“SCHEMES AT LISBON”

Course at Instituto Superio Técnico, Universidade de Lisboa, November-December 2024

Herwig HAUSER, University of Vienna

vs1, December 10, 2024

CLASS I: "The Geometry of a Ring"

(A) From Geometry to Rings

David Hilbert. The field $\mathbb{C}$ of complex numbers is algebraically closed: for all $P \in \mathbb{C}[x]$ a univariate non-constant polynomial, $P \notin \mathbb{C}$, there exists $a \in \mathbb{C}$ a root of P, $P(a) = 0$. Then $x - a$ divides P and P factors linearly,

$$P(x) = c \cdot \prod_{i=1}^{k} (x - a_i)^{m_i},$$

with c, and a_i in $\mathbb{C}$, and $m_i \geq 1$ the multiplicity of a_i as a root of P, $\sum_{i=1}^{m} m_i = \deg P$. The polynomial $\sqrt{P} := c \prod_{i=1}^{k}(x - a_i)$ has the same roots as P, now all simple. We call $\sqrt{P}$ the radical of P.

Several variables, several polynomials: $P_1, \dots, P_m \in \mathbb{C}[x_1, \dots, x_n]$,

$$X = V_{\mathbb{C}}(P_1, \dots, P_m) = \{a \in \mathbb{C}^n,\ P_i(a) = 0 \text{ for all } i\}.$$

$n = 1$: $\mathbb{C}[x]$ is a principal ideal domain: there exists a polynomial $P = \sum_{i=1}^{m} Q_i P_i$, linear combination of the P_i, such that $\langle P_1, ..., P_m \rangle = \langle P \rangle$ as ideals of $\mathbb{C}[x_1, \dots, x_n]$. This P is just the greatest common divisor of $P_1, ..., P_m$; it can be computed by Euclidean division.

$n \geq 1$: We now have a genuine system of polynomial equations,

$$\begin{gathered} P_1(x_1, \dots, x_n) = 0, \\ P_2(x_1, \dots, x_n) = 0, \\ \cdots\cdots \\ P_m(x_1, \dots, x_n) = 0. \end{gathered}$$

Does it have a common solution $a \in \mathbb{C}^n$? When is $V_{\mathbb{C}}(P_1, \dots, P_m) \neq \emptyset$ non-empty? An obvious necessary condition for this to hold is that the equations do not contradict each other: No linear combination of them gives a non-zero constant in $\mathbb{C}$,

$$\sum Q_i \cdot P_i = c \in \mathbb{C} \setminus \{0\}.$$

Said differently, the ideal I must not contain a non-zero constant $c \in \mathbb{C}$, say, there is a strict inclusion

$$I = \langle P_1, \dots, P_m \rangle \subsetneq \mathbb{C}[x_1, \dots, x_n].$$

Supported in part by the Austrian Science Fund FWF, Project P34765.

The Lemma of Zorn (or: the Noetherianess of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$) show that there exists a maximal ideal $\mathfrak{m} \subsetneq \mathbb{C}[x]$ which contains I,

$$I \subseteq \mathfrak{m}.$$

As $V_{\mathbb{C}}(\mathfrak{m}) \subseteq V_{\mathbb{C}}(I)$ it will then be sufficient to check whether $V_{\mathbb{C}}(\mathfrak{m}) \neq \emptyset$.

Nullstellensatz: In this situation, Hilbert's Nullstellensatz confirms that the above necessary condition is also sufficient: If $I \subsetneq \mathbb{C}[x_1, ..., x_n]$ is a proper ideal then its zero-set $V_{\mathbb{C}}(I)$ in $\mathbb{C}^n$ is not void,

$$V_{\mathbb{C}}(I) \neq \emptyset.$$

Moreover: If $\mathfrak{m} \subseteq \mathbb{C}[x]$ is a maximal ideal, then $V_{\mathbb{C}}(\mathfrak{m}) = \{a\}$ is a unique point a in $\mathbb{C}^n$, and $\mathfrak{m}$ is of the form $\mathfrak{m} = \mathfrak{m}_a = \langle x_i - a_i,\, i = 1, ..., n\rangle$.

We note that if $\sqrt{I}$ denotes the radical of I,

$$\sqrt{I} = \{f \in \mathbb{C}[x],\, f^k \in I \text{ for some } k \in I\},$$

then $V_{\mathbb{C}}(\sqrt{I}) = V_{\mathbb{C}}(I)$. And

$$I \subseteq \sqrt{I} \subseteq \bigcap_{I \subseteq \mathfrak{m}} \mathfrak{m},$$

the intersection ranging over all maximal ideal $\mathfrak{m}$ containing I (the last inclusion being in fact an equality, see Class V). Then, Hilbert's theorem tells us in addition: The ideal I_X of $\mathbb{C}[x_1, ..., x_n]$ of all polynomials P vanishing on $X = V_{\mathbb{C}}(I)$ equals the radical $I_X = \sqrt{I}$ of I. The inclusion $\sqrt{I} \subset I_X$ is obvious, and the actual equality is deduced from the earlier statement that $V_{\mathbb{C}}(I) \neq \emptyset$ if $1 \notin I$.

Bijection: This discussion yields a first basic observation. There is a bijection

$$\Phi : \{\text{points } a \text{ of } \mathbb{C}^n\} = \mathbb{C}^n \longleftrightarrow \{\text{maximal ideals } \mathfrak{m} \text{ of } \mathbb{C}[x_1, \ldots, x_n]\},$$

$$a \to \mathfrak{m}_a = \langle x_i - a_i,\, i = 1, ..., n\rangle.$$

For $X = V_{\mathbb{C}}(I) \subseteq \mathbb{C}^n$ an algebraic subset, $I = \sqrt{I}$, this extends to a bijection

$$\Phi_X : \{\text{points } a \text{ of } X\} = X \longleftrightarrow \{\text{maximal ideals } \mathfrak{m} \text{ of } \mathbb{C}[x_1, \ldots, x_n]/I\},$$

$$a \to \mathfrak{m}_a = \langle x_i - a_i,\, i = 1, ..., n\rangle.$$

The factor ring $\mathbb{C}[x_1, \ldots, x_n]/I$ is called the *coordinate ring* $\mathbb{C}[X]$ of X, and the set

$$\{\text{maximal ideals } \mathfrak{m} \text{ of } \mathbb{C}[x_1, \ldots, x_n]/I\} = \operatorname{Specmax} \mathbb{C}[X]$$

is the *spectrum* of maximal ideals of $\mathbb{C}[X]$.

Topology: For $X = V_{\mathbb{C}}(I) \subseteq \mathbb{C}^n$, a subset $A \subseteq X$ is *closed* if $A = V_{\mathbb{C}}(J)$, for some ideal J of $\mathbb{C}[X]$. Their complements are called *open* and define the *Zariski-topology* on X. From $A = V_{\mathbb{C}}(J) \cong \{\mathfrak{m} \subseteq \mathbb{C}[X] \text{ maximal},\, J \subseteq \mathfrak{m}\}$ we get a ring-theoretic description of the topology. We call $X = V_{\mathbb{C}}(I) \subseteq \mathbb{C}^n$ equipped with the Zariski-topology a *closed complex affine subvariety* of $\mathbb{C}^n$, or just *subvariety* for short (we do not consider open or locally closed subvarieties at the moment).

For $f \in \mathbb{C}[X]$ a polynomial, the set $X_f = \{\mathfrak{m} \subseteq \mathbb{C}[X] \text{ maximal},\, f \notin \mathfrak{m}\}$, is called *principal open* in X. These sets form a basis of the Zariski-topology of X.

Morphisms: Let be given algebraic subvarieties $X \subseteq \mathbb{C}^n$, $Y \subseteq \mathbb{C}^m$, $Z \subseteq Y$ (all closed). A morphism $f : X \to Y$ is defined as the restriction to X of a polynomial map $F = (F_1, ..., F_m) : \mathbb{C}^n \to \mathbb{C}^m$, sending X into Y. Then:

If $a \in X$ is a point, its image $b = f(a) \in Y$ is again a point;

If $b \in Y$ is a point, it pre-image $f^{-1}(b) \subseteq X$ is a subvariety of X;

If $Z \subseteq Y$ is a subvariety, its pre-image $f^{-1}(Z) \subseteq X$ is a subvariety of X.

If $V \subseteq X$ is a subvariety, its image $f(V) \subseteq Y$ is not necessarily a subvariety (it is only a constructible subset, i.e., a union of finitely many locally closed subsets, by a theorem of Chevalley). Take $X = V(\langle x, y) \cup V(xz - 1) \subseteq \mathbb{C}^3$, and $f : X \to Y = \mathbb{C}^2$ the restriction to X of the projection of $\mathbb{C}^3$ to $\mathbb{C}^2$ in z-direction, $(x, y, z) \to (x, y)$. The image is $(\mathbb{C}^2 \setminus V(y)) \cup \{0\}$, the xy-plane without x-axis, but augmented by the origin.

In terms of ideals: The map $f : X \to Y$ induces a ring homomorphism

$$\alpha_f : \mathbb{C}[Y] = \mathbb{C}[y]/J \to \mathbb{C}[X] = \mathbb{C}[x]/I,$$

$$y_j \to f_j(x),$$

$$P(y) \to P(f_1(x), \dots, f_m(x)).$$

Now, if $b = f(a) \in Y$ is the image of some point $a \in X$, we get for the maximal ideals the equality

$$\mathfrak{m}_b = \alpha_f^{-1}(\mathfrak{m}_a),$$

using that the pre-image of $\mathfrak{m}_a$ under α is maximal again (in the present context, not in general). And if $Z = V(G) \subseteq Y$ is a subvariety, for some ideal G of $\mathbb{C}[Y]$, its pre-image is given by

$$f^{-1}(Z) = V(G \circ f) = V(\alpha_f(G)),$$

say, by the image of G under α (or, more accurately: by the ideal of $\mathbb{C}[X]$ generated by $\alpha(G)$).

Prime ideals: These correspond to *irreducible* subvarieties of $\mathbb{C}^n$ or of X. They have better properties than maximal ideals, in particular, their pre-image under ring maps is again prime (easy check). Recall that $\mathfrak{p}$ is prime if $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. As we will use them a lot in the sequel, let us get some intuition (in what follows, $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$).

Examples: (1) If $\mathfrak{p} \subseteq \mathbb{C}[x]$ is a principal prime ideal (one or several variables), then $\mathfrak{p} = \langle P \rangle$, with P an irreducible polynomial.

(2) If $I \subseteq \mathbb{C}[x]$ is an ideal generated by irreducible polynomials, it does not follow that I is prime. Example: $I = \langle xy - zw, xy + zw \rangle = \langle xy, zw \rangle$.

(3) $I = \sqrt{I} \subseteq \mathbb{C}[x]$, then $I = \mathfrak{p}_1 \cap ... \cap \mathfrak{p}_k$ has a unique prime decomposition, corresponding to the *irreducible components* of $X = V_{\mathbb{C}}(I)$ (see below).

(4) $I = \langle x - \frac{3}{2} \rangle \subseteq \mathbb{Q}[x]$ is a maximal ideal and in particular prime (one variable x). Take the inclusion map

$$\alpha : \mathbb{Z}[x] \hookrightarrow \mathbb{Q}[x].$$

Then $\alpha^{-1}(I) = I \cap \mathbb{Z}[x] = \langle 2(x - \frac{3}{2}) \rangle = \langle 2x - 3 \rangle \subsetneq \langle x, 3 \rangle \subsetneq \mathbb{Z}[x]$ is not prime.

(5) $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$, $I = \langle x - i \rangle \subseteq \mathbb{C}[x]$, $\alpha^{-1}(I) = \langle x^2 + 1 \rangle \subseteq \mathbb{R}[x]$ maximal ideal.

(6) Number theory: prime ideals in $\mathbb{Z}$ are the zero ideal $\langle 0 \rangle$, and the maximal ideals $\mathbb{Z}p = \langle p \rangle$, p prime. The prime ideals in $\mathbb{Z}[x]$ are $\langle 0 \rangle$,$\langle p \rangle$, $p \in \mathbb{Z}$ prime, $\langle f \rangle$ with $f \in \mathbb{Z}[x]$ irreducible, and $\langle p, f \rangle$, with $p \in \mathbb{Z}$ prime and f irreducible modulo p.

(7) Take now the ring of Laurent polynomials $\mathbb{C}[x, \frac{1}{x}]$ in one variable x. Relate the prime ideals of $\mathbb{C}[x, \frac{1}{x}]$ to the prime ideals of $\mathbb{C}[x]$.

(B) From Rings to Geometry

Alexander Grothendieck. The preceding considerations may incite one to explore - by temptation and curiosity - what happens when the finitely generated $\mathbb{C}$-algebras $R = \mathbb{C}[x]/I$ are replaced by arbitrary, commutative rings R with $1 = 1_R$, and the zero-sets $V_{\mathbb{C}}(I) \subseteq \mathbb{C}^n$ isomorphic to the sets $\operatorname{Specmax} R$ of all maximal ideals of R by the sets $\operatorname{Spec} R$ of all prime ideals of R,

$$\operatorname{Spec} R = \{a = [\mathfrak{p}],\ \mathfrak{p} \subseteq R \text{ a prime ideal}\}.$$

Here, R itself does not count as a prime ideal (by convention), and $\mathfrak{p} = 0$ does count if R is an integral domain. A priori, there are no further assumptions on R. The notation $a = [\mathfrak{p}]$ - following Mumford - shall suggest that we consider the prime ideal $\mathfrak{p}$ as a *point* of $X = \operatorname{Spec} R$.

See Class V for some background about the genesis of this definition.

It is by no means plausible to expect that this abstract and completely formal generalization has a chance to produce an interesting theory with important applications. At first, it will be just a convenient way to communicate, say, a new language. We call

$$X = X_R = R = (\operatorname{Spec} R, R) = \operatorname{Spec} R$$

the *affine scheme* associated to R. The notation X or $\operatorname{Spec} R$ is not unique, the main point is that all information is contained in the ring R, and that $\operatorname{Spec} R$ - when considered just as a set or a topological space - is only subordinate to the scheme and part of the information. However, an affine scheme is often just denoted by $X = \operatorname{Spec} R$, for convenience and to emphasize the geometric viewpoint (indeed, X will immediately be equipped with a topology and then really deserves the name "geometric object").

Topology: Let $X = \operatorname{Spec} R$, and declare a subset $A \subseteq X$ to be *closed* if there exists a subset or an ideal $I \subseteq R$ such that

$$A = V(I) = \{\mathfrak{p} \in X, I \subseteq \mathfrak{p}\}.$$

This equips X with a topology, the *Zariski-topology*, since

$$\emptyset = V(1),\ X = V(0),\ V(I) \cup V(J) = V(IJ), \text{ and } \bigcap V(I_j) = V(\textstyle\sum I_j).$$

Points. If $a = [\mathfrak{p}]$ is a point of the affine scheme $X = \operatorname{Spec} R$, we associate to it the factor ring $R/\mathfrak{p}$. As $\mathfrak{p}$ is prime, this is an integral domain, so that we can consider its quotient field

$$\kappa_a = \kappa_{\mathfrak{p}} = \operatorname{Quot}(R/\mathfrak{p}),$$

called the *residue field* of X at a. A point $a = [\mathfrak{p}]$ is closed if and only if $\mathfrak{p} = \mathfrak{m}$ is a maximal ideal: indeed, we have $\{a\} = V(\mathfrak{m})$ in this case, whereas, for $\mathfrak{p} \subsetneq \mathfrak{m}$, the closed point $b = [\mathfrak{m}]$ belongs to the closure $\overline{\{a\}}$ of $\{a\}$. For closed points $a = [\mathfrak{m}]$ we have $\kappa_a = \kappa_{\mathfrak{m}} = R/\mathfrak{m}$, since this is already a field. If R is an integral domain and $\mathfrak{p} = 0$ the zero-ideal defining the *generic point* $\xi = [0]$ of X, then $\kappa_\xi = \kappa_0 = \operatorname{Quot} R$.

In contrast, for K an arbitrary field or ring, a K*-rational point* of X is a ring map $\alpha : R \to K$. Think of $R = \mathbb{R}[x_1, ..., x_n]/\langle f \rangle$, with f a polynomial, and $K = \mathbb{C}$, then $\alpha : \mathbb{R}[x]/\langle f \rangle \to \mathbb{C}$ is defined by the choice of a complex root $a \in \mathbb{C}^n$ of $f(x_1, ..., x_n)$, say, $f(a) = 0$.

Morphisms: We first note that any ring homomorphism $\alpha : S \to R$ induces a well-defined map

$$f_\alpha : X = \operatorname{Spec} R \to Y = \operatorname{Spec} S,$$

$$\mathfrak{p} \to \mathfrak{q} = \alpha^{-1}(\mathfrak{p}),$$

associating to a prime ideal $\mathfrak{p}$ of R its pre-image $\mathfrak{q} = \alpha^{-1}(\mathfrak{p})$ under α. As mentioned earlier, it is again a prime ideal. We call the pair (α, f_α) *morphism* from X to Y. The induced map $f_\alpha : X \to Y$ is continuous with respect to the Zariski-topology. The functor Spec goes from the category of rings to the category of topological spaces, with arrows reversed.

Examples: (1) $R = \mathbb{C}[x_1, ..., x_n]$, $n \geq 1$, $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec} R$, then the zero-ideal $\mathfrak{p} = 0$ is prime; the generic point $\xi = [0]$ is dense in X, say $\overline{\{\xi\}} = X$, since every prime ideal $\mathfrak{p}$ of R contains 0. In contrast, $\mathfrak{p} = \mathfrak{m}$ a maximal ideal defines a closed point, $V(\mathfrak{m}) = \{[\mathfrak{m}]\}$, and as $\mathfrak{m} = \mathfrak{m}_a = \langle x_1 - a_1, ..., x_n - a_n \rangle$ for some $a \in \mathbb{C}^n$, we recover $\mathbb{C}^n$ as the underlying set of closed points of $\operatorname{Spec}(\mathbb{C}[x_1, ..., x_n])$.

But there are other points, as e.g. $\mathfrak{p} = \langle f \rangle$, f an irreducible polynomial, corresponding to an (irreducible) hypersurface in $\mathbb{C}^n$, now considered as a point of $\operatorname{Spec} R$. For $n \geq 2$, the point $[\mathfrak{p}]$ is not closed, since $\mathfrak{p}$ is not maximal in $\mathbb{C}[x_1, ..., x_n]$.

(2) $R = K$ a field, $\operatorname{Spec} R = \{[0]\}$, a single point. Different fields define different points.

(3) $R = k[t]/\langle t^2 \rangle$, $\operatorname{Spec} R = \{[\langle t \rangle]\}$, one-point scheme, but with *non-reduced* structure (R contains nilpotent elements). $\operatorname{Spec} R$ is sometimes called the *disembodied tangent vector* (this is also the reason why the letter t for time is used)

(4) $R = \mathbb{Z}$, $\xi = [0]$ generic point, $[\langle p \rangle]$ closed points, for $p \in \mathbb{Z}$ prime. $R = \mathbb{Z}[t]$ with prime ideals as described earlier.

(5) R a ring, $\mathfrak{p} \subseteq R$ prime, $f \in R$, then the localization $R_{\mathfrak{p}}$ and the ring $R[\frac{1}{f}]$ define again affine schemes. If $X = \operatorname{Spec} R$, $a = [\mathfrak{p}] \in X$, then $\operatorname{Spec} R_{\mathfrak{p}}$ is written as (X, a), called the *germ* of X in a. In another direction, $\operatorname{Spec} R[\frac{1}{f}] = X \backslash V(f)$ defines a *principal open* subset. It is, by construction, again an affine scheme, since we can define these subsets by rings.

Exercises Class I. (1) Show that $K[x]$, K a field, x a single variable, is a principal ideal domain.

(2) Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

(3) Try to describe all prime ideals in $\mathbb{R}[x, y]$, $\mathbb{Z}[x]$, and $\mathbb{Z}/k\mathbb{Z}$, $k \in \mathbb{Z}$. Hint: You may want to consult Mumford's Red Book.

(4) What are the prime ideals of $\mathbb{Z} \times \mathbb{Z}$?

(5) Show that the inverse image of a prime ideal under a ring homomorphism is again prime.

(6) Show that the closed sets defined on $\operatorname{Spec} R$ define indeed a topology.

(7) Determine the prime ideals of $R[\frac{1}{f}]$, for $f \in R$ not a zero divisor, in terms of the prime ideals of R.

(8) Determine the prime ideals of $R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R$ in terms of the prime ideals of R.

(9) Show that the Zariski-topology on $\operatorname{Spec} R$ is in general not Hausdorff.

(10) Construct a ring R with $\operatorname{Spec} R$ consisting of 10 closed points.

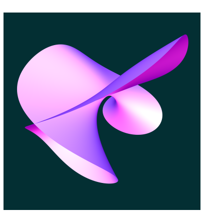
(11) Determine the residue fields of all points of $\operatorname{Spec} \mathbb{Z}$.

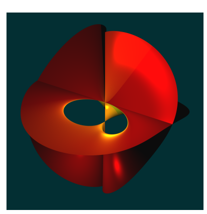
(12) What are the residue fields of the points of $\operatorname{Spec}(\mathbb{R}[x]/\langle x^2+1\rangle)$ and $\operatorname{Spec}(\mathbb{C}[x]/\langle x^2+1\rangle)$.

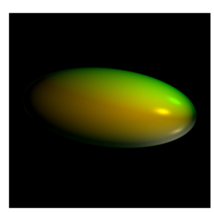
(13) Describe, for $R = K[x, y]/\langle x^2 - y^2 \rangle)$, the localizations $R_\mathfrak{p}$, for $\mathfrak{p} \subseteq R$ prime. What happens if K has characteristic 2?

(14) Determine the closed points of $\operatorname{Spec}(K[x, y, \frac{1}{x}]/\langle y - x^2\rangle)$.

(15) Show that the continuous map $f_\alpha : X = \operatorname{Spec} R \to Y = \operatorname{Spec} S$ associated to a ring homomorphism $\alpha : S \to R$ need not determine α.







CLASS II: "Constructions with Rings"

Throughout, let R and S be rings, with associated spectra $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ of prime ideals, equipped with the Zariski-topology.

Let $\alpha : S \to R$ be a ring homomorphism, with induced map $f_\alpha : X \to Y$, $[\mathfrak{p}] \to [\mathfrak{q}]$, where the pre-image $\mathfrak{q} = \alpha^{-1}(\mathfrak{p})$ of $\mathfrak{p}$ under α is again prime and declared as the image of the prime $\mathfrak{p}$ (recall that $\mathfrak{p} = R$ and $\mathfrak{q} = S$ do not count as prime ideals). We write $a = [\mathfrak{p}]$ and $b = [\mathfrak{q}] = [\alpha^{-1}(\mathfrak{p})]$. Then f_α is continuous, and a homeomorphism if α is a ring isomorphism (but: f_α does not determine entirely the ring map α in general).

Lemma 1. *If $\alpha : S \to R$ is surjective then $f_\alpha : X \to Y$ is injective with closed image $Z = f_\alpha(X)$ in Y: There is an ideal J in S with $Z = V(J)$, and $V(J) = \operatorname{Spec} S/J$ holds.*

Proof. As α is surjective, we get an isomorphism $\widetilde{\alpha} : S/J \to R$. So we may assume from the beginning that $R = S/J$. Now observe that the correspondence

$$V(J) = \{\mathfrak{q} \subseteq S \text{ prime}, J \subseteq \mathfrak{q}\} \longleftrightarrow \{\mathfrak{p} \subseteq S/J, \mathfrak{p} \text{ prime}\} = \operatorname{Spec} S/J,$$

$$\mathfrak{q} \to \overline{\mathfrak{q}} = \mathfrak{q}/J,$$

$$\mathfrak{p} + J \leftarrow \mathfrak{p},$$

is in fact a bijection (quick check). Therefore,

$$Z = \{\alpha^{-1}(\mathfrak{p}), \mathfrak{p} \subseteq S/J \text{ prime}\} \cong V(J) \subseteq \operatorname{Spec} S,$$

with $V(J) = \{\mathfrak{q} \subseteq S \text{ prime}, J \subseteq \mathfrak{q}\}$. ↺

Example: The projection $K[x,y] \to K[x]$, $x \to x$, $y \to 0$, induces a closed embedding of the affine line into the affine plane, $\mathbb{A}^1 = \mathbb{A}^1 \times 0 \subseteq \mathbb{A}^2$, taken there as the horizontal axis.

Lemma 2. *If $\alpha : S \to R$ is injective then $f_\alpha : X \to Y$ is dominant, i.e., has dense image in Y.*

Proof. We may assume that $S \hookrightarrow R$ is a subring. Therefore,

$$\operatorname{Im}(f_\alpha) = \{\mathfrak{q} \subseteq S \text{ prime}, \mathfrak{q} = \alpha^{-1}(\mathfrak{p}) = \mathfrak{p} \cap S, \mathfrak{p} \subseteq R \text{ prime}\}.$$

So assume that $\operatorname{Im}(f_\alpha) \subseteq V(J)$ for some ideal J of S. Then $J \subseteq \mathfrak{q} = \mathfrak{p} \cap S$ for all $\mathfrak{p} \subseteq R$ prime. It follows that $J \subseteq \bigcap_{\mathfrak{p} \subseteq R} \cap S$, the intersection ranging over all primes in R. But a short calculation shows that $\bigcap_{\mathfrak{p} \subseteq R}$ consists precisely of the nilpotent elements of R, say, equals the radical $\sqrt{0}^R = \{f \in R, f^k = 0 \text{ for some } k \in \mathbb{N}\}$ of the zero-ideal 0 of R (also called the *nil-radical*). We conclude that J is contained in the nil-radical $\sqrt{0}^S$ of S. As $V(\sqrt{0}^S) = V(0) = Y$, we get $V(J) = Y$ so that $\operatorname{Im}(f_\alpha)$ is dense in Y. ↺

Example: The ring extension $K[x] \hookrightarrow K[x, \frac{1}{x}]$ induces an open embedding of the punctured affine line $\mathbb{A}^1 \setminus \{0\} = \operatorname{Spec}(K[x, \frac{1}{x}])$ into the affine line $\mathbb{A}^1 = \operatorname{Spec}(K[x])$.

Lemma 3. *Let $f \in R$ be an element, and consider the ring of quotients*

$$R_f = R[\tfrac{1}{f}] = \{\tfrac{g}{f^k}, g \in R, k \in \mathbb{N}\}.$$

(i) If f is nilpotent, $f^m = 0$ for some $m > 0$, then $R_f = 0$.

(ii) The map $R \to R_f$, $g \to \frac{g}{1}$, is injective if and only if f is not a zero-divisor in R.

(iii) Assume that f is not a zero-divisor in R. Then $\operatorname{Spec}(R_f)$ equals $\operatorname{Spec} R \setminus V(f)$ and is open dense in $\operatorname{Spec} R$.

Proof. We leave (i) and (ii) to the interested reader. Assertion (iii) follows from $R \hookrightarrow R_f$ and the bijection

$$\operatorname{Spec}(R_f) = \{\mathfrak{q} \subseteq R_f \text{ prime}\} \longleftrightarrow \{\mathfrak{p} \in R \text{ prime}, f \notin R\},$$
$$\mathfrak{q} \to \mathfrak{p} = \mathfrak{q} \cap R,$$
$$\mathfrak{q} = \mathfrak{p}_f = \mathfrak{p}[\tfrac{1}{f}] = R_f \cdot \mathfrak{p} \leftarrow \mathfrak{p} \subseteq R.$$

Then just use the equality

$$\{\mathfrak{p} \subseteq R \text{ prime}, f \notin R\} = \{\mathfrak{p} \in R \text{ prime}\} \setminus \{\mathfrak{p} \in R \text{ prime}, f \in R\} = \operatorname{Spec} R \setminus V(f).$$

This gives $\operatorname{Spec}(R_f) = \operatorname{Spec} R \setminus V(f)$ as claimed. That this set is dense in Y follows from Lemma 2 since $R \to R_f$ is injective. ↺

We call in this case $X_f = \operatorname{Spec}(R_f)$ a *principal open subset* of $X = \operatorname{Spec} R$, and immediately note that this is again an affine scheme, with ring R_f. So we can also call it *open affine subscheme* of X. Finite intersections of principal open subschemes are again principal open since $X_f \cap X_g = X_{fg}$ (to see this, use the above bijection and the fact that $f, g \notin \mathfrak{p}$, for $\mathfrak{p}$ prime, implies that $fg \notin \mathfrak{p}$). Conversely, arbitrary unions of principal open subsets are again open (but not necessarily affine schemes), since

$$\bigcup_{f \in F} X_f = \bigcup_{f \in F} X \setminus V(f) = X \setminus \bigcap_{f \in F} V(f) = X \setminus V(F).$$

We conclude that the sets X_f, $f \in R$, form a basis of the Zariski-topology on X. And, indeed, for $U = X \setminus V(I)$ open in X, we have $U = \{\mathfrak{p} \subseteq R \text{ prime}, I \not\subseteq \mathfrak{p}\} = \bigcup_{f \in I} X_f$.

Example: Consider the open subset $Y = \mathbb{A}^2 \setminus \{0\} = \mathbb{A}^2 \setminus V(x, y)$ in $X = \mathbb{A}^2 = \operatorname{Spec}(K[x, y])$, where the origin $0 = [\mathfrak{m}_0]$ of $\mathbb{A}^2$ is defined through the maximal ideal $\mathfrak{m}_0 = \langle x, y\rangle$ of $K[x, y]$. Then Y is the union of the two principal open sets $X_x = \mathbb{A}^2 \setminus V(x) = \mathbb{A}^2 \setminus (0 \times \mathbb{A}^1)$ and $X_y = \mathbb{A}^2 \setminus V(y) = \mathbb{A}^2 \setminus (\mathbb{A}^1 \times 0)$, but cannot be defined as the spectrum of a single ring (this has to be proven, it is not completely obvious). Hence it is not an affine scheme.

Lemma 4. *Let $F \subseteq R$ be an arbitrary subset of elements $f \in R$, and set $X = \operatorname{Spec} R$.*
(i) The collection $\{X_f,\ f \in F\}$ forms an open affine covering $\bigcup_{f \in F} X_f = X$ of X if and only if the ideal $\langle F\rangle$ of R generated by the elements of F contains 1, i.e., is the whole ring.
(ii) If $\{X_f,\ f \in F\}$ forms an open covering of X, there exists a finite subset F' of F with $\bigcup_{f \in F'} X_f = X$, i.e., the Zariski-topology of X is quasi-compact.

Proof. From the proof of Lemma 3 we know that $\bigcup_{f \in F} X_f = X \setminus V(F)$. But $V(F)$ is empty if and only if $\langle F\rangle = R$. This proves (i). As for (ii), just notice that $1 \in \langle F\rangle$ implies that there are $f_1, ..., f_k \in F$ and $g_1, ..., g_k \in R$ such that $1 = \sum_{i=1}^k g_i f_i$. Then set $F' = \{f_1, ..., f_k\}$ and you are done. ↺

Lemma 5. *Let $\mathfrak{p} \subseteq R$ be a prime ideal, and denote by*

$$R_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1} R = \{\tfrac{g}{f},\, g \in R,\, f \in R \setminus \mathfrak{p}\}$$

the localization of R at $\mathfrak{p}$, with induced map $\mathrm{ext}_{\mathfrak{p}} : R \to R_{\mathfrak{p}},\ f \to \frac{f}{1}$.
(i) $R_{\mathfrak{p}}$ is a local ring, i.e., has a unique maximal ideal, namely

$$\mathfrak{m}_{\mathfrak{p}} = R_{\mathfrak{p}} \cdot \mathfrak{p} = \mathfrak{p}_{\mathfrak{p}} = \{\tfrac{g}{f},\, g \in \mathfrak{p},\, f \in R \setminus \mathfrak{p}\}.$$

(ii) There is a bijection

$$\{\mathfrak{q} \subseteq R \text{ prime}, \mathfrak{q} \subseteq \mathfrak{p}\} \leftrightarrow \{\mathfrak{r} \subseteq R_{\mathfrak{p}} \text{ prime}\} = \mathrm{Spec}(R_{\mathfrak{p}}),$$

$$\mathfrak{q} \to \mathfrak{r} = R_{\mathfrak{p}} \mathfrak{q},$$

$$\mathfrak{q} = \mathrm{ext}_{\mathfrak{p}}^{-1}(\mathfrak{r}) \leftarrow \mathfrak{r}.$$

(iii) If R is an integral domain, the map $\mathrm{ext}_{\mathfrak{p}} : R \to R_{\mathfrak{p}}$ is injective. Therefore, in this case, by Lemma 2, the scheme $\mathrm{Spec}(R_{\mathfrak{p}})$ is dense in $\mathrm{Spec}\, R$.

Proof. This should now be a routine for the ambitious reader. It is momentarily omitted. But, to avoid confusion, note that $\{\mathfrak{q} \subseteq R \text{ prime}, \mathfrak{q} \subseteq \mathfrak{p}\} \neq V(\mathfrak{p}) = \{\mathfrak{q} \subseteq R \text{ prime}, \mathfrak{p} \subseteq \mathfrak{q}\}$. ○

For $a = [\mathfrak{p}]$ a point of $X = \mathrm{Spec}\, R$, the scheme $\mathrm{Spec}(R_{\mathfrak{p}})$ is denoted by (X, a) and called the *germ* (or *stalk*) of X at a. The underlying topological space is not the topological germ $X_a = \lim_{a \in U \subseteq X} U$, for $U \subseteq X$ open, of X at a in the sense of classical topology. Nevertheless, (X, a) shares many nice properties with X_a.

Example. Let $X = V(xy) = \mathrm{Spec}(K[x,y]/\langle xy \rangle)$ in $\mathbb{A}^2 = \mathrm{Spec}(K[x,y])$ be the union of the two coordinate axes. Let $\mathfrak{p} = \mathfrak{m}_{(0,0)} = \langle x, y \rangle$ and $\mathfrak{q} = \mathfrak{m}_{(0,1)} = \langle x, y-1 \rangle$ define the two closed points $a = (0,0)$ and $b = (0,1)$ on X. Then

$$(X, a) = \mathrm{Spec}((K[x,y]/\langle xy \rangle)_{\mathfrak{p}}) = \mathrm{Spec}(K[x,y]_{\langle x,y \rangle}/\langle xy \rangle)$$

is not an integral domain (since $x \cdot y = 0$), but

$$(X, b) = \mathrm{Spec}((K[x,y]/\langle xy \rangle)_{\mathfrak{q}}) = \mathrm{Spec}(K[x,y]_{\langle x,y-1 \rangle}/\langle xy \rangle \cong \mathrm{Spec}(K[x]_x)$$

is an integral domain. Use here that localization commutes with taking factor rings, and that y is invertible in $K[x,y]_{\mathfrak{q}} = K[x,y]_{\langle x,y-1 \rangle}$.

Lemma 6. *If $S \to R$ is a finite ring homomorphism, i.e., R is finitely generated as an S-module, then $f_\alpha : X = \mathrm{Spec}\, R \to Y = \mathrm{Spec}\, S$ is quasi-finite, i.e., has finite fibers $f_\alpha^{-1}(b)$, for all $b \in Y$.*

Proof. Left out at the moment. ○

If $\alpha : S \to R$ is a finite ring homomorphism, one says that $f_\alpha : X = \mathrm{Spec}\, R \to Y = \mathrm{Spec}\, S$ is a *finite morphism*. Thus finite implies quasi-finite, the converse being wrong in general. In fact, Zariski's Main Theorem in the version of Grothendieck says: quasi-finite maps are the restriction of finite morphisms to open subsets. There is also a theorem of Chevalley saying that quasi-finite *proper* morphisms are already finite morphisms.

Example: Let $R = S[x]/\langle P\rangle$, for P a monic polynomial, $X = \operatorname{Spec} R = V(P) \subseteq Y \times \mathbb{A}^1 = \operatorname{Spec}(S[x])$, with $Y = \operatorname{Spec} S$. Then $f_\alpha : X \to Y$, the restriction to X of the projection $Y \times \mathbb{A}^1 \to Y$, has finite fibers. Compare this with the Noether Normalization Lemma.

Gluing: Let $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, and $U \subseteq X$, $V \subseteq Y$ be open. Assume we wish to glue X and Y along U and V. First, wlog, we may restrict to $U = X_f$ and $V = Y_g$ principal open, $X_f = \operatorname{Spec}(R_f)$ and $Y_p = \operatorname{Spec}(S_g)$. Hence, to glue, it suffices to prescribe a ring isomorphism $S_g \to R_f$.

Example. If we wish to glue $X = \mathbb{A}^1$ with itself, but called now $Y = \mathbb{A}^1$ for clarity, along the open subsets $U = X \setminus \{0\}$ and $V = Y \setminus \{0\}$, we have two natural options. Write $U = X_x$ and $V = Y_y$ with $U = \operatorname{Spec}(K[x, \frac{1}{x}])$ and $V = \operatorname{Spec}(K[y, \frac{1}{y}])$. Either we take the "identity" map $\alpha = \operatorname{Id} : K[x, \frac{1}{x}] \to K[y, \frac{1}{y}]$, sending x to y and $\frac{1}{x}$ to $\frac{1}{y}$. This produces $\mathbb{A}^1$ again, since, by continuity, 0 in X will be identified with 0 in Y (more precisely, α sends $K[x]$ isomorphically to $K[y]$). So this is not so interesting

The second option takes

$$\alpha : K[x, \tfrac{1}{x}] \to K[y, \tfrac{1}{y}],\ x \to \tfrac{1}{y},\ \tfrac{1}{x} \to y.$$

Thinking of $K = \mathbb{R}$ the real numbers, we get as a result - geometrically - a circle.

Fiber products. If X and Y are relative affine schemes over Z, i.e., are equipped with morphisms $X \to Z$ and $Y \to Z$, given by rings $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, $Z = \operatorname{Spec} T$ and ring homomorphisms $T \to R$ and $T \to S$, the *fiber product* $X \times_Z Y$ of X and Y over Z is defined as the affine scheme given by the tensor product $R \otimes_T S$,

$$X \times_Z Y = \operatorname{Spec}(R \otimes_T S),$$

together with the two projection maps $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$, which are induced by the natural homomorphisms $R \to R \otimes_T S$, $r \to r \otimes 1$, and $S \to R \otimes_T S$, $s \to 1 \otimes s$. To be precise, one should write the fiber product as a triple $(X \times_Z Y, p_X, p_Y)$.

The fiber product of affine schemes satisfies the usual universal property: For all affine schemes W over Z with morphisms $q_X : W \to X$ and $q_Y : W \to Y$ over Z there exists exactly one morphism $h : W \to X \times_Z Y$ over Z which makes all involved diagrams commutative. This follows immediately from the universal property of the tensor product of rings by the inversion of all arrows.

Special cases. In particular, we obtain the schema-theoretic notions of the *cartesian product* $X \times Y$ of two schemes $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ as the fiber product

$$X \times Y := X \times_{\operatorname{Spec}(\mathbb{Z})} Y = \operatorname{Spec}(R \otimes_{\mathbb{Z}} S)$$

over $\operatorname{Spec} \mathbb{Z}$ (if X and Y are defined over a field K, i.e., R and S are both K-algebras, one could also take the fiber product over $\operatorname{Spec} K$). Similarly, the intersection $V \cap W$ of two (closed or principal open) affine subschemes $V = \operatorname{Spec} S$ and $W = \operatorname{Spec} T$ of X given by ring maps $R \to S$ and $R \to T$ inducing (closed, respectively open) inclusion maps $V \hookrightarrow X$ and $W \hookrightarrow X$, is given by

$$V \cap W := V \times_X W = \operatorname{Spec}(S \otimes_R T).$$

Finally, the (schema-theoretic) *pre-image* $X_Z = f^{-1}(Z)$ of a closed or principal open affine subscheme $Z = \operatorname{Spec} T$ of Y under a morphism $f : X \to Y$ is defined as the fiber product with respect to the maps $X \to Y$ and $Z \hookrightarrow Y$ and is then given by

$$X_Z = f^{-1}(Z) := X \times_Y Z = \operatorname{Spec}(R \otimes_S T).$$

It is easy to check that the underlying topological space X_Z of X_Z is just the set-theoretic pre-image $f^{-1}(Z)$ of Z (taken as a topological space). Therefore, the notation $X_Z = f^{-1}(Z)$ is meaningful and appropriate.

These constructions can be expressed by the associated rings as follows: Let $X = \operatorname{Spec} R$ with closed subschemes $V = \operatorname{Spec}(R/I) \subseteq X$ and $W = \operatorname{Spec}(R/J) \subseteq X$, and $Y = \operatorname{Spec} S$ with $Z = \operatorname{Spec}(S/G) \subseteq Y$ closed in Y; the ring homomorphism $\alpha : S \to R$, which defines $X \to Y$, turns R, R/I and R/J into S-modules. Then

$$V \cap W = \operatorname{Spec}(R/I \otimes_R R/J) = \operatorname{Spec}(R/(I + J)) \subseteq X,$$

$$X_Z = \operatorname{Spec}(R \otimes_S S/G) = \operatorname{Spec}(R/RG) \subseteq X.$$

Here $RG = R\alpha(G)$ denotes the ideal generated by the image of G in R.

If, on the other hand, $V = X_f$ and $W = X_g$ are principal open in X, and $Z = Y_h$ is principal open in Y, with $f, g \in R$ and $h \in S$, then

$$V \cap W = X_f \cap X_g = \operatorname{Spec}(R_f \otimes_R R_g) = \operatorname{Spec}(R_{fg}) \subseteq X,$$

$$X_Z = X_{Y_h} = \operatorname{Spec}(R \otimes_S S_h) = \operatorname{Spec}(R_{\alpha(h)}) \subseteq X.$$

The fiber of a morphism $X \to Y$ over a point b of Y can also be defined in this way: Let first $b = [\mathfrak{n}] \in Y$ be a closed point with *residue field* $\kappa_b = S/\mathfrak{n}$ (it can thus also be considered as the closed affine subscheme $\{b\} = \operatorname{Spec}(S/\mathfrak{n}) = \operatorname{Spec}(\kappa_b)$ of Y with surjective homomorphism $S \to \kappa_b$), then the *scheme-theoretic fiber* X_b of f over b is defined by

$$X_b = \operatorname{Spec}(R \otimes_S \kappa_b) = \operatorname{Spec}(R \otimes_S S/\mathfrak{n}) = \operatorname{Spec}(R/R\mathfrak{n}) = V(R\mathfrak{n}) \subseteq X.$$

Here $R\mathfrak{n} = R\alpha(\mathfrak{n})$ denotes the ideal generated by $\alpha(\mathfrak{n})$ in R. Then X_b is a closed subscheme of X; it will generally not consist of a single point. The underlying topological space of X_b is just the (set-theoretic) pre-image $f^{-1}(b)$ of b under the continuous map $f : X \to Y$, $a = [\mathfrak{p}] \to b = [\alpha^{-1}(\mathfrak{p})]$. We therefore write $f^{-1}(b)$ or X_b for the fiber of f over b.

Exercises Class II. (1) Show that $R \to R_f, g \to \frac{g}{1}$, is injective if and only if f is not a zero-divisor in R.

(2) Describe $X = \mathrm{Spec}(K[x,y,z]_{\langle x-y \rangle})$.

(3) Let $\mathfrak{p} \subseteq R$ be prime. Is $R \to R_\mathfrak{p}$ always injective? If not, characterize the cases when it is injective.

(4) Describe a morphism $f : X \to Y$ of affine schemes which is not dominant (image not dense in Y).

(5) Show that the radical $\sqrt{I}$ of an ideal of R is the intersection of all prime ideals $\mathfrak{p}$ of R which contain I (this is an abstract version of Hilbert's Nullstellensatz).

(6) Prove the assertions of Lemmata 3, 5 and 6.

(7) Show that taking the localization $R_\mathfrak{p}$ of a ring R commutes with passing to a factor ring R/I. More explicitly: $(R/I)_{\overline{\mathfrak{p}}} \cong R_\mathfrak{p}/R_\mathfrak{p}I$ for $I \subseteq R$ an ideal, $\mathfrak{p} \subseteq R$ prime, $\overline{\mathfrak{p}}$ the image of $\mathfrak{p}$ in R/I, and $R_\mathfrak{p}I$ the ideal of $R_\mathfrak{p}$ generated by I.

(8) Try to construct ring-theoretically the projective plane $\mathbb{P}^2$ by gluing three affine planes $\mathbb{A}^2$ along dense open subsets.

(9) Write down with all details the universal property of the fiber product of affine schemes.

(10) Let X be defined in $\mathbb{A}^2$ by the equation $y - x^2 = 1$, and consider $f : X \to Y = \mathbb{A}^1$, the restriction of the first projection $\mathbb{A}^2 \to \mathbb{A}^1$. What are the pre-images of the closed points $b \in Y$? What is the pre-image of the generic point ξ of Y.

(11) Let $U \subsetneq X$, with X as in exercise 10, be principal open. Show that $f_{|U} : U \to \mathbb{A}^1$ is quasi-finite but not a finite morphism.

(12) Determine the intersection of the plane curves $y - x^k$ with $y = 0$ scheme-theoretically, i.e., find for all three schemes the underlying ring.

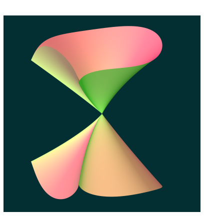
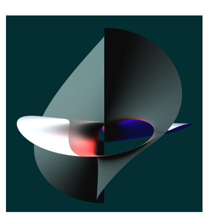
(13) Let $\mathfrak{p}$ be prime in R. Show that its residue field $\kappa_\mathfrak{p}$ can be written as

$$\kappa_\mathfrak{p} = \mathrm{Quot}(R/\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p},$$

where $\mathfrak{p}R_\mathfrak{p}$ denotes the ideal of $R_\mathfrak{p}$ generated by $\mathfrak{p}$ (more precisely, by the image of $\mathfrak{p}$ in $R_\mathfrak{p}$ under $R \to R_\mathfrak{p}$).

(14) Show that $\mathrm{Spec}(K[x, \frac{1}{x}])$ is isomorphic to the affine scheme $\mathrm{Spec}(K[x,y]/\langle xy-1 \rangle)$.

(15) Show that the intersection of two closed subschemes of an affine scheme X can be defined as a fibre product.


CLASS III: "The Geometry of Schemes"

Throughout, let R be a commutative ring with 1 and let $X = (R, \operatorname{Spec} R) = \operatorname{Spec} R$ be the associated affine scheme, where $\operatorname{Spec} R = \{a = [\mathfrak{p}],\, \mathfrak{p} \subseteq R \text{ prime}\}$ is equipped with the Zariski-topology. Recall the main operations:

> $I \subseteq R$ ideal, R/I the factor ring associated to I, $V(I) = \{a = [\mathfrak{p}],\, \mathfrak{p} \subseteq R$ prime, $I \subseteq \mathfrak{p}\}$, the vanishing or zero-set of I in X, $V(I) = \operatorname{Spec}(R/I) \subseteq X$ closed subset/subscheme.
>
> $f \in R$ element, $R_f = \{1, f, f^2, ...\}^{-1}R = R[\frac{1}{f}] = \{\frac{g}{f^k},\, g \in R,\, k \in \mathbb{N}\}$, the ring of quotients associated to f, $\operatorname{Spec}(R_f) = \{a = [\mathfrak{p}],\, \mathfrak{p} \subseteq R \text{ prime},\, f \notin \mathfrak{p}\} = X \setminus V(f) \subseteq X$, principal open subset/subscheme of X; $ext_f : R \to R_f,\, g \to \frac{g}{1}$, the associated ring map, injective if and only if f not a zero-divisor in R.
>
> $\mathfrak{p} \subseteq R$ prime ideal, $R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R = \{\frac{g}{f},\, g \in R,\, f \notin \mathfrak{p}\}$, localization or local ring of R at $\mathfrak{p}$, unique maximal ideal $\mathfrak{p}_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p} = \{\frac{g}{f},\, g \in \mathfrak{p},\, f \notin \mathfrak{p}\} \subseteq R_\mathfrak{p}$, residue field $\kappa_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ of X at $\mathfrak{p}$; $\operatorname{Spec}(R_\mathfrak{p}) = (X, a)$, the germ of X in the point $a = [\mathfrak{p}]$.
>
> Note the abuse of notation between R_f and $R_\mathfrak{p}$.

Important cases. Finitely generated K- or $\mathbb{Z}$-algebras $R = K[x_1, ..., x_n]/I$ and $R = \mathbb{Z}[x_1, ..., x_n]/I$, K a field; $R_f = K[x_1, ..., x_n, \frac{1}{f}] = K[x_1, ..., x_n]_f$, f a polynomial; $K[x_1, ..., x_n]_{\langle x_1, ..., x_n\rangle}$, the ring of rational functions defined at $0 \in \mathbb{A}^n$; formal power series rings $R = K[[x_1, ..., x_n]]/I$; the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions on $\mathbb{C}^n$ at 0.

Irreducible components. A closed subset $Y \subseteq X$ of a topological space X is called *irreducible*, if Y is not a proper union of closed subsets of X (this is a topological definition). A scheme $X = \operatorname{Spec} R$ is *integral*, if R is an integral domain, i.e., has no zero-divisors (this is an algebraic definition). And X is *reduced*, if R is a reduced ring, i.e., contains no nilpotent elements). A closed subscheme $Y = \operatorname{Spec}(R/I)$ of X is reduced if and only if $I = \sqrt{I}$ is a radical ideal of R. Stacks project, Tag 01ON: An affine scheme is integral if and only if it is reduced and irreducible.

A topological space X is *Noetherian*, if every descending chain $X \supseteq X_1 \supseteq X_2 \supseteq ...$ of closed subsets terminates. An affine scheme $X = \operatorname{Spec} R$ is *Noetherian*, if R is a Noetherian ring, i.e., every ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ of ideals terminates, or, equivalently, if every ideal is finitely generated. The rings mentioned above are all Noetherian. The associated topological space $X = \operatorname{Spec} R$ is then Noetherian with respect to the Zariski-topology.

Theorem. (Emmanuel Lasker 1905, Emmy Noether 1921) *Let $I = \sqrt{I}$ be a radical ideal of a Noetherian ring R. Then there exist unique prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_k$ of R such that*

$$I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k,$$

provided that the decomposition is irredundant, say, $\bigcap_{j \in A} \mathfrak{p}_j \not\subseteq \mathfrak{p}_i$, for all $A \subsetneq \{1, ..., k\}$ and $i \in \{1, ..., k\}$, $i \notin A$.

Geometric version: Every reduced closed subscheme Y of an affine Noetherian scheme X admits a unique irredundant decomposition

$$Y = Y_1 \cup Y_2 \cup \cdots \cup Y_k$$

into closed integral schemes $Y_i = \operatorname{Spec} R/\mathfrak{p}_i$*, called the irreducible components of* Y*.*

There is also a version of the theorem for arbitrary ideals, say, non reduced closed subschemes, known as the *primary decomposition* of an ideal, and yielding the notion of *isolated* and *embedded irreducible components.*

Dimension. The (topological) *dimension* $\operatorname{top.dim} X$ of a topological space X is the maximal length of a descending chain $X \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_k$ of closed irreducible subsets X_i. It is finite if and only if X is Noetherian. The (algebraic) *Krull dimension* $\dim X = \dim R$ of an affine scheme $X = \operatorname{Spec} R$ is the maximal length of an ascending chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k \subseteq R$ of prime ideals $\mathfrak{p}_i$ of R. It is finite if R is Noetherian. Equivalently, it is the maximal length of a chain of surjective ring maps $R \to R_0 \to R_1 \to \cdots \to R_k$ with $R_i \cong R/\mathfrak{p}_i$ integral domains. The local dimension $\dim_a X$ of X at a point $a = [\mathfrak{p}]$ is defined as the Krull dimension $\dim R_\mathfrak{p}$ of the local ring of X at a.

Example. Closed points $a = [\mathfrak{m}]$ of $\operatorname{Spec} R$ have Krull dimension 0, since $R/\mathfrak{m}$ is a field, with unique prime ideal $\mathfrak{p}_0 = 0$. One can show that the dimension of $R = K[x_1, ..., x_n]/I$ with K a field is equal to the transcendence degree of R over K. In particular, $\dim \mathbb{A}^n_K = n$. The inequality $\geq$ is obvious by choosing a chain of prime ideals, the converse inequality $\leq$ is considerably harder and requires Krull's principal ideal theorem. If I is (minimally) generated by elements $f_1, ..., f_k$, then $\dim X = \dim V(I) \geq n - k$ (again, by Krull), but equality need not hold. If equality holds, we say that X is a *complete intersection* in $\mathbb{A}^n_K$.

In contrast, $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ have Krull dimension 1, and $\mathbb{Z}[x_1, ..., x_n]$ has Krull dimension $n+1$.

Tangent space. (This section is a bit technical) If $X = \operatorname{Spec} R$ is an affine scheme and $a = [\mathfrak{p}]$ is a point of X with prime ideal $\mathfrak{p} \subseteq R$, associated local ring $R_\mathfrak{p}$ whose maximal ideal is denoted by $\mathfrak{n}_a = \mathfrak{n} = \mathfrak{p} R_\mathfrak{p} \subseteq R_\mathfrak{p}$, and residue field $\kappa_a = \kappa_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p} R_\mathfrak{p}$, we consider the factor ring $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{p} R_\mathfrak{p}/\mathfrak{p}^2 R_\mathfrak{p}$ as a vector space over κ_a. We call the κ_a-vector space

$$\mathrm{T}_a X = (\mathfrak{n}/\mathfrak{n}^2)^* = \operatorname{Hom}_{\kappa_a}(\mathfrak{n}/\mathfrak{n}^2, \kappa_a)$$

of κ_a-linear maps from $\mathfrak{n}/\mathfrak{n}^2$ to κ_a the *Zariski tangent space* of X at a. Taking the dual turns out to be necessary to obtain later a covariant functor. One shows that $\mathfrak{n}/\mathfrak{n}^2 \cong \mathfrak{m}/\mathfrak{m}^2$ if $\mathfrak{p} = \mathfrak{m}$ is maximal, i.e., a is a closed point (see the lemma below), and thus $\mathrm{T}_a X = (\mathfrak{m}/\mathfrak{m}^2)^*$ in this case. This allows one to compute the tangent space directly from the ideal $\mathfrak{m}$ of a without transition through the localization $R_\mathfrak{m}$ of R.

Lemma 1. *In the above situation, and if* $\mathfrak{m}$ *is maximal in* R*, one has* $\mathfrak{n}/\mathfrak{n}^2 \cong \mathfrak{m}/\mathfrak{m}^2$*.*

Proof. (a lot of writing without too much content) Set $M = R \setminus \mathfrak{m}$, $N = (R/\mathfrak{m}^2) \setminus (\mathfrak{m}/\mathfrak{m}^2)$ mutliplicatively closed subsets of R and $R/\mathfrak{m}^2$. Then, using that passing to a ring of quotients commutes with passing to factor rings, we get

$$\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m} R_\mathfrak{m}/(\mathfrak{m} R_\mathfrak{m})^2 = \mathfrak{m} R_\mathfrak{m}/\mathfrak{m}^2 R_\mathfrak{m} = M^{-1}\mathfrak{m}/M^{-1}\mathfrak{m}^2 = N^{-1}(\mathfrak{m}/\mathfrak{m}^2).$$

It remains to show that $N^{-1}(\mathfrak{m}/\mathfrak{m}^2) = \mathfrak{m}/\mathfrak{m}^2$. Pick $g \in R \setminus \mathfrak{m}$, so that $\mathfrak{m} + Rg = R$ because $\mathfrak{m}$ is maximal. Write $gh = 1 - k \in \mathfrak{m}$ for suitable $h \in R$ and $k \in \mathfrak{m}$. If now $\frac{f}{g} \in M^{-1}(\mathfrak{m}/\mathfrak{m}^2)$ with some $f \in \mathfrak{m}$ and $g \in M$, then

$$\frac{f}{g} = \frac{fh}{gh} = \frac{fh}{1-k} = fh\sum_{i=0}^{\infty} k^i \equiv fh(1+k) \text{ modulo } \mathfrak{m}^2.$$

Here, the the sum $\sum_{i=0}^{\infty} k^i$ has to be seen heuristically, as we are only interested in it modulo $\mathfrak{m}^2$. We obtain from $fh(1+k) \in \mathfrak{m}$ that $\frac{f}{g} \in \mathfrak{m}$ modulo $\mathfrak{m}^2$, proving the claim. $\circlearrowleft$

The tangent space is considered as a vector space over κ_a, and not as a scheme. For schemes $X = \operatorname{Spec} R$ of *finite type* over an algebraically closed field K (i.e., defined by a finitely generated K-algebra R), one can show that the tangent space $\mathrm{T}_a X$ is canonically isomorphic to the K-vector space of K-derivations $\delta : R_{\mathfrak{m}} \to K$. If R is a Noetherian ring, then $\mathfrak{m}_a$ and thus $\mathfrak{n}_a$ are finitely generated ideals, so $\mathrm{T}_a X$ is then a finite dimensional κ_a-vector space. Its vector space dimension $\dim_{\kappa_a} \mathrm{T}_a X$ is called the *local embedding dimension* of X at a, and is denoted by $\mathrm{emb.dim}_a X$. For non-Noetherian rings, the dimension of $\mathrm{T}_a X$ can be infinite.

Examples. All this looks complicated, but reveals to be rather easy in concrete situations: Take $R = K[x_1, ..., x_n]$, $a = 0 = [\langle x_1, ..., x_n\rangle]$ the origin of $\mathbb{A}^n$ with residue field $\kappa_a = K$, then $\mathfrak{m} = \langle x_1, ..., x_n\rangle$ gives the n-dimensional K-vector space $\mathfrak{m}/\mathfrak{m}^2$ of linear homogeneous polynomials in n variables, hence $\dim_K \mathrm{T}_0\mathbb{A}^n = n$ as expected.

Now, if $X = \operatorname{Spec} R$, $R = K[x_1, ..., x_n]/\langle f\rangle$, with some non-constant polynomial f, and taking again $a = 0$, let $f(x) = c + \sum c_i x_i + (\textit{higher order terms})$ be the Taylor expansion of f at 0. Then $c_i = \partial_{x_i} f(0)$, and $T_0 X$ is isomorphic to the linear subspace of K^n defined by the equation $\sum c_i t_i = 0$ (here, t_i denote variables in K^n). And, more generally, for an arbitrary closed point $a \in X$ (and K algebraically closed), we get that $\mathrm{T}_a X$ is isomorphic to the subspace of K^n defined by $\sum \partial_{x_i} f(a) t_i = 0$. It has dimension $n-1$ if and only at least one partial derivative $\partial_{x_i} f$ of f does not vanish at a, and dimension n otherwise.

If $X \subseteq \mathbb{A}^n_K$ is a closed subscheme, K algebraically closed, with defining equations $f_1, ..., f_k \in K[x_1, ..., x_n]$, and if $a \in X$ is a closed point, then $\mathrm{T}_a X$ is defined as a linear subspace of $\mathrm{T}_a \mathbb{A}^n_K \cong K^n$ by the system of equations

$$\textstyle\sum_{i=1}^n \partial_{x_i} f_j(a) \cdot t_i = 0,$$

for $j = 1, ..., k$ and variables $t_1, ..., t_n$ on K^n.

Tangent maps. We now define tangent maps. The notation is cumbersome, the substance being though simple, since everything is functorial. Every ring homomorphism $\alpha : S \to R$ induces, for $\mathfrak{p} \subseteq R$ and $\mathfrak{q} \subseteq S$ prime, with $\alpha^{-1}(\mathfrak{p}) = \mathfrak{q}$, a local ring homomorphism $\alpha_{\mathfrak{p}} : S_{\mathfrak{q}} \to R_{\mathfrak{p}}$ (which in particular interprets $R_{\mathfrak{p}}$ as an $S_{\mathfrak{q}}$-module), sending $\mathfrak{n}_b = \mathfrak{n}_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$ into $\mathfrak{n}_a = \mathfrak{n}_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ and $\mathfrak{q}^2 S_{\mathfrak{q}}$ into $\mathfrak{p}^2 R_{\mathfrak{p}}$. The residue fields are $\kappa_b = \kappa_{\mathfrak{q}} = S_{\mathfrak{q}}/\mathfrak{n}_{\mathfrak{q}}$ and $\kappa_a = \kappa_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{n}_{\mathfrak{p}}$. One thus obtains a $\kappa_{\mathfrak{q}}$-linear map

$$\overline{\alpha}_{\mathfrak{p}} : \mathfrak{n}_b/\mathfrak{n}_b^2 = \mathfrak{q}S_{\mathfrak{q}}/\mathfrak{q}^2 S_{\mathfrak{q}} \to \mathfrak{n}_a/\mathfrak{n}_a^2 = \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}^2 R_{\mathfrak{p}},$$

where $\mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}^2 R_{\mathfrak{p}}$ is understood via $\overline{\alpha}_{\mathfrak{p}}$ as a $\kappa_{\mathfrak{q}}$-vector space. Let $f_\alpha : X = \operatorname{Spec} R \to Y = \operatorname{Spec} S$ be the map induced by α and let $a = [\mathfrak{p}] \in X$, $b = f_\alpha(a) = [\mathfrak{q}] \in Y$ be the corresponding points of the spectra. Then α induces, by dualizing $\overline{\alpha}_{\mathfrak{p}}$, a κ_b-linear map

$$\mathrm{T}_a f_\alpha : \mathrm{T}_a X \to \mathrm{T}_b Y,$$

the *tangent map* of f_α at a. It is defined by $\mathrm{T}_a f_\alpha(v) = w$, where for $v : \mathfrak{m}_a/\mathfrak{m}_a^2 \to \kappa_a$ the image $w : \mathfrak{m}_b/\mathfrak{m}_b^2 \to \kappa_b$ is given by $w(y) = v(\overline{\alpha}_\mathfrak{p}(y))$, for all $y \in \mathfrak{m}_b/\mathfrak{m}_b^2$.

Example. If $\alpha : K[y_1, ..., y_m] \to K[x_1, ..., x_n]$ is given by m polynomials $P_1, ..., P_m$ without constant term, and if $a = 0$ is the origin of $\mathbb{A}^n_K$, then $\mathrm{T}_0 f_\alpha : \mathrm{T}_0\mathbb{A}^n_K \to \mathrm{T}_0\mathbb{A}^m_K$ is given by the multiplication $\mathrm{T}_0 P : K^n \to K^m : v \to v \cdot DP(0)$ of vectors $v \in K^n$ with the Jacobian matrix $DP(0) = (\partial_{x_i} P_j)_{ij}(0)$ of the vector $P = (P_1, ..., P_m)$ at 0. In DP we write the components of P horizontally, the derivatives according to the variables vertically.

The chain rule, i.e., functoriality, is as always: $\mathrm{T}_a \mathrm{Id}_X = \mathrm{Id}_{\mathrm{T}_a X}$ and $\mathrm{T}_a(f \circ g) = \mathrm{T}_{g(a)} f \circ \mathrm{T}_a g$.

Regularity and Singularities. In this part, $X = \operatorname{Spec} R$ is always a Noetherian affine scheme, say, R is a Noetherian ring. It is not too hard to see that, for $a = [\mathfrak{m}]$ a closed point of X, the dimension $\dim_{\kappa_a} \mathfrak{m}/\mathfrak{m}^2$ equals the minimal number of generators of $\mathfrak{m}$ (this is done with Nakayama's Lemma). We say that a local ring R of Krull dimension $\dim R = d$ is *regular* if its maximal ideal $\mathfrak{m} = \mathfrak{m}_R$ can be generated by d elements, or, equivalently, if its tangent space $\mathrm{T}_a X$ has dimension d. An arbitrary (Noetherian) ring is regular if all its localizations $R_\mathfrak{p}$ at prime ideals $\mathfrak{p}$ are regular local rings.

Geometrically speaking, X is regular if all its germs (X, a) (defined by $R_\mathfrak{p}$ with $a = [\mathfrak{p}]$) are regular. And a point $a \in X$ is *singular*, if its local ring $R_\mathfrak{p}$ is not regular.

This somewhat abstract definition boils down to the classical Jacobian criterion for manifolds whenever R is a finitely generated K-algebra over an algebraically closed field K (perfect suffices). So let $R = K[x_1, ..., x_n]/I$ with $K = \overline{K}$, and $I = \langle f_1, ..., f_k \rangle$ an ideal of $K[x_1, ..., x_n]$. Let $a \in X = \operatorname{Spec} R$ and $d = \dim(X, a)$ the local Krull dimension of R (i.e., the dimension of the ring $R_\mathfrak{p}$, for $a = [\mathfrak{p}]$). Then a is a regular point of X if and only if the rank of the evaluation $Df(a)$ of the Jacobian matrix $Df = (\partial_{x_i} f_j)$ at a equals $n - d$. Here, if $a \in K^n$ is a closed point, $\mathfrak{p} = \mathfrak{m}$ maximal, the evaluation is understood in the usual sense. For arbitary points, the evaluation is defined by taking the entries of the Jacobian matrix modulo $\mathfrak{p}$, getting a value in the residue field $\kappa_\mathfrak{p}$.

We immediately obtain

Lemma 2. *Let $X = \operatorname{Spec}(K[x_1, ..., x_n]/I)$ be integral with $K = \overline{K}$ (or: K perfect). Then $\operatorname{Sing} X = \{a \in X,\ a$ singular in $X\}$, the* singular locus *of X, is a closed subscheme of X, defined by the vanishing of the respective minors of the Jacobian matrix. It is a proper subset of X if $I = \sqrt{I}$, i.e., if X is a reduced scheme.* ⟲

Zariski has shown that, for non-perfect fields K, the Jacobian criterion, defining the *smoothness* of X, is weaker than regularity. Over perfect fields, regularity and smoothness are used equivalently.

Examples. The surface defined by $f = x^2 + y^2 - z^3$ has an isolated singularity at 0, whereas the one defined by $g = x^2 - y^2 z$ has the whole z-axis as its singular locus. The surface defined by $h = (x^2 - y^3)^2 - (x^2 - z^2)^3$ has two singular plane curves as its singular locus, and they meet at 0 in $\mathbb{A}^3$. The scheme X defined by $K[x]/\langle x^2 \rangle$ is non-reduced and $\operatorname{Sing} X = X$.

Corollary. *Let G be an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$, i.e., a subgroup cut out by polynomial equations (e.g., $\mathrm{SL}_n(\mathbb{C})$, $U_n(\mathbb{C})$). Then G is smooth at all its points.*

Proof. Indeed, as G is considered as a *reduced* closed subscheme of the affine scheme $\mathrm{GL}_n(\mathbb{C}) = \mathbb{A}^{n^2} \setminus V(\det) = \mathrm{Spec}(\mathbb{C}[x_{ij}]_{\det})$ (which is principal open in $\mathbb{A}^{n^2}$), it has at least one smooth point, say $s \in G$. But multiplication by an arbitrary element t of G is an isomorphism (in the sense of affine schemes), and thus also ts is a smooth point. This proves the claim.

Étale Morphisms. We have seen that localization allows us to zoom in into an affine scheme at a chosen point. For instance, the localization of $X = V(x^2 - y^2)$ in $\mathbb{A}^2$ at $a = (0,0)$ shows again the two irreducible components (the two diagonals), whereas the localization at the point $b = (1,1)$ shows only one component (the first diagonal): To make it clear, $K[x,y]/\langle x^2 - y^2\rangle)_{\langle x,y\rangle} = (K[x,y]_{\langle x,y\rangle})/\langle x^2 - y^2\rangle)$ with zero divisors $x - y$ and $x + y$. In contrast, $(K[x,y]/\langle x^2 - y^2\rangle)_{\langle x-1,y-1\rangle} = (K[x,y]/\langle x - y\rangle)_{\langle x-1,y-1\rangle}$ since $x + y$ is now invertible in this ring (it belongs to $K[x,y] \setminus \langle x-1, y-1\rangle$).

More involved is the integral scheme $X = V(x^2 + x^3 - y^2)$, defined by an irreducible polynomial (the *alpha-curve* or the *node*). Localization at $a = (0,0)$ gives again an integral domain (not hard to show), so we miss the two *analytic* branches of X at 0. Here, the language of schemes comes in handy: If R is the defining ring, and $R_\mathfrak{p}$ its localization at $\mathfrak{p}$, with maximal ideal $\mathfrak{n} = \mathfrak{n}_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, we may pass to the *completion* of this local ring, defined as the inverse limit and denoted by

$$\widehat{R} = \lim_{k\to\infty} R_\mathfrak{p}/\mathfrak{n}^k.$$

In case of R a finitely generated K-algebra and $\mathfrak{m} = \langle x_1, ..., x_n\rangle$, we obtain a ring of formal power series $K[[x_1, ..., x_n]]$. And, indeed, in the example of the node, the completion shows clearly the two components, since $x^2 + x^3 - y^2$ is reducible in $K[[x,y]]$, namely $x^2 + x^3 - y^2 = (x\sqrt{1+x} - y)(x\sqrt{1+x} + y)$.

Passing to the completion allowed Grothendieck to define *étale morphisms* and the *étale topology*, a convenient framework for schemes where the inverse function theorem holds.

Exercises Class III. (1) Determine the irreducible components of $V(xz, yz)$.

(2) Same for $V(yz(x^2 + y - z))$. Determine then the pairwise intersections, as well as the triple intersections of the three components, both, set- and scheme-theoretically.

(3) Show that $X_g \subseteq X_f$ if and only if $g \in \sqrt{\langle f \rangle}$.

(4) Describe the localizations $\mathbb{Z}_{\mathbb{Z}p}$, for p prime. Compute the residue fields of $\operatorname{Spec} \mathbb{Z}$ at all its points.

(5) Show that in a Noetherian ring, every descending chain of prime ideals terminates.

(6) Find a ring of infinite Krull dimension.

(7) Show that localization (passage to a ring of quotients) commutes with taking factor rings.

(8) Compute the tangent space of $X = \operatorname{Spec} \mathbb{Z}[x, y, z]/\langle xyz \rangle$ at all its points.

(9) Show that $f : \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \to (x, y - h(x))$, is an invertible morphism for all polynomials $h \in K[x]$. Compute the tangent map at $a = (0, 0)$ and $b = (1, 1)$. What happens, if h also depends on y?

(10) Determine the geometry of the real surface $V_{\mathbb{R}}(x^2 - y^2 z)$ in $\mathbb{A}^3_{\mathbb{R}}$ by slicing it with planes parallel to the three coordinate planes.

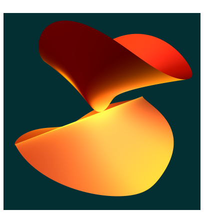
(11) Compute the Krull dimension of $K[x, y]/\langle xy \rangle$ at $a = (0, 0)$ and $a = (1, 0)$, and of the local rings $K[x, y]_{\langle x,y \rangle}$, $K[x, y]_{\langle x \rangle}$, $K[x, y]_0$.

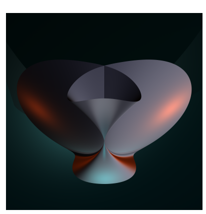
(12) Let X be the circle over the field $\mathbb{F}_3$ with three elements, $X = \operatorname{Spec}(\mathbb{F}_3[x, y]/\langle x^2 + y^2 - 1 \rangle)$. Show that it is irreducible (wrt the Zariski-topology).

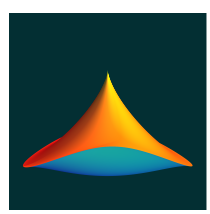
(13) Determine the singular points of $V(x^2 - y^2 z)$ and of $V((x^2 - y^3)^2 - (x^2 - z^2)^3)$.

(14) Let $X = \operatorname{Spec}(\mathbb{Z}[x, y]/\langle 27x^2 y \rangle)$. Which points are singular?

(15) Show that multiplication and inversion in an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ can be interpreted as morphisms in the sense of schemes. How do you define the group laws and the unit element 1 scheme-theoretically (giving rise to the notion of *group-schemes*)?







CLASS IV: "Sheaves and General Schemes"

The following two statements are essential for the later construction of sheaves from rings and for gluing affine schemes. Denote again by $\mathrm{ext}_f : R \to R_f$, $g \to \frac{g}{1}$, the canonical map of a ring into the ring of quotients defined by f. If f is not a zero divisor in R, then ext_f is injective. In addition, associate to every $g \in R$ a map

$$X = \mathrm{Spec}(R) \to \bigcup_{a \in X} \kappa_a,\, a \to g(a) = \overline{g}^a,$$

where $\kappa_a = \kappa_\mathfrak{p} = \mathrm{Quot}(R/\mathfrak{p})$ for $a = [\mathfrak{p}] \in X$ is the residue field of a and $\overline{g}^a$ denotes the residue class of g in κ_a (taking the union of residue fields may seem strange, but does no harm).

Lemma 1 *Algebraic version: Let F be a collection of elements f in R whose ideal $\langle F \rangle$ gives whole R. If $g \in R$ is an element whose images $\mathrm{ext}_f(g)$ in R_f are zero for all $f \in F$, then $g = 0$ in R.*
Geometric version: Let $X = \bigcup_{f \in F} X_f$ be a covering of $X = \mathrm{Spec}(R)$ by principal open sets X_f. If the restriction of $g : X \to \bigcup_{a \in X} \kappa_a$ to X_f is zero for all f , then f is already zero on X.

Proof. Choose a finite representation $1 = \sum_{f \in F} c_f \cdot f$, with $c_f \in R$, $c_f = 0$ for almost all f. We can thus assume F to be finite. Now choose $k \in \mathbb{N}$ such that $\mathrm{ext}_f(g) = 0$ can be written in R_f as $\mathrm{ext}_f(g) = \frac{g}{1} = \frac{0}{f^k}$, for all $f \in F$. Thus, by definition of the ring of quotients, there exists an $m \in \mathbb{N}$ with $f^m \cdot (g \cdot f^k - 1 \cdot 0) = 0$, i.e., $f^{m+k} \cdot g = 0$, for all $f \in F$. If ℓ is sufficiently large, $1 = (\sum_{f \in F} c_f \cdot f)^\ell$ lies in the ideal generated by all f^{m+k}. From this now follows $g = 1 \cdot g = 0$ in R. ○

Lemma 2. *Algebraically: Let F be a collection of elements f in R that generate whole R. For each $f \in F$, let an element $g_f \in R_f$ be given. Assume that $\mathrm{ext}_{f'}(g_f) = \mathrm{ext}_f(g_{f'})$ in $R_{ff'}$ holds for all $f, f' \in F$. Then there exists (exactly) one element g in R whose images $\mathrm{ext}_f(g)$ in R_f are equal to g_f for all $f \in F$.*
Geometrically: Let $X = \bigcup_{f \in F} X_f$ be a covering of $X = \mathrm{Spec}(R)$ by principal open sets X_f. If for each f maps $g_f : X_f \to \bigcup_{a \in X_f} \kappa_a$ are given that match on all pairwise intersections $X_f \cap X_{f'}$, then there exists (exactly) one map $g : X \to \bigcup_{a \in X} \kappa_a$ on X, whose restrictions to X_f coincide with g_f for all $f \in F$.

Proof. The uniqueness of g is given by Lemma 1. Again, we can assume that F is finite. So write $g_f = h_f / f^k$ with $h_f \in R$ and $k \in \mathbb{N}$, for all $f \in F$. Because of $\mathrm{ext}_{f'}(g_f) = \mathrm{ext}_f(g_{f'})$ in $R_{ff'}$ for $f, f' \in R$ it follows that

$$(ff')^m \cdot (h_f \cdot f'^k - h_{f'} \cdot f^k) = 0$$

for a common $m \in \mathbb{N}$ and all $f, f' \in F$. Write $g_f = \widetilde{h}_f / f^\ell$ with $\ell = k + m$ and $\widetilde{h}_f = h_f \cdot f^{\ell - k} \in R$. We get

$$\widetilde{h}_f \cdot f'^\ell = h_f \cdot f^{\ell-k} \cdot f'^\ell = h_f \cdot f^m \cdot f'^{k+m}$$

and, symmetrically,

$$\widetilde{h}_{f'} \cdot f^\ell = h_{f'} \cdot f^\ell \cdot f'^{\ell-k} = h_{f'} \cdot f^{k+m} \cdot f'^m,$$

thus, because of $\operatorname{ext}_{f'}(g_f) = \operatorname{ext}_f(g_{f'})$,

$$\widetilde{h}_f \cdot f'^\ell = \widetilde{h}_{f'} \cdot f^\ell.$$

Further we may decompose $1 = \sum_{f' \in F} c_{f'} \cdot f'^\ell$ for suitable $c_{f'} \in R$. This is possible because the powers f'^ℓ, $f' \in F$, also generate R. Now set $g := \sum_{f' \in F} c_{f'} \cdot \widetilde{h}_{f'} \in R$. We claim that this g satisfies $\operatorname{ext}_f(g) = g_f$ for all f. In fact, $g_f = \frac{h_f}{f^k} = \frac{\widetilde{h}_f}{f^\ell} = \frac{g}{1}$ holds in R_f if and only if $f^n \cdot (\widetilde{h}_f \cdot 1 - f^\ell \cdot g) = 0$ for an $n \in \mathbb{N}$, i.e., if

$$f^n \cdot \Big(\widetilde{h}_f \cdot (\textstyle\sum_{f' \in F} c_{f'} \cdot f'^\ell) - f^\ell \cdot (\sum_{f' \in F} c_{f'} \cdot \widetilde{h}_{f'})\Big) = 0.$$

This follows from the previous identity $\widetilde{h}_f \cdot f'^\ell = \widetilde{h}_{f'} \cdot f^\ell$. $\circlearrowleft$

Sheaves. The letters $\mathcal{F}$ and $\mathcal{G}$ are dominant here because of the French school of the fifties (Bourbaki, ...), calling them $\mathcal{F}$*aisceaux.* A *sheaf* $\mathcal{F}$ (of sets, rings, ideals, ...) on a topological space X is a collection $\mathcal{F}(U)$ (of sets, rings, ideals, ...), for all $U \subseteq X$ open, together with *restriction maps* $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all open $V \subseteq U \subseteq X$ such that $\rho_{U,U} = \operatorname{Id}_{\mathcal{F}(U)}$ and $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$ holds for all $W \subseteq V \subseteq U \subseteq X$ open and for which the following *sheaf condition* is satisfied (combination of Lemma 1 and 2):

> If $U \subseteq X$ is open with open covering $U = \bigcup_{i \in I} U_i$, and if elements $f_i \in \mathcal{F}(U_i)$ are given whose restrictions (given by ρ) coincide on the intersections $U_i \cap U_j$, then there is exactly one $f \in \mathcal{F}(U)$ whose restrictions to U_i are equal to f_i for all i.

Usually, for $f \in \mathcal{F}(U)$ and $V \subseteq U$ open, one writes $f_{|V}$ instead of $\rho_{U,V}(f)$. The elements of $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ on U (there is some reason for this name).

Examples. If X and Y are topological spaces, then the continuous maps $f : U \subseteq X \to Y$ on open subsets U define a sheaf of sets on X. The holomorphic functions $f : U \subset \mathbb{C}^n \to \mathbb{C}$ on open subsets U of $\mathbb{C}^n$ define a sheaf $\mathcal{O}_{\mathbb{C}^n}$ of rings on $\mathbb{C}^n$. If K is a field and $U \subseteq K^n$ is open with respect to the Zariski topology, then the *regular functions* on U, given by rational functions $f = \frac{P}{Q} \in K(x_1, ..., x_n)$ with polynomials $P, Q \in K[x_1, ..., x_n]$ such that Q does not vanish on U, define a sheaf of K-algebras on K^n.

If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X, then a *sheaf morphism* $\alpha : \mathcal{F} \to \mathcal{G}$ is a collection of maps $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$, for $U \subseteq X$ open, which is compatible with the restriction maps, $\rho^{\mathcal{G}}_{U,V} \circ \alpha(U) = \alpha(V) \circ \rho^{\mathcal{F}}_{U,V}$, for $V \subseteq U$ open. If $\mathcal{F}$ and $\mathcal{G}$ have the algebraic structure of a ring, module, etc., then all $\alpha(U)$ should be compatible with these structures.

If a basis U_i, $i \in I$, of the topology of X is given, then the $\mathcal{F}(U_i)$ already determine $\mathcal{F}$: the elements of $\mathcal{F}(U)$, for $U \subseteq X$ open, are uniquely prescribed by their restrictions to the sets U_i which are contained in U, due to the sheaf condition.

One can push the story a bit further (if one wants): If $U = \bigcup_{i \in I} U_i$ is an open cover of an open subset U of X, then $\mathcal{F}(U) = \lim_{\leftarrow i \in I} \mathcal{F}(U_i)$ can be interpreted as the inverse limit of all $\mathcal{F}(U_i)$, where $\{\mathcal{F}(U_i)\}_{i \in I}$ is considered as a direct system via the restriction maps $\rho_{ij} : \mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$. The elements of $\mathcal{F}(U)$ are thus understood as tuples $(s_i)_{i \in I}$ of elements $s_i \in \mathcal{F}(U_i)$ whose restrictions $(s_i)_{|U_i \cap U_j}$ and $(s_j)_{|U_i \cap U_j}$ match for all i and j in I.

Conversely, for an open cover $\{U_i\}_{i\in I}$ of X, let sheaves $\mathcal{F}_i$ on U_i be given together with isomorphisms $\varphi_{ij} : (\mathcal{F}_i)_{|U_i\cap U_j} \to (\mathcal{F}_j)_{|U_i\cap U_j}$, which are compatible on triple intersections $U_i \cap U_j \cap U_k$ and equal to the identity $\mathrm{Id}_{\mathcal{F}_i}$ for $i = j$. Here, $(\mathcal{F}_i)_{|U_i\cap U_j}$ denotes the sheaf on $U_i \cap U_j$ defined by $(\mathcal{F}_i)_{|U_i\cap U_j}(V) = \mathcal{F}_i(V)$, $V \subseteq U_i \cap U_j$ open. Then there exists exactly one sheaf $\mathcal{F}$ on X whose restrictions $\mathcal{F}_{|U_i}$ on U_i are isomorphic to $\mathcal{F}_i$ via isomorphisms that are compatible with the φ_{ij}.

Stalks. Let a be a point of X. The sets $\mathcal{F}(U)$, $U \subseteq X$ open neighborhood of a, form a direct system: For neighborhoods U, V we obtain with $W = U \cap V$ the restriction maps $\mathcal{F}(U) \to \mathcal{F}(W)$ and $\mathcal{F}(V) \to \mathcal{F}(W)$. We set

$$\mathcal{F}_a = \lim\nolimits_{\to} \mathcal{F}(U),$$

where the limit runs over all open neighborhoods U of a. So

$$\mathcal{F}_a = \coprod\nolimits_U \mathcal{F}(U)/\sim$$

is the disjoint union of all $\mathcal{F}(U)$ modulo the following equivalence relation: $f \in \mathcal{F}(U)$ is equivalent to $g \in \mathcal{F}(V)$ if and only if there exists an open neighborhood W of a in $U \cap V$ such that $f_{|W} = g_{|W}$ in $\mathcal{F}(W)$. Then $\mathcal{F}_a$ is called the *stalk* of $\mathcal{F}$ at a, and its elements are *germs* at a. The natural map $\mathcal{F}(U) \to \mathcal{F}_a$, $a \in U$, is denoted by $f \to f_a$. If $s \in \mathcal{F}_a$, then every $f \in \mathcal{F}(U)$ with $f_a = s$, for U an open neighborhood of a, is called a representative of s on U. Two representatives of s always agree on a sufficiently small neighborhood of a.

Structure sheaf of an affine scheme. Let R be a ring and $X = \operatorname{Spec} R$ the associated affine scheme, with underlying topological space X (note the double notation). We construct from R a sheaf of rings $\mathcal{O}_X$ on X, the *structure sheaf* of the scheme X. According to the remarks above, it is sufficient to specify $\mathcal{O}_X(U)$ for a choice of basis $\{U\}$ of the topology on X. For this purpose, we choose the principal open subsets X_f, $f \in R$, as the basis. If $U = X_f \subseteq X$, then set

$$\mathcal{O}_X(U) = \mathcal{O}_X(X_f) = R_f = R[\tfrac{1}{f}],$$

the ring of quotients with denominators f^k. For inclusions $V = X_g \subseteq U = X_f$, one has $g \in \sqrt{(f)}$, i.e., $g^m = a \cdot f$ for an $a \in R$ and an $m \in \mathbb{N}$. This induces the restriction maps $\rho_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$, given by

$$R_f \to R_g,\ \tfrac{h}{f^k} = \tfrac{a^k h}{(af)^k} \to \tfrac{a^k h}{g^{mk}}.$$

In Lemmata 1 and 2 above it was shown that the rings R_f, $f \in R$, fulfill the sheaf condition. This means that the construction of the sheaf $\mathcal{O}_X$ on X is guaranteed. It is completely determined by the ring R. We obtain, for $a = [\mathfrak{p}] \in X$, because of $R_{\mathfrak{p}} = \lim\limits_{\to\ f\in R\setminus\mathfrak{p}} R_f$, that the stalks of $\mathcal{O}_X$ in a are the localizations

$$\mathcal{O}_{X,a} = (R_f)_{\mathfrak{p}} = R_{\mathfrak{p}},$$

where $f \in R$ was chosen so that $a \in X_f$ and therefore $f \notin \mathfrak{p}$. Note that $\bigcap_{\mathfrak{p}} R_{\mathfrak{p}} = R$ holds, i.e., the ring R is the intersection of all local rings $\mathcal{O}_{X,a}$. Seeing the ring as the intersection of its localizations was the starting point in Chevalley's lecture entitled "Schemes" at the

Séminaire Cartan-Chevalley from 1955/56. For all $a \in X_f$, the germ $f_a \in \mathcal{O}_{X,a}$ is invertible, since f (more precisely: $\mathrm{ext}_f(f)$) is invertible in R_f.

For every open subset U the rings $\mathcal{O}_X(U)$ are given by

$$\mathcal{O}_X(U) = \lim_{\leftarrow X_f \subseteq U} \mathcal{O}_X(X_f) = \lim_{\leftarrow X_f \subseteq U} R_f.$$

Using the same reasoning as above, $\mathcal{O}_X(U) = \bigcap_{a \in U} \mathcal{O}_{X,a}$.

Construction of general schemes by gluing rings. As an alternative to the classical definition of general schemes via sheaves as locally ringed spaces locally isomorphic to affine schemes we briefly sketch the construction of schemes by gluing affine schemes. This is in perfect analogy with the two definitions of differentiable manifolds, either as topological spaces equipped with an atlas, or as a collection of open subsets of $\mathbb{R}^n$ together with transition functions between them (defining the gluing).

Let $\{R_i\}_{i \in \Lambda}$ be a collection of rings with associated affine schemes $X_i = \mathrm{Spec}\, R_i$. For each pair (i, j) of indices in Λ, let a ring map

$$R_i \to R_{ij} = (R_i)_{f_{ij}}$$

be given, with $f_{ij} \in R_i$, together with ring isomorphisms

$$\alpha_{ij} : R_{ji} \to R_{ij},$$

for which $\alpha_{ji} = \alpha_{ij}^{-1}$ and $\alpha_{ik} = \alpha_{ij} \circ \alpha_{jk}$ on the rings $R_{ijk} = (R_i)_{f_{ij} f_{ik}}$. Then the specification of R_i, R_{ij} and α_{ij} defines a (general) scheme X with underlying topological space

$$X = \coprod X_i / \sim,$$

where $a \in X_i$ is aquivalent to $b \in X_j$ if and only if $a \in U_{ij} = \mathrm{Spec}\, R_{ij}$, $b \in U_{ji} = \mathrm{Spec}\, R_{ji}$, and the equality $\varphi_{ij}(a) = b$ holds for the homeomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$ induced by α_{ij}.

Morphisms $f : X \to Y$ are defined by the existence of open affine covers $X = \bigcup X_i$ and $Y = \bigcup Y_j$ with principal open subsets X_{ij} of X_i and Y_{ji} of Y_j and by morphisms $f_{ij} : X_{ij} \to Y_{ji}$, which are compatible with all restriction maps. We leave out the (tedious but straightforward) details

General schemes as locally ringed spaces. We now give the usual definition of schemes, starting from a topological space X, equipped with a sheaf. The reader will see that other readers might be repelled by the notational and semantic complexity. A *locally ringed space* is a pair $X = (T, \mathcal{F})$ consisting of a topological space T together with a sheaf $\mathcal{F}$ on T whose stalks $\mathcal{F}_a$ are local rings at all points $a \in T$. A morphism $(T, \mathcal{F}) \to (S, \mathcal{G})$ of locally ringed spaces is a continuous mapping $f : T \to S$ together with a collection of ring homomorphisms $\alpha_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$, $V \subseteq S$ open, which are compatible with the restriction maps and satisfy the following property: For every $a \in f^{-1}(V)$ and every $h \in \mathcal{G}(V)$ with $h(f(a)) = 0$, one has $\alpha_V(h)(a) = 0$. Here, $h \in \mathcal{G}(V)$ is again regarded as a map to V as before. It is also said that $(T, \mathcal{F})$ and $(S, \mathcal{G})$ are isomorphic *via* $f : T \to S$. We obtain local ring homomorphisms $\alpha_a : \mathcal{G}_b \to \mathcal{F}_a$, for all $a \in T$ and $b = f(a) \in S$.

A *(general) scheme* is a pair $X = (T, \mathcal{O}) = (T_X, \mathcal{O}_X)$ consisting of a topological space T and a sheaf $\mathcal{O} = \mathcal{O}_X$ of rings on T that admits an open covering $T = \bigcup_{i \in I} U_i$ for which each pair $(U_i, \mathcal{O}_{|U_i})$ is isomorphic (as a locally ringed space) to an affine scheme $X_i = (\operatorname{Spec} R_i, \mathcal{O}_{X_i})$, where $\mathcal{O}_{X_i}$ denotes the structure sheaf defined by R_i on $X_i = \operatorname{Spec} R_i$. In particular, $U_i = \operatorname{Spec}(\mathcal{O}(U_i)) \cong X_i = \operatorname{Spec}(R_i)$. In contrast to the situation with affine schemes, the topological space T is now also given and not an object derived from a ring. Nevertheless, we will again write $T = X$, i.e., $X = (X, \mathcal{O}_X)$. The open subsets U_i are identified with the schemes X_i and are called *open affine cover* of X. Grothendieck in EGA also requires that X is separated, i.e., that the diagonal $\Delta_X : X \to X \times X$ is a closed embedding.

Using the concepts of sheaf theory, we then obtain the entire category of schemata and their basic constructions, i.e., in particular morphisms, open and closed subschemata and fiber products. These concepts can often be defined on affine charts, in which case the independence of the choice of charts must be shown.

Within the category of general schemes, we can now characterize the affine ones as follows Let $X = (T, \mathcal{F})$ be a topological space together with a sheaf $\mathcal{F}$ of rings. For $h \in R = \mathcal{F}(T)$ set $T_h = \{a \in T,\, h \text{ invertible in } \mathcal{F}_a\}$. Then X is an *affine scheme* if and only if: (i) $\mathcal{F}(T_h) = R_h$ for all h, (ii) the stalks $\mathcal{F}_a$ are local rings for all a, (iii) the induced map $f : T \to \operatorname{Spec} R$, given by $a \to \mathfrak{p}_a$, where $\mathfrak{p}_a$ is the pre-image under $R \to \mathcal{F}_a$ of the maximal ideal $\mathfrak{m}_a$ of $\mathcal{F}_a$ in R, is a homeomorphism. Clearly, the structure sheaf of an affine scheme fulfills these conditions. Conversely, for a pair $(T, \mathcal{F})$ with (i) to (iii), set $R = \mathcal{F}(T)$ and denote by $\mathcal{O}_X$ the structure sheaf associated to R on $X = \operatorname{Spec} R$. Then check that the ringed spaces $(T, \mathcal{F})$ and $(X, \mathcal{O}_X)$ are isomorphic via f.

Exercises Class IV. (1) Glue $V(xy)$ with $V(xy)$ along the y-axis, sending points close to zero to points close to infinity.

(2) Show that the intersection of all localizations $R_\mathfrak{p}$, $\mathfrak{p}$ a prime of R, coincides with R.

(3) Describe $\mathbb{A}^n \times \mathbb{P}^m$ ring theoretically.

(4) Let R_i° denote the ring of elements $\frac{P}{Q}$ of $K[x_0, x_1, ..., x_n, \frac{1}{x_i}]$ with homogeneous polynomials P and Q of the same degree. Show that R_i° is isomorphic to

$$K[y_0, y_1, ..., y_{i-1}, 1, y_{i+1}, ..., y_n].$$

(5) Find a better proof for Lemma 2.

(6) Give explicitly the chart maps for $\mathbb{P}^2$.

(7) Find a reference where the two definition of sheaves, one through sections on open subsets, one through the collection of stalks, are explained and shown to be equivalent.

(8) Show that an open subset of an affine scheme has a natural structure of a general scheme.

(9) Do this for $\mathbb{A}^3 \setminus \{0\}$.

(10) Express the definition of a sheaf on a topological space X as a functor on the category of open subsets of X, with morphisms the inclusions.

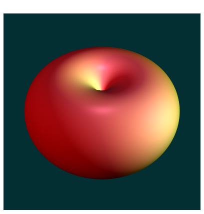
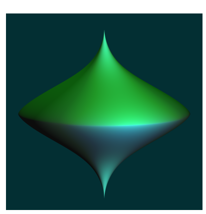
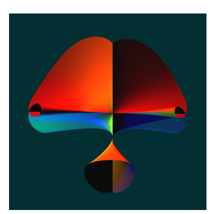
(11) Are the Möbius transformations of $\mathbb{P}^1$ also scheme morphisms?

(12) Why and how do we want to see the elements of a ring R as functions on the associated topological space $X = \operatorname{Spec} R$. Think of $R = K[x_1, ..., x_n]/I$. It is quite clear what the image of a closed point should be if $K = \overline{K}$, but for non-closed points one has to take into account residue fields.

(13) Define three embeddings of $\mathbb{P}^1$ into $\mathbb{P}^2$.

(14) How would you define the fiber product of general schemes? And how would you construct them?

(15) Consider a scheme X defined by $R = \mathbb{Z}[x_1, ..., x_n]/I$ for some ideal I. The inclusion map $\mathbb{Z} \to R$ gives a morphism $X \to \operatorname{Spec} \mathbb{Z}$. What are its fibers? Illustrate this in the example $I = \langle y^2 - x^3 - x \rangle$ of an elliptic curve.

CLASS V: "The Origin of Schemes & Examples"

Let us go back some 70 years and acompany the Bourbaki group in France in the year 1955.

A first Bourbaki meeting of part of the "*tribu*" took place end of February, beginning of March, in Southern France at La Ciotat, a touristic place in the Calanques not too far from Marseille. The group was small (Cartan, Dixmier, Koszul, Samuel, Serre) and discussed the forthcoming plan of how to prepare a treatise on Algebraic Geometry. There exist "Notes" of this meeting, here it is "La Ciotat-Tribu, nr. 35", and the prospective chapters were: I Algebraic Varieties, II the rest of chaper I, III Divisors, IV Intersections.

Three months later, end of May 1955, another meeting was held, this time in Chicago at the University there, where André Weil was working in that period. The unique topic was Algebraic Geometry, and the proposal of the La Ciotat meeting was largely ignored (or even rejected) by the participants, which aside from Samuel, were disjoint from those who were at the earlier meeting. The new plan was: I Schemes, II Theory of multiplicities for schemes, III Varieties, IV Calculation of Cycles, V Divisors, VI Projective Geometry, etc. The members present at that meeting were Dieudonné, Weil, Chevalley, Samuel, Borel, Lang.

On page 3 of the Notes of this meeting it reads:

Ou l'on explique ce que c'est qu'un schéma.

The geometric objects composing a scheme were called in the "Notes" of the Chicago meeting *tâches*, in english, *spots*. These are local rings, obtained by localizing a given ring R at a prime ideal $\mathfrak{p}$ (R was called *affine algebra* and of the form $A[x_1, ..., x_n]$, the elements x_i not necessarily being algebraically independent, but requiring that $A[x_1, ..., x_n]$ is an integral domain). And then, an affine scheme was defined as the collection of tâches which could be associated to a single affine algebra, having the same field of fractions.

It is interesting to observe here the psychological predisposition these mathematicians had: It was still mandatory to define everything within a given "universe", say, on a solid fundament, in this case, finitely generated A-algebras, A a Dedekind ring (A Noetherian integral domain of Krull dimension ≤ 1, as e.g. fields K or the rings $\mathbb{Z}$ and $K[t]$) and prescribing a common field of fractions. As such, the local rings had to be integral domains.

This viewpoint is confirmed by Chevalley's lectures in his joint seminar with Henri Cartan at École Normale Supérieure from fall 1955. On December 12th, exposé 5, Chevalley (re)-defines schemes (now publicly), with a slight modification. Citation: Fix a finite field extension $K \subseteq L$. An *affine algebra* is a finitely generated K-subalgebra of L. A *locality* (in French: *localité*) of L is any ring which is the localization of an affine algebra at a prime ideal. So this corresponds, in modern language, and given an affine scheme $X = \operatorname{Spec} R$, to the local rings $R_\mathfrak{p}$, for $\mathfrak{p}$ prime in R, say, the germs (X, a) of X at points $a = [\mathfrak{p}]$.

Chevalley then proceeds to define "dominance" (corresponding to injective ring homomorphisms, see our first class), then "apparentment" of two localities, meaning that they are dominated both by a third locality, and finally "specialization": This term is also used nowadays and just concerns an inclusion of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$, yielding R, $R_\mathfrak{p}$ and $R_\mathfrak{q}$. Then $R_\mathfrak{q}$

is called a specialization of $R_{\mathfrak{p}}$. Geometrically, $\mathfrak{p}$ defines an irreducible subscheme $Y = V(\mathfrak{p})$ inside $X = \operatorname{Spec} R$, and $Z = V(\mathfrak{q}) \subseteq$ is closed in Y. Thinking of germs, with $a = [\mathfrak{p}]$ and $b = [\mathfrak{q}]$, we have (X, b) being a specialization of (X, a). Think of $X = \mathbb{A}^2$, $\mathfrak{p} = \langle x - y \rangle$, $\mathfrak{q} = \langle x, y \rangle$, then (X, a) is the germ of $\mathbb{A}^2$ along the diagonal $Y = V(x - y)$, and (X, b) is the germ of $\mathbb{A}^2$ at the origin $\{0\} = V(x, y)$, or, if you wish, the germ of (X, a) at b.

The next items of Chevalley's exposé are the definition of the *dimension* of a locality (using transcendence degree) and of the *height* of a locality (using chains of prime ideals - already used and studied by Krull some 25 years earlier). He also gives a characterization of height in terms of minimal systems of generators of a primary ideal.

In section 2, *affine schemes* are defined as the collection of all localities of an affine algebra R. And Chevalley immediately remarks that the localities determine R, since $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, for $\mathfrak{p}$ varying over all primes of R (one may even just take maximal ideals). This description actually holds for arbitrary rings.

Lemma. *Let R be a (commutative) ring. Then $R = \bigcap_{\mathfrak{p} \subset R} R_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \subset R} R_{\mathfrak{m}}$, seeing $R_{\mathfrak{p}}$ as a subring of the ring of quotients* $\operatorname{Quot} R = M^{-1}R$, *where $M = \bigcup_{\mathfrak{p}} (R \setminus \mathfrak{p})$ is the set of non-zero divisors of R.*

Proof. Let $g \in \bigcap_{\mathfrak{m} \subset R} R_{\mathfrak{m}}$. There then exists for every maximal ideal $\mathfrak{m}$ an $h \in R \setminus \mathfrak{m}$ with $g = \frac{f}{h}$ for some $f \in R$. Hence $gh = f \in R$. Consider the ideal J of R of those elements $g \in R$ with $gh \in R$. It follows from the above that J is not contained in any maximal ideal of R. Zorn's Lemma implies that $J = R$. We finish with $g = g \cdot 1 \in R$. ↺

Observe here that one can only define the intersection $\bigcap_{\mathfrak{p} \subset R} R_{\mathfrak{p}}$ if all the localities $R_{\mathfrak{p}}$ are included in a common field (or ring), in Chevalley's case the extension L of K. Knowing this, one can indeed recover R from the localities (which, apparently, was important for the algebraic geometers from that time).

In our days, we see it from the opposite side, first comes the ring R and then, from it, all its localizations, respectively, prime ideals are extracted. And, suddenly, the hypothesis of a common a priori universe becomes redundant. This step - to pass to arbitrary rings first and to associate to them a geometric object - was carried out by Grothendieck himself, used already in his talk 1958 at the ICM in Edinburgh (see the bottom of page 105) and worked out systematically (together with Dieudonné) in 1960 in EGA 1 (Cartier's lecture on schemes from 2015 at IHP provides more details). All this was strongly influenced by Serre's "Faisceaux Algébriques Cohérents" from 1955, as well as Nagata's paper from 1956.

The next step in Chevalley's exposition is the introduction of the Zariski-topology on the set of localities (see again Cartier), and to show that the closure of a locality is the collection of all its specializations (Prop. 5). Section 3 is devoted to the definition of general schemes (as a union of affine schemes) and of projective space $\mathbb{P}^n$. And section 4 introduces "completeness" of schemes, showing that projective schemes are complete (analog of compactness in topology).

The reader may now be curious to look her- or himself at Chevalley's notes, and also at his second talk, Exposé 6 (where, among other things, the universal property of the Cartesian product of two schemes is described). After this, go directly to EGA 1, chap. 1.

Examples. To round up these notes, we will look at two instances where the language of schemes comes in handy. The first concerns the use of generic points and is very simple.

Proposition. *Let $Z \subseteq X$ be closed and irreducible, and $f : X \to Y$ a morphism of (affine) schemes. Then the (closure of the) image $f(Z)$ of Z is irreducible.*

Proof. As Z is closed and irreducible, it is the closure of a point $a = [\mathfrak{p}]$ of $X = \operatorname{Spec} R$, say $Z = \overline{\{a\}}$. Clearly, the image of a point under a morphism is again a point, hence $f(a) = b = [\mathfrak{q}]$ with $\mathfrak{q}$ prime in S, for $Y = \operatorname{Spec} S$. We get, by continuity, $f(Z) = f(\overline{\{a\}}) = \overline{\{f(a)\}} = \overline{\{b\}}$, and this is irreducible since equal to $V(\mathfrak{q})$. ◌

It follows from the proposition that the image of $X = \mathbb{A}^1$ under any non-constant morphism $\mathbb{A}^1 \to Y = V(xy) \subseteq \mathbb{A}^2$ is contained in only one component of Y, either the x- or the y-axis.

(2) We consider only Noetherian schemes here, so all dimensions are finite. The next example will be an instance where universal properties are used to define objects (here: blowups), and then used again to prove statements about blowups, but now using also the universal property of fiber products and of the Zariski-closure of a subset of a scheme.

A *Cartier divisor* of a scheme X is a closed subscheme Z of X of codimension 1 which, on a suitable open affine cover $X = \bigcup X_i$ of X, can be defined in each chart $X_i = \operatorname{Spec} R_i$, by a non-zero divisor of R_i, say $Z \cap X_i = V(f_i)$. If X is smooth and irreducible, codimension 1 alone suffices, but for singular schemes the notion is more subtle, see the author's notes "Blowups and Resolution" for examples. The complement of a Cartier divisor is dense in X.

Let X be a scheme and Z an arbitrary closed subscheme. A *blowup of X with center Z* is a morphism $\pi : \widetilde{X} \to X$ whose *exceptional divisor* $E = \pi^{-1}(Z)$ is a Cartier divisor in $\widetilde{X}$, and which is minimal in the following sense: If $\tau : \widetilde{X}' \to X$ is another morphism whose exceptional divisor $E' = \pi^{-1}(Z)$ is Cartier in $\widetilde{X}'$, then there exists a unique morphism $\sigma : \widetilde{X}' \to \widetilde{X}$ such that $\tau = \pi \circ \sigma$.

$$
\begin{array}{ccccccc}
F & \hookrightarrow & \widetilde{X}' & \overset{\sigma}{\dashrightarrow} & \widetilde{X} & \hookleftarrow & E \\
& \searrow & & \tau\searrow & \downarrow \pi & & \downarrow \\
& & & & Z \hookrightarrow X & \hookleftarrow & Z
\end{array}
$$

There exists an explicit construction of the blowup of X with center Z: If $Z = V(I) \subseteq X$, with $I \subseteq R$ the defining ideal of Z, where $R = \operatorname{Spec} R$, then

$$\widetilde{X} = \operatorname{Proj}\left(\bigoplus_{k=0}^{\infty} (It)^k\right),$$

where the *Rees algebra* $\mathcal{R}(X, Z) = \bigoplus_{k=0}^{\infty} (It)^k$ is understood as a graded ring, the k-th piece being $(It)^k$ (t is here a dummy variable to distinguish between I^k and $(It)^k$). We stipulate here that $(It)^0 = R$, so the elements of R have degree 0. This somewhat monstruous definition has a simple interpretation when $X = \mathbb{A}^n$ is affine space and $Z = \mathbb{A}^k = \mathbb{A}^k \times 0^{n-k}$ is a coordinate subspace. Then $\widetilde{X}$ is obtained by gluing (suitably) k copies of the rings $K[x_1, ..., x_n]$ along their ring extensions $K[x_1, ..., x_n, \frac{1}{x_j}]$, for $j = 1, ..., k$. Geometrically, k copies of $\mathbb{A}^n$ are glued along the principal open subschemes $\mathbb{A}^n \setminus V(x_j)$.

Now, if $X = \operatorname{Spec} R$ is a closed subscheme of $\mathbb{A}^n$, things seem to be more complicated, since the ring $R = K[x_1, ..., x_n]/I$ is no longer generated by algebraically independent elements. Here, the base change property of blowups is a wonderful tool to overcome the difficulty. In all generality, it goes as follows.

Proposition. *Let $\pi : \widetilde{X} \to X$ be the blowup of X along the closed subscheme Z, and let $\varphi : Y \to X$ be a morphism, the base change. Denote by $p : \widetilde{X} \times_X Y \to Y$ the projection from the fiber product to the second factor. Let $S = \varphi^{-1}(Z) \subseteq Y$ be the pre-image of Z under φ, and let $\widetilde{Y}$ be the Zariski-closure of the pre-image $p^{-1}(Y \setminus S)$ of $Y \setminus S$ in $\widetilde{X} \times_X Y$. Then the restriction $\tau : \widetilde{Y} \to Y$ of p to $\widetilde{Y}$ equals the blowup of Y along S (Fig. 1, replacing there $\widetilde{X} \times_X Y$ by $Y \times_X \widetilde{X}$).*

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[dr] & \widetilde{Y} \arrow[r, hook] \arrow[dr, "\tau"] & \widetilde{X} \times_X Y \arrow[r, "q"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "\pi"] \\
 & S \arrow[r, hook] & Y \arrow[r, "\varphi"] & X
\end{tikzcd}$$

Figure 1: Constructing the blowup τ of Y along S from the blowup π of X along Z.

Proof. The assertion is most elegantly proven by combining the universal property of blowups with the universal property of fiber products. To show that $F = \tau^{-1}(S)$ is a Cartier divisor in $\widetilde{Y}$, consider the projection $q : Y \times_X \widetilde{X} \to \widetilde{X}$ onto the second factor. The diagram

$$\begin{array}{rcl}
q : Y \times_X \widetilde{X} & \longrightarrow & \widetilde{X} \\
\downarrow p & & \downarrow \pi \\
\varphi : Y & \longrightarrow & X
\end{array}$$

commutes. This implies that $F = p^{-1}(S) \cap \widetilde{Y}$, where

$$p^{-1}(S) = S \times_Z \widetilde{X} = p^{-1} \circ \varphi^{-1}(Z) = q^{-1} \circ \pi^{-1}(Z) = q^{-1}(E).$$

As E is a Cartier divisor in $\widetilde{X}$ there exists an open affine cover $X = \bigcup \widetilde{X}_i$, with $\widetilde{X}_i = \operatorname{Spec} \widetilde{R}_i$, such that $E \cap \widetilde{X}_i$ can be defined by a principal ideal of $\widetilde{R}_i$ generated by a non-zero divisor. It follows that F is also locally defined in $\widetilde{Y}$ by a principal ideal, namely, by the pull-back of these principal ideals of $\widetilde{R}_i$ by the ring maps associated to q. To show that F is Cartier, it remains to check that these pull-backs are generated by non-zero divisors of wtR_i.

Note first that $p^{-1}(S) = S \times_Z E \subseteq Y \times_X \widetilde{X}$. As $\widetilde{Y}$ is the Zariski-closure of $p^{-1}(Y \setminus S) = (Y \setminus S) \times_X \widetilde{X}$, the irreducible components of $\widetilde{Y}$ are the Zariski-closures of the irreducible components of $(Y \setminus S) \times_X \widetilde{X}$. Hence, if $F = p^{-1}(S) \cap \widetilde{Y}$ would be defined locally in $\widetilde{Y}$ by a zero divisor, its intersection with $(Y \setminus S) \times_X \widetilde{X}$ would be a union of components of $(Y \setminus S) \times_X \widetilde{X}$. But F equals $p^{-1}(S) \cap \widetilde{Y} = (S \times_Z \widetilde{X}) \cap \widetilde{Y}$ where $(S \times_Z \widetilde{X})$ has empty intersection with $(Y \setminus S) \times_X \widetilde{X} = \emptyset$, so this is not possible. Hence F is locally defined by non-zero divisors. This proves that F is Cartier in $\widetilde{Y}$.

To show that $\tau : \widetilde{Y} \to Y$ fulfills the universal property, let $\psi : Y' \to Y$ be a morphism such that $\psi^{-1}(S)$ is a Cartier divisor in Y'. Since $\psi^{-1}(S) = \psi^{-1}(\varphi^{-1}(Z)) = (\varphi \circ \psi)^{-1}(Z)$, there exists by the universal property of the blowup $\pi : \widetilde{X} \to X$ a unique map $\rho : Y' \to \widetilde{X}$

such that $\varphi \circ \psi = \pi \circ \rho$. By the universal property of fiber products, there exists a unique map $\sigma : Y' \to \widetilde{X} \times_X Y$ such that $q \circ \sigma = \rho$ and $p \circ \sigma = \psi$.

The argument is depicted in the following diagram (Fig. 2).

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[dr, "\pi"] & \\
Y' \arrow[urr, bend left, "\rho"] \arrow[r, "\sigma"] \arrow[drr, bend right, "\psi"'] & \widetilde{X} \times_X Y \arrow[ur, "q"] \arrow[dr, "p"'] & & X \\
 & & Y \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Figure 2: The use of the universal properties of blowups and fiber products.

It remains to show that $\sigma(Y')$ lies in $\widetilde{Y}$, the Zariski-closure of $\mathfrak{p}^{-1}(Y \setminus S)$ in $\widetilde{X} \times_X Y$. Since $\psi^{-1}(S)$ is a Cartier divisor in Y', its complement $Y' \setminus \psi^{-1}(S)$ is dense in Y'. From

$$Y' \setminus \psi^{-1}(S) = \psi^{-1}(Y \setminus S) = (p \circ \sigma)^{-1}(Y \setminus S) = \sigma^{-1}(p^{-1}(Y \setminus S))$$

it follows that $\sigma(Y' \setminus \psi^{-1}(S)) \subseteq p^{-1}(Y \setminus S)$. But $\widetilde{Y}$ is the closure of $p^{-1}(Y \setminus S)$ in $\widetilde{X} \times_X Y$, so that, by the density of $Y' \setminus \psi^{-1}(S)$ in Y' and the continuity of σ, the inclusion $\sigma(Y') \subseteq \widetilde{Y}$ holds as required. $\circlearrowleft$

Special cases. Base changes and fiber products facilitate various operations with morphisms.

(a) Let $\pi : X' \to X$ be the blowup of X along a subvariety Z, and let Y be a closed subvariety of X. Denote by Y' the Zariski closure of $\pi^{-1}(Y \setminus Z)$ in X', i.e., the *strict transform* of Y under π. The restriction $\tau : Y' \to Y$ of π to Y' is the blowup of Y along $Y \cap Z$. In particular, if $Z \subseteq Y$, then τ is the blowup $\widetilde{Y}$ of Y along Z.

(b) Let $U \subset X$ be an open subvariety, and let $Z \subseteq X$ be a closed subvariety, so that $U \cap Z$ is closed in U. Let $\pi : X' \to X$ be the blowup of X along Z. The blowup of U along $U \cap Z$ equals the restriction of π to $U' = \pi^{-1}(U)$. Thus blowups can be defined locally.

(c) Let $a \in X$ be a point. Write (X, a) for the germ of X at a, and $(\widehat{X}, a) = \operatorname{Spec} \widehat{R}_{\mathfrak{p}}$ for the formal neighbourhood. There are natural maps

$$(X, a) \to X \qquad \text{and} \qquad (\widehat{X}, a) \to X$$

corresponding to the localization and completion homomorphisms $\mathcal{O}_X \to \mathcal{O}_{X,a} \to \widehat{\mathcal{O}}_{X,a}$. Take a point a' above a in the blowup X' of X along a subvariety Z containing a. This gives local blowups of germs and formal neighborhoods

$$\pi_{a'} : (X', a') \to (X, a), \qquad \widehat{\pi}_{a'} : (\widehat{X}', a') \to (\widehat{X}, a).$$

(d) If $X_1 \to X$ is an isomorphism between varieties sending a subvariety Z_1 to Z, the blowup X_1' of X_1 along Z_1 is canonically isomorphic to the blowup X' of X along Z. This also holds for local isomorphisms.

(e) If $X = Z \times Y$ is a cartesian product of two varieties, and a is a given point of Y, the blowup $\pi : X' \to X$ of X along $Z \times \{a\}$ is isomorphic to the cartesian product $\mathrm{Id}_Z \times \tau : Z \times Y' \to Z \times Y$ of the identity on Z with the blowup $\tau : Y' \to Y$ of Y in a.

Links

Grothendieck Audio 1973: www.youtube.com/watch?v=yysW-egOCR4

Dieudonné Video 1972: www.youtube.com/watch?v=qhkPtQWR_oY

Deligne Video 1996: www.youtube.com/watch?v=PeMAyPGjL68

Cartier Video 2015: www.youtube.com/watch?v=etdvnswIhMw

Neverendingbooks Birth of Schemes: www.neverendingbooks.org/the-birthplace-of-schemes

Bourbaki meeting March 1955: https://archives-bourbaki.ahp-numerique.fr/files/original/fe0a74e7fa0906cce661b9f2a96895e2.pdf

Bourbaki meeting May 1955: https://archives-bourbaki.ahp-numerique.fr/items/show/866#?c=0& m=0& s=0& cv=0

References

Séminaire Cartan-Chevalley 1955/56: www.numdam.org

Grothendieck, A.: The cohomology theory of abstract algebraic varieties. Proc. ICM Edinburgh, 1958, 103-118. Cambridge Univ. Press 1960

Grothendieck, A.: Éléments de Géométrie Algébriques: 1 Le Langage des Schémas. Publ. Math. l'IHÉS, 4 (1960), 5-228. www.numdam.org

Serre, J.-P.: Faisceaux Algébriques Cohérents. Ann. Math. 61 (1955), 1970-278

Nagata, M.: A general theory of algebraic geometry over Dedekind domains. Amer. Math. J. 78 (1956), 78-116

Hauser, H.: Das Kleine Buch über Schemata. Demnächst.

Epilogue. This concludes our brief exposition of "A Gentle Approach to the Theory of Schemes". It is not aimed at giving a full understanding, but it should convince the reader that even for a non-expert it is possible, without too much effort, to become familiar with the main definitions and constructions. If this could be achieved, this note has done its job. A much more extensive introduction to schemes by the author is forthcoming.

At that point we would like to thank the people from the Mathematics Departement of Instituto Superior Técnico, especially José Mourão and João Pimentel, for their hospitality and support during the author's stay at Lisbon. The interest and feedback of the audience - both students and researchers - wer instrumental to make this course a very enjoyable endeavour.

Herwig Hauser
Faculty of Mathematics
University of Vienna, Austria
herwig.hauser@univie.ac.at