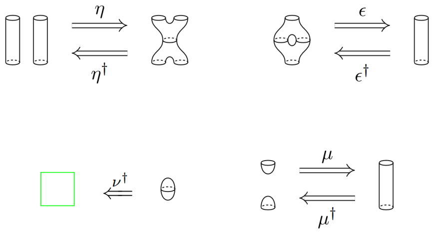
Non-compact 3d TQFTs from non-semisimple modular categories

Theodoros Lagiotis
University of Edinburgh



Lisbon TQFT 18/12/2024

Outline

1. Introduction &Motivation
   - The RT TQFT
   - Towards non-semisimplicity
2. Non-semisimplicity in the extended setting
   - Non-semisimplicity and non-compactness
3. Presentation of $\mathrm{Bord}^{nc}_{3,2,1}$
   - Generators for $\mathrm{Bord}_{3,2,1}$
   - Generators for $\mathrm{Bord}^{nc}_{3,2,1}$
4. Results
   - Non-compact TQFT
   - 3-manifold invariants
   - Modified trace

Table Of Contents

The RT TQFT

- Input data is a semisimple modular tensor category $\mathcal{C}_0$.
- Initially invariants of links, later promoted to a TQFT. Cobordisms can be decorated by objects and morphisms of $\mathcal{C}_0$.
- Distinguished object $\mathcal{F} := \bigoplus_i x_i^\vee \otimes x_i$ plays an important role in the construction.

$$\mathcal{Z}_{RT}(\Sigma_g) = \mathrm{Hom}_{\mathcal{C}}(1, \mathcal{F}^{\otimes g}).$$

- In 2015, [BDSPV] classified once extended 3d TQFTs $\mathrm{Bord}_{3,2,1} \to$ '2Vect' by semisimple MTCs by providing generators and relations for $\mathrm{Bord}_{3,2,1}$.

Towards non-semisimplicity

- Hennings: First to obtain 3-manifold invariants from non-ss data.
- Lyubashenko: Gave a non-semisimple version of the original definition of an MTC, and used it to define 3-manifold invariants and (projective) mapping class group representations of surfaces.
- Kerler-Lyubashenko: 'TQFT-like' construction (double categories, partially defined, only connected surfaces).
- DGGPR TQFT, input is again a potentially non-ss MTC, $\mathcal{C}$. Importantly, there is an admissibility condition on the cobordisms. They need to carry projective labels.

Coend Hopf algebra

When $\mathcal{C}$ is a non-semisimple MTC, the distinguished object takes the form of a coend:

$$\mathcal{F} := \int^{x \in \mathcal{C}} x^{\vee} \otimes x.$$

In both ss and non-ss cases, it is a Hopf algebra object. In fact, it carries extra structure, that of an integral $\Lambda \colon \mathbb{1} \to \mathcal{F}$ and a cointegral $\Lambda^{co} \colon \mathcal{F} \to \mathbb{1}$.

This data will be central for our construction.

Non-semisimplicity in the extended setting?

Thinking of classification questions:

‘Can we define extended 3d TQFTs from non-semisimple MTCs?’

However, bearing in mind the classification result of [BDSPV], we should rather ask:

‘How to adapt the extended setting to allow for non-semisimplicity?’

Table Of Contents

Non-semisimplicity and non-compactness

Let $Z\colon \mathrm{Bord}_{3,2,1} \to \mathcal{D}$ be a once extended 3d TQFT.
Then, $Z(S^1 \times -)$ is a fully extended 2d TQFT, and therefore $Z(S^1)$ is 2-dualizable.
However, for all the '2-vector space' candidate targets that were considered by [BDSPV], the 2-dualizable objects are semisimple.

Idea: Relax the 2-dualizability requirement by altering the source and target 2-categories of the TQFT.

Non-semisimplicity and non-compactness

- To alter the source, turn to **Lurie's non-compact TQFT**. $\mathrm{Bord}^{nc}_{3,2,1}$ has the same objects and 1-morphisms as $\mathrm{Bord}_{3,2,1}$, but 2-morphisms have incoming boundary in every connected component.
  This gets rid of 2-dualizability. (More explicit with generators)
- Choose a **suitable target**, whose 1-dualizable objects are not semisimple:
  The symmetric monoidal bicategory **Rex**, with **objects** small linear categories that admit finite colimits, **1-morphisms** right exact functors (finite colimit preserving functors), and **2-morphisms** natural transformations.
  Intuitively, everything is determined by projectives if $\mathcal{C}$ is finite.

Table Of Contents

Generators for Cob_2

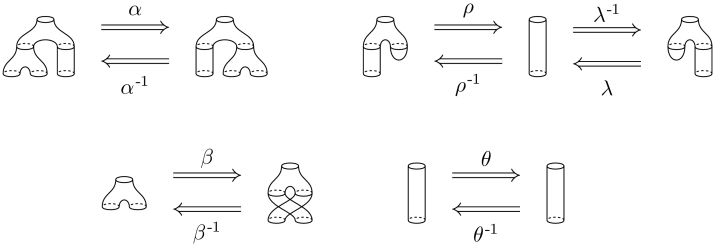
Cob_2 is the free symmetric monoidal category generated by a commutative Frobenius algebra.
Generating object: S^1
Generating morphisms:

They satisfy (co)associativity, (co)unit, Frobenius and commutativity relations.

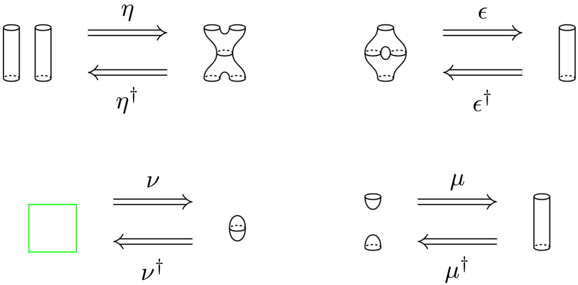
TQFTs $\text{Cob}_2 \to \mathcal{A}$ are classified by commutative Frobenius algebras in $\mathcal{A}$.

Invertible 2-morphism generators

Now, passing to 2-morphism generators of $\mathrm{Bord}_{3,2,1}$:



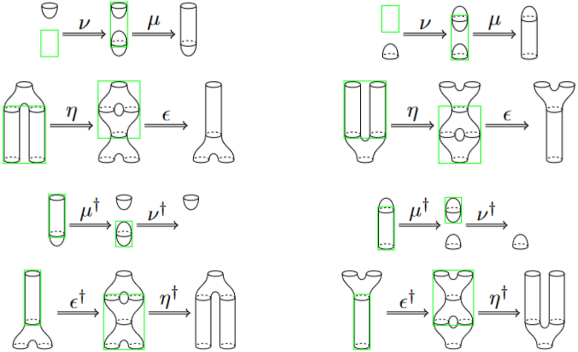
Non-invertible 2-morphism generators



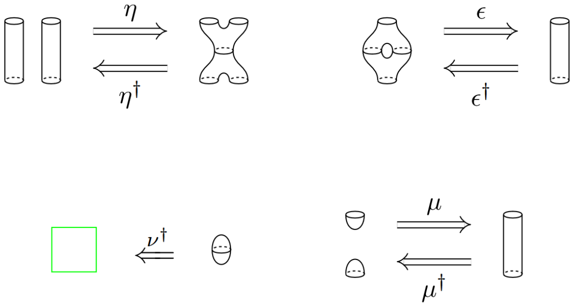
Claim[BDPSV]: The presentation above, subject to relations, generates the symmetric monoidal bicategory $\mathrm{Bord}_{3,2,1}$.

Some relations - Zigzag identities

The following composites are equal to the identity 2-morphism:



Non-invertible 2-morphism generators



Conjecture: The presentation obtained by discarding ν and the relations involving it, generates $\mathrm{Bord}^{nc}_{3,2,1}$.

Table Of Contents

Assignment $\mathcal{Z}_{nc}$ on objects and 1-morphisms

Using the data of $\mathcal{C}$, we define the (for now) assignment $\mathcal{Z}_{nc} \colon \mathrm{Bord}^{nc}_{3,2,1} \to \mathrm{Rex}$:

Generators of $\mathrm{Bord}^{nc}_{3,2,1}$	Data used to define
S^1	$\mathcal{C}$
	$\otimes \colon \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$
	$\int^{y \in \mathcal{C}} - \otimes y^\vee \boxtimes y \colon \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$
	$- \otimes \mathbb{1} \colon \mathrm{Vec}_k \to \mathcal{C}$
	$\mathrm{Hom}_{\mathcal{C}}(-, \mathbb{1})^* \colon \mathcal{C} \to \mathrm{Vec}_k$

We already see that $\mathcal{Z}_{nc}(\Sigma_g) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}^{\otimes g}, \mathbb{1})^*$.

Assignment $\mathcal{Z}_{nc}$ on 2-morphisms

For the assignments on 2-morphisms we make use of the equivalence $\mathcal{C} \boxtimes \mathcal{C} \cong \mathcal{C}^{\mathcal{F}}$.

Generators of $\mathrm{Bord}^{nc}_{3,2,1}$	Data used to define
ϵ	Counit of $\mathcal{F}$ (evaluation)
η	Coaction (coevaluation)
$\epsilon^\dagger$	Integral Λ
$\eta^\dagger$	Cointegral Λ^{co}, m, S
μ	Involves $\varepsilon_{\mathbb{1}} : P_{\mathbb{1}} \to \mathbb{1}$ and $\eta_{\mathbb{1}} : \mathbb{1} \to P_{\mathbb{1}}$
$\mu^\dagger$	Unit of $\mathrm{Hom}_{\mathcal{C}}(-, \mathbb{1})^* \dashv - \otimes \mathbb{1}$
$\nu^\dagger$	Counit of $\mathrm{Hom}_{\mathcal{C}}(-, \mathbb{1})^* \dashv - \otimes \mathbb{1}$

Non-compact TQFT

Theorem (L.)

The assignment $\mathcal{Z}_{nc}$ is a well defined symmetric monoidal 2-functor.

The proof of this theorem amounts to checking that all the relations of the presentation are preserved by $\mathcal{Z}_{nc}$.

Theorem (L.)

The (projective) mapping class group representations ρ_{nc} are dual to Lyubashenko's ρ_L.

3-manifold invariants

Given a closed, connected 3-manifold M, we can evaluate the noncompact TQFT on $M \setminus D^3$:

$$\mathcal{Z}_{nc}(M \setminus D^3) \colon \mathcal{Z}_{nc}(S^2) \to k.$$

The vector spaces $\mathcal{Z}_{nc}(S^2)$ and k are isomorphic, but not identical, so this map doesn't quite produce an invariant.
However, there is also the non-zero map $\mathcal{Z}_{nc}(D^3) \colon \mathcal{Z}_{nc}(S^2) \to k$. Both $\mathcal{Z}_{nc}(D^3)$ and $\mathcal{Z}_{nc}(M \setminus D^3)$ are elements of the one dimensional vector space $\mathcal{Z}_{nc}(S^2)^*$.

Theorem (L. - In progress)

The number $\mathcal{Z}_{nc}(M \setminus D^3)/\mathcal{Z}_{nc}(D^3)$ is the Lyubashenko invariant of M.

Modified trace

For an MTC $\mathcal{C}$, a *modified trace* on the tensor ideal $\mathrm{Proj}(\mathcal{C})$, is a k-linear map $\int^{P\in\mathrm{Proj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(P,P) \to k$, satisfying a nondegeneracy condition and some compatibility with the monoidal product. Modified traces were used in the construction of [DGGPR].

However, the space $\int^{P\in\mathrm{Proj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(P,P)$ is canonically isomorphic to $\mathcal{Z}_{nc}(T^2)$.

Theorem (L. - In progress)

The map

$$\int^{P\in Proj(\mathcal{C})} Hom_{\mathcal{C}}(P,P) \xrightarrow{\sim} \mathcal{Z}_{nc}(T^2) \xrightarrow{\mathcal{Z}_{nc}(\nu^{\dagger}\circ\epsilon)} k$$

is a modified trace.

Thank you for your attention!