String Theory and Integrability

LisMath seminar

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28/11/2018

Outline

1. Integrable systems: an introduction
2. The KdV and KP hierarchy
3. Conclusions: τ functions and string theory

Motivations - Beyond perturbation theory

In the field of mathematical physics, we can solve a restricted class of problems exactly.

The goal of modern physics is to go beyond perturbation theory, and to study the physics encoded in the non-perturbative properties of the physical observables.

Exactly solvable systems provide precious insight on non perturbative properties of the solution

Motivations - Integrability

What does it mean to solve a system? For definiteness - let us consider the case of a differential equation for a function $f(x)$.

A first definition would be

Attempt at definition

A differential equation for a function $f(x)$ is solved if an expression of $f(x)$ in terms of elementary functions is known.

This is too restrictive - in many cases, we have defined new functions starting from differential equations, and studied their properties!

Integrability - the setting I

We will study integrability in the context of Hamiltonian mechanics. Short recap of necessary ingredients:

- A phase space, that we take to be as $\mathbb{R}^{2n}$, on which we choose coordinates (q^i, p^i), $i = 1, ..., n$;
- Poisson brackets between functions $F, G : \mathbb{R}^{2n} \to \mathbb{R}$:

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p^i} - \frac{\partial G}{\partial q^i} \frac{\partial F}{\partial p^i} \right).$$

- An Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$, generating time evolution of q^i, p^i and any function of those:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q(t), p(t)) = \{F(q, p), H(q, p)\}.$$

Integrability - Differential equations

We can apply this formalism to systems of differential equations in one variable by looking for an Hamiltonian that generates the differential equation.

Example:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}q(t) = -\omega^2 q(t).$$

This is generated by the Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 :$$

We have $\dot{q} = \{q, H\} = p$ and $\dot{p} = \{p, H\} = \omega q$, that gives the starting differential equation.

Integrability - the setting II

We now give basic definitions that will bring us to the definition of integrability.

Definition

A function on the phase space F is said to be a *constant of motion* if, for every t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q(t), p(t)) = \{F(q(t), p(t)), H\} = 0.$$

Definition

Two functions F, G on the phase space are said to be in *involution* if we have

$$\{F, G\} = 0.$$

Integrability - the main definition

We are now ready to give the definition of integrability (in this context).

(Liouville) Integrability

An Hamiltonian system for a phase space of dimension $2n$ is said to be *integrable in the Liouville sense* if there exist n functions F_j $(j = 1, ..., n)$ such as:

- the F_j are independent,
- the F_j are conserved quantities,
- The F_j are in involution.

Integrability - Liouville-Arnold theorem

Why this definition? This comes from the result by Liouville and Arnold:

Liouville-Arnold theorem

An integrable Hamiltonian system (in the Liouville sense) admits canonical coordinates $(S_j(q,p), \theta_j(q,p))$ (where canonical means $\{S_j, \theta_k\} = \delta_{j,k}$) such as the Hamiltonian, written in those coordinates, is function of the S_j alone.

Equations for S_j and θ_j are then very simple!

$$\frac{\mathrm{d}}{\mathrm{d}t} S_j = \{S_j, H(S_j)\} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \theta_j = \{\theta_j, H(S_j)\} = \text{const.}$$

Integrability - a simple example

Let's do an example! Hamiltonian system with $n = 1$ and Hamiltonian

$$H = \frac{p^2}{2} + V(q).$$

A constant of motion is always available for Hamiltonian systems, as $\{H, H\} = 0$. One dimensional systems are always integrable!

Suppose initial condition $(q(0), p(0)) = (q_0, p_0)$

Integrability - a simple example

As H is conserved, the motion is going to take place on a level set, and we can use that to eliminate one of the variables:

$$E = \frac{p^2}{2} + V(q) \implies p = \pm 2\sqrt{E - V(q)}.$$

Then, from the motion equations, we obtain (choosing for example the plus sign)

$$\dot{q} = p = 2\sqrt{E - V(q)} \implies t = \int_{q_0}^{q(t)} \frac{\mathrm{d}x}{2\sqrt{E - V(x)}}.$$

We have solved the system by quadratures, and we have a formula giving $q(t)$ implicitly. E is the action variable, t is the angle variable.

Integrability - Lax pairs

A technique that is very useful for generating constants of motion is the technique of Lax pairs: it consists in rewriting the motion equations as

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [M, L],$$

where M, L are $2n \times 2n$ matrices whose entries are functions of the (q, p).

Constants of motion are given by

$$F_k = \mathrm{tr} L^k : \quad \frac{\mathrm{d}}{\mathrm{d}t} F_k = k \,\mathrm{tr}\Big(L^{k-1} [M, L] \Big) = 0.$$

Integrability - towards the infinite dimensional case

We have introduced the concept of integrability in a finite dimensional situation, but infinite dimensional cases are also important.

This is because many partial differential equations of interest can be described by infinite dimensional Hamiltonian systems.

Unfortunately, an analogue of the Liouville-Arnold theorem is not well estabilished in literature. Motivated by the earlier discussion, we will keep looking for constants of motion, keeping the previous definition unchanged (replacing n with ∞).

Integrability - conclusions

To conclude:

- A system is integrable when we can find a certain number of constants of motion.
- In the case in which we find those constants, we can solve the problem in quadratures.
- For infinite dimensional systems, we look for an infinite number of constants of motion.

1. Integrable systems: an introduction

2. The KdV and KP hierarchy

3. Conclusions: τ functions and string theory

Integrable hierarchies - the KdV equation

Our starting point is the Korteweg - de Vies (KdV) equation

$$\partial_t \phi + \partial_x^3 \phi - 6\phi\partial_x \phi = 0.$$

Physical curiosity: this equation describes scattering of waves in shallow water.

We will introduce a general formalism that will allow us to generate infinite constants of motion for the KdV equation, and then we will explore how to use this general formalism in problems that are of interest in string theory.

Integrable hierarchies - Lax formulation

The KdV equation can be expressed in terms of Lax pairs. In the infinite dimensional case, the Lax equation is always the same:

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [M, L].$$

Instead of being matrices, M and L are now operators!

For KdV equation,

$$L = \partial_x^2 + \phi, \quad M = \partial_x^3 + \frac{3}{2}\phi\partial_x + \frac{3}{4}\partial_x\phi.$$

Integrable hierarchies - differential operators

Before proceeding, we need to define the concept of differential and pseudo differential operators. Let $D = \partial_x$: the definition of D^n for n natural is obvious (and also encompasses the case $n = 0$). We can then define a differential operator A of order n:

$$A = \sum_{p=0}^{n} a_p(x) D^{n-p}.$$

We notice that the composition of an operator D^n with an operator of order 0 is given by the Leibniz rule:

$$D^n \circ f = \sum_{t=0}^{n} \binom{n}{t} (D^t f) D^{n-t}.$$

Integrable hierarchies - pseudo differential operators

The composition formula can actually be used to define pseudo differential operators: we can extend the formula as (now α is an arbitrary real number)

$$D^{\alpha} \circ f = \sum_{t=0}^{\infty} \binom{\alpha}{t} (D^t f) D^{\alpha - t}.$$

We then define a pseudo differential operator of order α as a formal object of the form

$$A = \sum_{j=0}^{\infty} a_j(x) D^{\alpha - j},$$

and use the previous definition to define composition.

Integrable hierarchies - further definitions

We can also define the root of a differential operator of order n, $A^{1/p}$ ($p \in \mathbb{N}$) as the operator B such as $B^p = A$, using the extended composition rule. We make the root unique by requiring that operators of the form $D^p + ...$ have roots of the form $D^{p/n} + ...$

As an example, let $A = D + u$: its square root can be expressed as

$$A^{1/2} = \sum_{j=0}^{\infty} r_j(x) D^{1/2-j}.$$

The composition rule gives

$$A = \sum_{j,l,t=0}^{\infty} \binom{\frac{1}{2} - j}{t} r_j (D^t r_l) D^{1-j-l-t}.$$

Integrable hierarchies - roots

The coefficients are then obtained by matching term by term. For the coefficient of D^1, $j = l = t = 0$, we have

$$\binom{\frac{1}{2}}{0} r_0^2 = 1 \implies r_0 = 1.$$

For the coefficient of D^0, we have

$$\binom{\frac{1}{2}}{0} r_0 r_1 + \binom{\frac{1}{2}}{1} r_0 r_0' + \binom{-\frac{1}{2}}{0} r_1 r_0 = u \implies r_1 = \frac{u}{2}.$$

Then we fix the rest of the coefficients by requiring that the coefficients of D^{-n} vanish for $n > 1$.

Integrable hierarchies - last definitions

We conclude this long string of definitions by defining, for a pseudo differential operator A of natural order n,

$$A_+ = \sum_{j=0}^{n} a_j D^{n-j}, \quad A_- = A - A_+.$$

Furthermore, we define the residue of A as the coefficient of D^{-1}, whenever it exists, and we have a natural commutator between pseudo differential operators

$$[A, B] = A \cdot B - B \cdot A.$$

Integrable hierarchies - towards a generalization

Let us now consider a more general problem, that will contain the KdV equation as a particular case and allow us to generate constants of motion for the equation.

We will consider the differential operator ($p \geq 2$)

$$L = D^p + \sum_{i=0}^{p-2} u_i(x) D^i.$$

We will look for deformations of L in a set of infinite parameters $t_1, t_2, ..., t_n, ...$ of the form (we always identify $t_1 = x$)

$$L(x, t_2, ...) = D^p + \sum_{i=0}^{p-2} u_i(x, t_2, ...) D^i$$

Integrable hierarchies - the KdV hierarchy

We obtain a generalization of the KdV equation by considering the flow of L in an infinite number of times, $t_1, t_2, ..., t_n, ...$, regulated by the commutators

$$\frac{\partial}{\partial t_n} L = [H_n, L].$$

The operators H_n are required to generate commuting flows: we will take

$$H_n = (L^{n/p})_+.$$

We then get an hierarchy of differential equations.

Integrable hierarchies - constants of motion

By commutativity of partial derivatives, we get the non trivial condition

$$[H_n, [H_m, L]] = [H_m, [H_n, L]].$$

As the Jacobi identity holds for the commutator, we get

$$[H_n, [H_m, L]] + [H_m, [L, H_n]] = 0 = -[L, [H_n, H_m]].$$

As this is valid for any L, we have

$$[H_n, H_m] = 0.$$

This implies that

$$\frac{\partial}{\partial t_n} H_m = 0,$$

so all H_n are constants of motion, in involution!

Integrable hierarchies - the KdV equation

We obtain the KdV hierarchy by specializing to a particular case: initial condition

$$L(0) = D^2 + \phi(x),$$

so we have

$$H_3 = D^3 + \frac{3}{2}\phi(x)D + \frac{3}{4}D\phi(x),$$

and the evolution is

$$\frac{\partial L}{\partial t_3} = \frac{\partial \phi}{\partial t_3} = [H_3, L] = \phi\frac{\partial \phi}{\partial x} + \frac{1}{6}\frac{\partial^3 \phi}{\partial x^3}.$$

Integrable hierarchies - the KP hierarchy

There exists a further generalization that has been extensively studied, that is the KP hierarchy. It is given by the evolution of an operator Q, defined as

$$Q = D + \sum_{j=0}^{\infty} q_j D^{-j},$$

with evolution given by

$$\frac{\partial}{\partial t_n} Q = [H_n, Q], \quad H_n = Q_+^n.$$

The p^{th} KdV hierarchy is a particular case, in which we impose that Q^p is a purely differential operator.

Integrable hierarchies - the dressing operator

An alternative way to express the solution of the hierarchy is to express it in terms of the dressing operator:

$$Q(t) = K(t) \cdot D \cdot K(t)^{-1},$$

with the normalization

$$K = 1 + \sum_{j=1}^{\infty} k_j D^{-j}.$$

The dressing operator transforms the trivial KP operator D (that is interpreted as a solution to the KP flow where the initial condition is $Q(0) = D$) in a non trivial KP operator that solves the KP flow.

Integrable hierarchies - the Baker-Akhiezer function

To get towards a solution, we introduce the Baker-Akhiezer (BA) function as the solution of the linear system. Let

$$\xi(t,\lambda) = \sum_{r=1}^{\infty} t_r \lambda^r.$$

Using this ξ function, we can define the BA function using the coefficients of the dressing operators k_j as

$$\Psi(t,\lambda) = \exp(\xi(t,\lambda)) \sum_{j=0}^{\infty} \frac{k_j}{\lambda^j}.$$

Integrable hierarchies - the τ function

From the BA function, we can introduce the τ function as (Sato's theorem)

$$\Psi(t,\lambda) = \frac{\tau\left(t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, t_3 - \frac{1}{3\lambda^3}, \dots\right)}{\tau(t_1, t_2, t_3, \dots)} \exp(\xi(t,\lambda)).$$

Matching order by order in λ, we get τ in terms of the BA coefficients k_i. Examples:

$$k_1 = -\frac{\partial \tau}{\partial t_1}\tau^{-1}, \quad k_2 = \frac{1}{2}\left(\frac{\partial^2 \tau}{\partial t_1^2} - \frac{\partial \tau}{\partial t_2}\right)\tau^{-1}, \dots$$

The τ function can be seen as a solution of an infinite number of differential equations, in an infinite number of variables.

Integrable hierarchies - interpretation of τ

Different τ functions will give different solutions of the KP hierarchy, corresponding to different initial conditions.

As a trivial example: $\tau = 1$ is a τ function for the initial condition $K = 1$, that is the trivial problem.

Other τ solutions are obtained through various methods. Those methods are not all! Not all τ functions are known.

Integrable hierarchies - the soliton solutions

Soliton solutions for the KP hierarchy are generated by transforming the trivial solution.

Consider the vertex operator

$$X(p,q) = \exp\left(\sum_{j=1}^{\infty}(p^j - q^j)t_j\right) \times \\ \times \exp\left(-\sum_{j=1}^{\infty}\frac{1}{j}(p^{-j} - q^{-j})\frac{\partial}{\partial t_j}\right).$$

With arbitrary parameters $c_1, ..., c_N$, the $N-$soliton solution is given by the τ function

$$\tau = \exp(c_1 X(p_1, q_1)) ... \exp(c_N X(p_N, q_N))1.$$

Integrable hierarchies - one soliton solution

Let us look at the simplest non trivial solution of the KdV hierarchy. The KdV $N-$solitons τ function is obtained by taking the particular case $q = -p$ and setting all even time parameters to zero. We can verify the result

$$X(p,q)^2 = 0 \implies \exp(cX(p,q)) = 1 + cX(p,q).$$

The one soliton τ function that only depends on the times $t_1 = x$ and $t_3 = t$ is given by

$$\tau = 1 + c\exp\left(2px + 2p^3 t\right).$$

Integrable hierarchies - one soliton solution

From the relation of τ with the dressing coefficients, in the case of the $p-$th order KdV hierarchy, we recover

$$\frac{\partial^2}{\partial t_1 \partial t_n} \log \tau = \text{res}\, L^{n/p}.$$

For the KdV equation, we have a KdV hierarchy with $p = 2$ and general form of the Lax operator given by

$$L = D^2 + \phi(x,t).$$

The residue of $L^{1/2}$ and the τ function are then related by

$$\frac{1}{2}\phi(x,t) = \text{res}\, L^{1/2} = \partial_x^2 \log \tau.$$

Integrable hierarchies - one soliton solution

From the explicit form of the τ function, we get

$$\phi(x,t) = 2\partial_x^2 \log \tau.$$

This can be explicitly verified to solve the KdV equation (with rescaled variables)

$$\partial_t \phi(x,t) = \frac{1}{4}\partial_x^3 \phi(x,t) + \frac{3}{2}\phi(x,t)\partial_x \phi(x,t).$$

In the particular case $c = 1$, the solution assumes a nice form:

$$\phi(x,t) = 2p^2 \operatorname{sech}[p(p^2 t + x)]^2.$$

Integrable hierarchies - conclusions

- We have found that we can generalize the KdV equation to a tower of differential equations in an infinite number of variables.
- This hierarchy naturally contains an infinite number of constants of motion.
- Solutions to this tower of equations are expressed in terms of an unique function, the τ function.
- From the τ function, we can obtain particular solutions to the KdV equation, and to the KdV hierarchy in general.

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A connection between strings and integrable hierarchies

There is a conjecture that, if verified, allows us to compute τ functions for the KdV hierarchy using exact results in minimal string theory.

In a typical string theory, we have observables $\mathcal{O}_n$, and we are interested on connected correlation functions on surfaces of genus g as

$$\langle \mathcal{O}_{n_1} \ldots \mathcal{O}_{n_s} \rangle_g .$$

The partition function

All of those correlation functions can be collected in the string generating functional (or partition function)

$$\tau(t) = \exp \sum_{g=0}^{\infty} \langle \exp \sum_{n \text{ odd}} t_n \mathcal{O}_n \rangle_g$$

The t_n are called coupling constants.

This partition function is a generating functional of the correlators, as

$$\frac{\partial}{\partial t_{n_1}} \dots \frac{\partial}{\partial t_{n_s}} \log \tau(t) \Big|_{t=0} = \langle \mathcal{O}_{n_1} \dots \mathcal{O}_{n_s} \rangle = \sum_{g=0}^{\infty} \langle \mathcal{O}_{n_1} \dots \mathcal{O}_{n_s} \rangle_g .$$

Usually, partition functions in string theory are computed genus by genus.

The partition function - the string coupling

Familiar partition functions in string theory usually present a string coupling g_S, that is very useful in practical computations. This can be reintroduced by rescaling the couplings as

$$t_n \to g_S{}^{\frac{n}{3}-1} t_n :$$

then the string coupling appears in the expansion of the partition function as

$$\tau(t) = \exp \sum_{g=0}^{\infty} g_S{}^{2g-2} \langle \exp \sum_{n \text{ odd}} t_n \mathcal{O}_n \rangle_g$$

Witten's conjecture

In the various string theory models, we have prescriptions to compute the partition function.

In the case of topological string theory, there exists a conjecture by Witten, that has then been rigorously proven by Kontsevich, stating that

Witten's conjecture

The partition function $\tau(t)$ for topological string theory is a τ function of the KdV hierarchy.

This means that solutions for topological string theory can give new τ functions for the KdV hierarchies!

Consequences of the conjecture

This proven conjecture relates two fields of string theory, namely matrix models and topological string theory.

This is because for matrix models it is already estabilished that, in their continuum limit, the partition function is a τ function of the KdV hierarchy.

Topological string theory (Intersection theory on moduli spaces)	$\leftrightarrow$	Matrix models (random surfaces and quantum gravity)

Consequences of the conjecture

Another consequence, of a more immediate interest, is that we can find τ functions from results in topological string theory!

As an example, consider Kontsevich's representation of the partition function in topological string theory:

$$\tau(Z) = \rho(Z)^{-1} \int \mathrm{d}Y \cdot \exp \mathrm{Tr} \left[-\frac{1}{2} Z Y^2 + \frac{\mathrm{i}}{6} Y^3 \right].$$

Here Z is an Hermitian matrix, and the integral is taken over Hermitian matrices Y, and

$$\rho(Z) = \int \mathrm{d}Y \exp -\frac{1}{2} \mathrm{Tr}\, Z Y^2, \quad t_n = -\frac{1}{n} \mathrm{Tr}\, Z^{-n}.$$

With the identifications of the t_n, $\tau(Z)$ is a τ function of KdV!

Consequences of the conjecture

Furthermore, consider the Virasoro operators

$$L_{-n} = \sum_{k=n+1}^{\infty} k t_k \frac{\partial}{\partial t_{k-n}} + \frac{1}{2} \sum_{i+j=n} i j t_i t_j,$$

$$L_0 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k},$$

$$L_n = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{2} \sum_{i+j=n} \frac{\partial^2}{\partial t_i t_j}.$$

From topological string theory, we have $L_n \cdot \tau = 0$, with $n \geq -1$ for τ partition function. This is a procedure to compute τ functions!

Conclusions

- In all string theories (and in most physical theories), a fundamental quantity to compute is the partition function.
- In matrix models (in the double scaling limit), the partition function is a τ function of the KdV hierarchy.
- In topological string theory, a result states that the partition function is a τ function of the KdV hierarchy. This also relates the two theories.
- From exact results in string theory, we can compute τ functions for the KdV hierarchy, and enrich our understandings of the hierarchy.

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