrow$ Get inclusion

$$\begin{tikzcd}
\mathrm{Cl}(TPM)_2 \otimes \mathrm{Cl}(TPM)_1^{\mathrm{op}} \arrow[d, "\iota_*"] \\
\mathrm{Cl}(TLM)
\end{tikzcd}$$

The stringor bundle



need an associative bimodule isomorphism
$\iota^*\mathcal{S}_{23} \otimes_{\mathrm{Cl}(TPM)_2} \iota^*\mathcal{S}_{12} \longrightarrow \iota^*\mathcal{S}_{13}$

Problem: These bimodules are not isomorphic!

$$\begin{tikzcd}
\mathrm{Cl}(TPM) \arrow[d] & \iota^*\mathcal{S} \arrow[d] & {} \arrow[d, dash, dotted] & {} \arrow[d, dash, dotted] \\
PM \arrow[d] & PM^{[2]} \arrow[l, shift left] \arrow[l, shift right] & PM^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & PM^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \\
M & & &
\end{tikzcd}$$

need to complete Cl(TPM) to a von Neumann algebra bundle!

The stringor bundle

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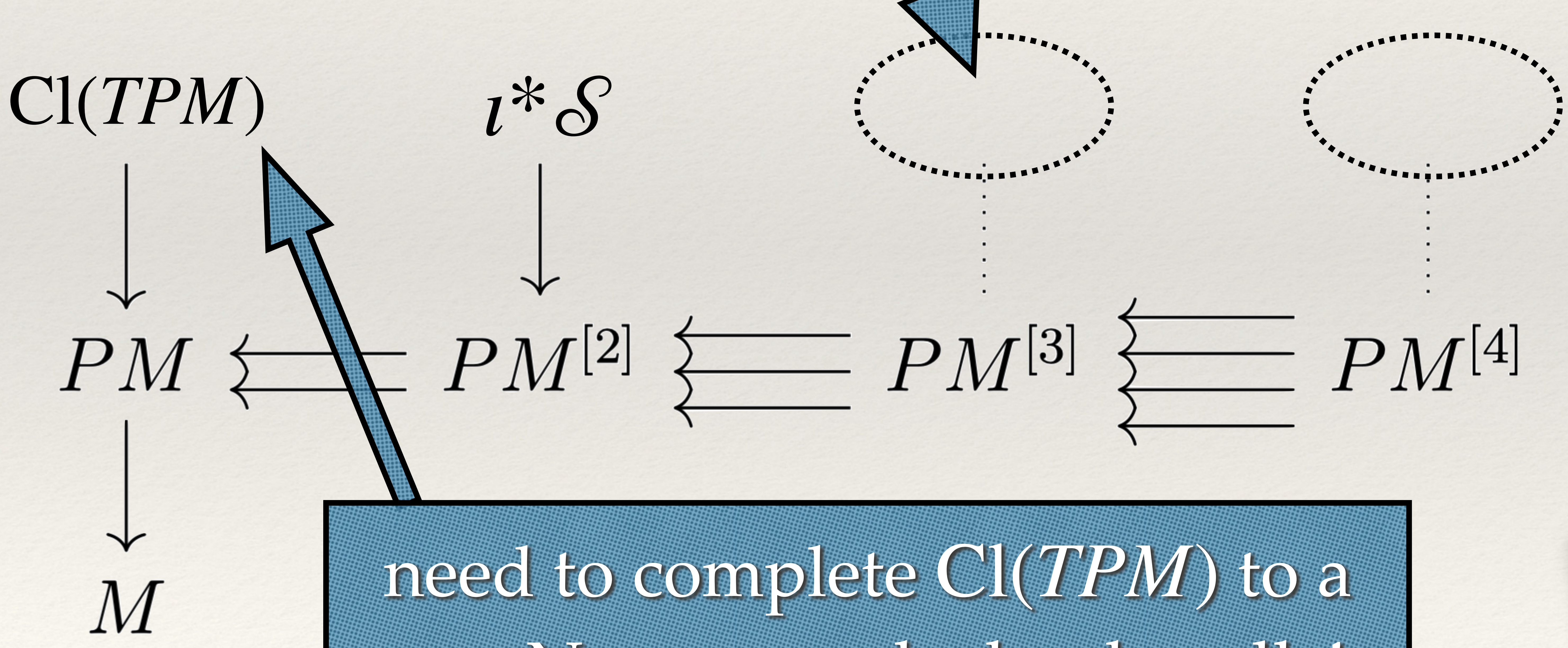
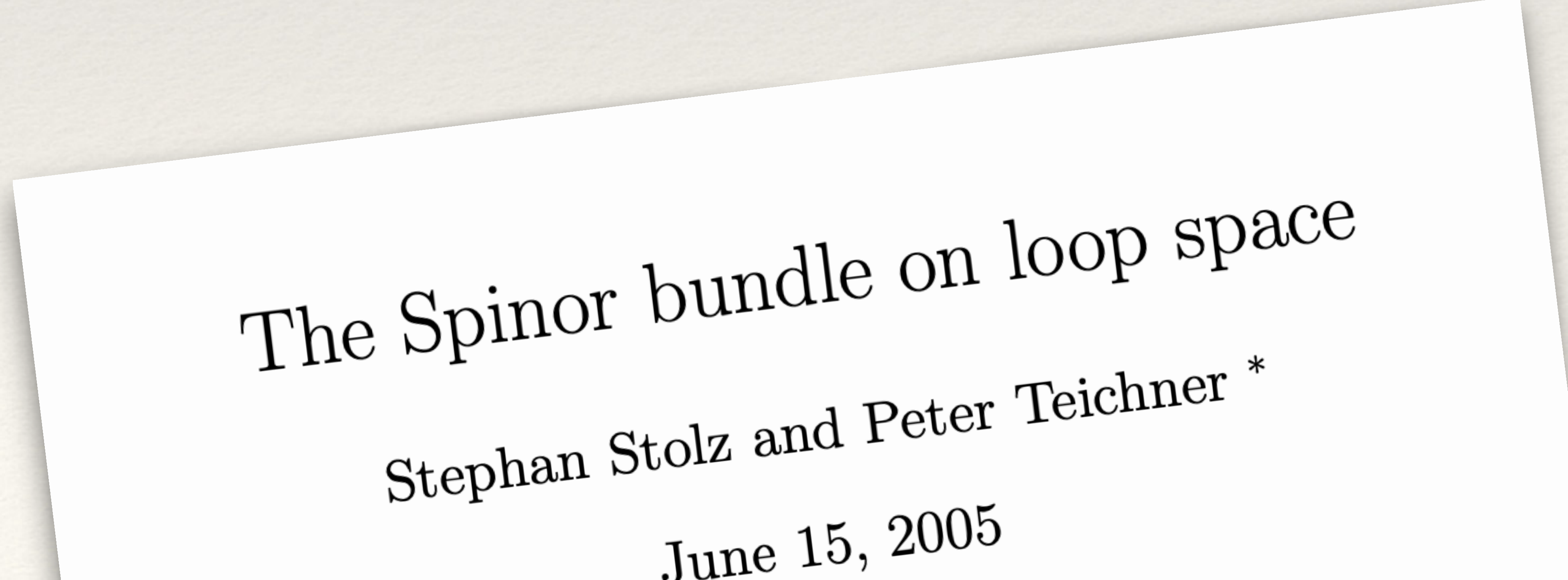


$$\begin{tikzcd}
\mathrm{Cl}(TPM) \arrow[d] & \iota^*\mathcal{S} \arrow[d] & & \\
PM \arrow[d] & PM^{[2]} \arrow[l, shift left] \arrow[l, shift right] & PM^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & PM^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \\
M & & &
\end{tikzcd}$$

need to complete Cl(TPM) to a von Neumann algebra bundle!

Conjecture. [S-T] After passing to vN-completions, a fusion product exists if and only if M is string.



The Spinor bundle on loop space

Stephan Stolz and Peter Teichner *

June 15, 2005

Von Neumann algebra bundles

What we did: Developed a theory of vN-algebra bundles and Hilbert bimodule bundles.

Upshot: They form a prestack of bicategories and may be stackified to yield the stack of *2-Hilbert bundles*.

Slogan: A 2-Hilbert space is a von Neumann algebra.

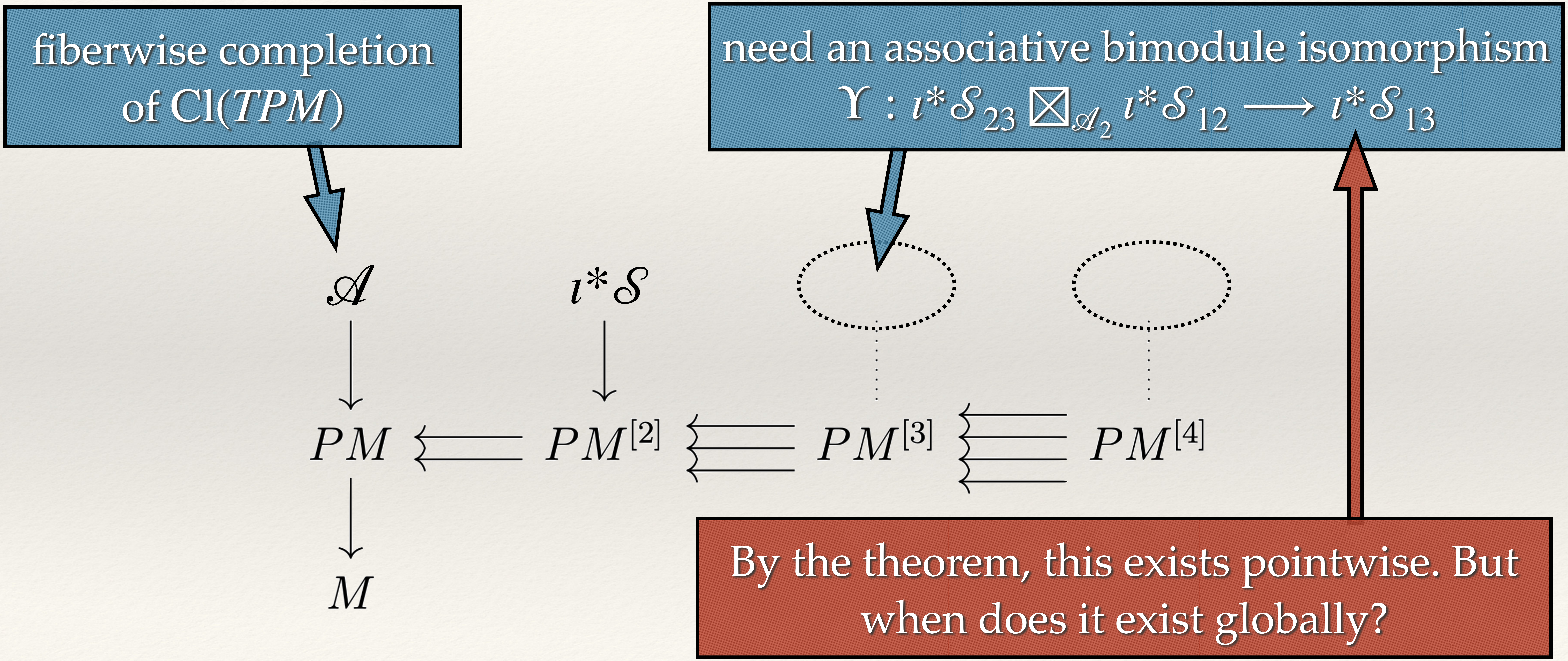
Given any γ_1, we may complete $\mathrm{Cl}(T_{\gamma_2}PM)$ in $\mathcal{S}_{\gamma_1 \circledast \gamma_2}$. This gives many different von Neumann completions!

Theorem 1. All these completions are canonically isomorphic and one gets a bundle $\mathcal{A}$ of vN-algebras over PM.

Theorem 2. For $(\gamma_1, \gamma_2, \gamma_3) \in PM^{[3]}$,

$$\mathcal{S}_{\gamma_2 \circledast \gamma_3} \boxtimes_{\mathcal{A}_{\gamma_2}} \mathcal{S}_{\gamma_1 \circledast \gamma_2} \cong \mathcal{S}_{\gamma_1 \circledast \gamma_3}$$

The stringor bundle



fiberwise completion of Cl(TPM)

need an associative bimodule isomorphism $\Upsilon : \iota^*\mathcal{S}_{23} \boxtimes_{\mathcal{A}_2} \iota^*\mathcal{S}_{12} \longrightarrow \iota^*\mathcal{S}_{13}$

$$\begin{tikzcd}
\mathscr{A} \arrow[d] & \iota^*\mathcal{S} \arrow[d] & {} \arrow[d, dash, dotted] & {} \arrow[d, dash, dotted] \\
PM \arrow[d] & PM^{[2]} \arrow[l, shift left] \arrow[l, shift right] & PM^{[3]} \arrow[l] \arrow[l, shift left] \arrow[l, shift right] & PM^{[4]} \arrow[l, shift left=0.5] \arrow[l, shift right=0.5] \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \\
M & & &
\end{tikzcd}$$

By the theorem, this exists pointwise. But when does it exist globally?

The stringor bundle

Results:

- Kristel-Waldorf [’20] constructed a fusion product on $\mathcal{S}$ from a *fusive* loop space spin structure.
- Fusive loop structures were known to be equivalent to string structures from the work of Waldorf [’15].

need an associative bimodule isomorphism

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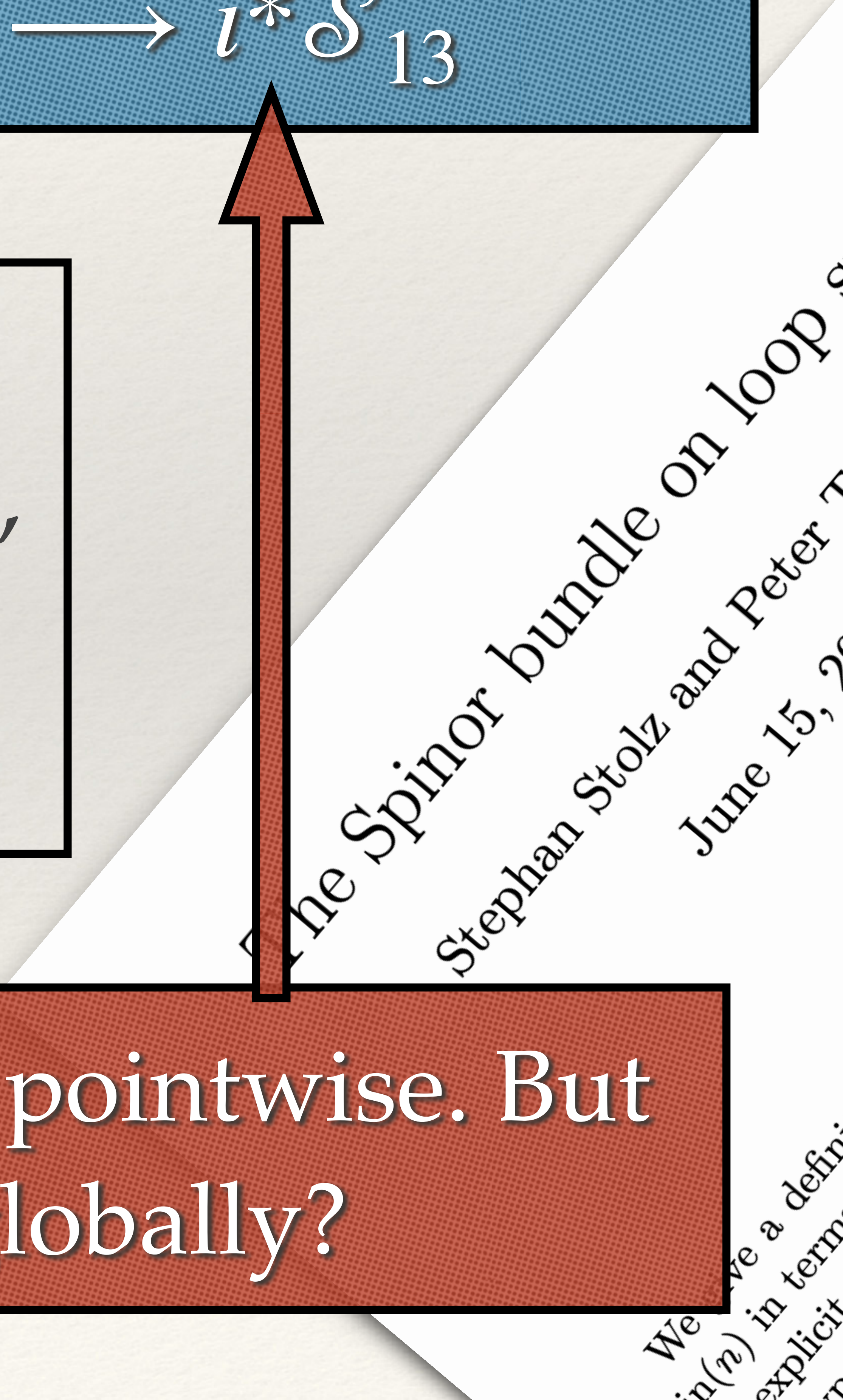
Open: Converse direction.

need an associative bimodule isomorphism
$$\Upsilon : \iota^* \mathcal{S}_{23} \boxtimes_{\mathcal{A}_2} \iota^* \mathcal{S}_{12} \longrightarrow \iota^* \mathcal{S}_{13}$$

Conjecture. [S-T] After passing to vN-completions, a fusion product exists if and only if M is string.

By the theorem, this exists pointwise. But when does it exist globally?



he Spinor bundle on loop

Stephan Stolz and Peter T

June 15, 2

The fusion 2-gerbe

The fusion line bundle
$\mathfrak{Fus} = \mathrm{Hom}(\mathscr{S}_{23} \boxtimes_{\mathscr{A}_2} \mathscr{S}_{12}, \mathscr{S}_{13})$

$$\begin{tikzcd}
 & & \mathfrak{Fus} \arrow[d] & \mu \arrow[d] & \text{cocycle condition} \arrow[d, dotted, no head] \\
P_xM \arrow[d] & P_xM^{[2]} \arrow[l] & P_xM^{[3]} \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] & P_xM^{[4]} \arrow[l, shift left=4.5] \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[l, shift right=4.5] & P_xM^{[5]} \arrow[l, shift left=6] \arrow[l, shift left=3] \arrow[l] \arrow[l, shift right=3] \arrow[l, shift right=6] \\
M & & & &
\end{tikzcd}$$

This gives a bundle 2-gerbe $\mathbb{F}\mathrm{us}(\mathscr{S})$ over M.

Lemma. A fusion product Υ gives a trivialization of $\mathbb{F}\mathrm{us}(\mathscr{S})$. Conversely, if $\mathbb{F}\mathrm{us}(\mathscr{S})$ is trivial, then there exists a line bundle $\mathscr{L}$ on LM such that $\mathscr{S} \otimes \mathscr{L}$ admits a fusion product.

The fusion 2-gerbe

Theorem. [L '23] The characteristic 4-class of the fusion gerbe is

$$\mathrm{CC}\Big((\mathbb{F}\mathrm{us}(\mathscr{S})\Big) = \frac{1}{2}p_1(M).$$

Corollary. A fusion product exists if and only if M admits a spin structure.

🙂

$$\mathbb{F}\mathrm{us}(\mathscr{S}) = \left(\begin{tikzcd}
 & & \mathfrak{Fus} \arrow[d] & \mu \arrow[d] & \text{cocycle condition} \arrow[d, dotted, no head] \\
P_xM \arrow[d] & P_xM^{[2]} \arrow[l, shift left] \arrow[l, shift right] & P_xM^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & P_xM^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & P_xM^{[5]} \arrow[l, shift left=4] \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[l, shift right=4] \\
M & & & &
\end{tikzcd} \right)$$

The fusion 2-gerbe

Theorem. [L '23] The characteristic 4-class of the fusion gerbe is

$$\mathrm{CC}\big((\mathbb{F}\mathrm{us}(\mathscr{S}))\big) = \tfrac{1}{2}p_1(M).$$

Corollary. A fusion product exists if and only if M admits a spin structure.

🙂

Still a mystery: How does the fusion product influence or help with the construction of the loop space Dirac operator?

The stringor representation

With Kristel and Waldorf, we constructed a representation of the string group on a certain von Neumann algebra A, the hyperfinite type III_1 factor.

$$\mathrm{String}(d) \xrightarrow{\sigma} \mathrm{Aut}(A)$$

automorphism 2-group of A

Note: Everything in this construction is strict.

This should be viewed as a unitary representation of $\mathrm{String}(d)$ an a 2-Hilbert space.

Associated 2-vector bundles

Let σ be a strict representation of a strict topological 2-group $\mathscr{G}$ on a 2-Hilbert space.

$$\mathscr{G} \xrightarrow{\sigma} \mathrm{Aut}(A)$$

Construction. [Kristel-Waldorf '23] There exists an associated 2-Hilbert bundle construction, which is a morphism of 2-stacks

$$\left\{ \begin{matrix} \text{principal} \\ \mathscr{G}\text{-2-bundles} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{2-Hilbert} \\ \text{bundles} \end{matrix} \right\}.$$

The stringor bundle as an associated bundle

$$\begin{tikzcd}
\mathrm{String}(M) \arrow[d] & \circlearrowleft & \mathrm{String}(d) \arrow[d] \\
\mathrm{Spin}(M) \arrow[d] & \circlearrowleft & \mathrm{Spin}(d) \\
M & &
\end{tikzcd}$$

Definition. A string structure on M is a lift of structure groups to $\mathrm{String}(d)$.

Theorem. [Waldorf ’15] The datum of a string structure on M is equivalent* to that of a fusive loop space spin structure for M.

* on the level of isomorphism classes

The stringor bundle

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{string}\\ \text{structures on } M\end{array}\right\} \arrow[r, "\text{associated bundle}", "\text{construction}"'] \arrow[d, "\text{[Waldorf '14]}"'] & \left\{\begin{array}{c}\text{2-Hilbert}\\ \text{bundles}\end{array}\right\} \\
\left\{\begin{array}{c}\text{fusive loop spin}\\ \text{structures on } M\end{array}\right\} \arrow[r, "\text{[Kristel-Waldorf '23]}"] & \left\{\begin{array}{c}\text{spinor bundles with}\\ \text{fusion product on } LM\end{array}\right\} \arrow[u, "\text{degression}"']
\end{tikzcd}$$

Theorem. [Kristel-L.-Waldorf ’24] The diagram commutes*.

Thank you for your attention