Kinematic Formulas in Convex Geometry

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Let γ_0, γ_1 be two piecewise differentiable curves in $\mathbb{R}^2$ and $\overline{SO(2)} = SO(2) \ltimes \mathbb{R}^2$ be the group of orientation-preserving isometries in $\mathbb{R}^2$.

- Question: Letting γ_0 be fixed and letting γ_1 traverse in the plane, how much do γ_0 and γ_1 intersect?

That is, we want to compute

$$\int_{\overline{SO(2)}} \#(\gamma_0 \cap \overline{g}\gamma_1) d\mu(\overline{g})$$

for a suitable measure μ.

A natural condition for μ is right-invariance, that is, for all $\overline{h} \in \overline{SO(2)}$,

$$\int_{\overline{SO(2)}} \#(\gamma_0 \cap \overline{g}\,\overline{h}\gamma_1) d\mu(\overline{g}) = \int_{\overline{SO(2)}} \#(\gamma_0 \cap \overline{g}\gamma_1) d\mu(\overline{g}).$$

As $\overline{SO(2)}$ is a locally compact Hausdorff topological group, we require μ to be a right Haar measure. It can be computed explicitely through differential forms:

$$\mu(A) = \int_A |da \wedge db \wedge d\phi|,$$

where (a, b, ϕ) are local coordinates for $\overline{SO(2)}$.

Poincaré's Formula

$$\int_{SO(2)} \#(\gamma_0 \cap g\gamma_1) d\mu(g) = 4|\gamma_0||\gamma_1|,$$

where $|\gamma_i|$ is the length of γ_i.

Similarly, for the space of lines in $\mathbb{R}^2$, $\overline{Gr}_1(\mathbb{R}^2)$, we can ask how much does a curve γ intersect $\overline{Gr}_1(\mathbb{R}^2)$.
A line is defined by their distance to the origin p and the angle ϕ its outward normal vector makes with the x-axis.

The measure

$$\mu_1(A) = \int_A |dp \wedge d\phi|$$

is $\overline{SO(2)}$-invariant, and we can obtain

Crofton's Formula

$$\int_{\overline{Gr_1}} \#(\gamma \cap E) d\mu_1(E) = 2|\gamma|.$$

Can these formulas be generalized to $\mathbb{R}^n$?

Convex bodies

Let $\mathcal{K}^n$ be the set of convex bodies (non-empty, compact convex sets) in $\mathbb{R}^n$. How do we measure how similar two sets $K, L \in \mathcal{K}^n$ are?

Hausdorff metric

Let B^n be the unit ball. The function $\delta : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R}$ given by

$$\begin{aligned}\delta(K,L) &:= \max\left\{\max_{x\in K}\min_{y\in L}\|x-y\|, \max_{x\in L}\min_{y\in K}\|x-y\|\right\}. \\ &= \min\left\{\varepsilon \geq 0 : K \subset L + \varepsilon B^n, L \subset K + \varepsilon B^n\right\}\end{aligned}$$

is called the Hausdorff metric.

Notice that $K + \varepsilon B^n = \{x \in \mathbb{R}^n : d(K,x) \leq \varepsilon\}$.

Convex Bodies

Polytope

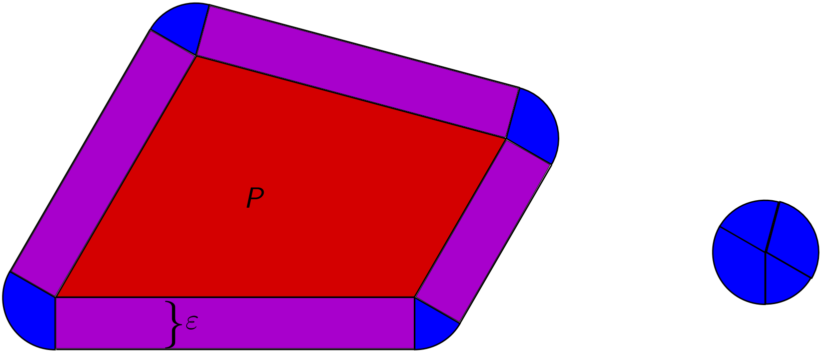
A polytope $P \subset \mathbb{R}^n$ is a bounded set that can be represented as the intersection of finitely many closed halfspaces ($\iff$ P is the convex hull of a finite subset of $\mathbb{R}^n$).

Polytopes are dense in $\mathcal{K}^n$

Let $K \in \mathcal{K}^n$ and $\varepsilon > 0$. Then there is a polytope $P \in \mathcal{K}^n$ with $P \subset K \subset P + \varepsilon B^n$, hence $\delta(K, P) \leq \varepsilon$.

In fact, we can cover K by finitely many balls of radius ε and centers $p_i \in K$. Then $P = \text{conv}(\{p_1, \ldots, p_n\})$ is the required polytope.

ε-thickening of a polytope in $\mathbb{R}^2$



$$\lambda_2(P + \varepsilon B^n) = \lambda_2(P) + \varepsilon \times \text{perimeter of P} + \pi\varepsilon^2$$

Steiner's Formula

Let $V_n = \lambda_n$ be the Lebesgue measure in $\mathbb{R}^n$. For every convex body K and $\rho \geq 0$, we have

$$V_n(K + \rho B^n) = \sum_{i=0}^{n} \rho^{n-i} \kappa_{n-i} V_i(K),$$

where $\kappa_j := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}$ is the volume of B^j

Idea of proof: Prove the formula first when $K = P$ is a polytope. Replacing ρ by $1, 2, \ldots, n+1$, we get a system of $n+1$ linear equations. Invert the system to write each $V_i(P)$ as a linear combination of $V_n(P + tB^n), t = 1, 2, \ldots, n+1$. Show that $K \mapsto V_n(K + tB^n)$ is continuous and extend the V_i by density.

Remark

For polytopes, we have the explicit formula

$$V_i(P) = \sum_{F \in \mathcal{F}_i(P)} \gamma(P, F) \lambda_i(F)$$

where $\mathcal{F}_m(P)$ are the m-dimensional faces of P and

$$\gamma(P, F) := \frac{\lambda_{n-i}(N(P, F) \cap B^n)}{\kappa_{n-i}},$$

where $N(P, F)$ is the set of outward normal vectors to F.

In general, $V_0 \equiv 1$ and V_{n-1} is half of the $(n-1)$-dimensional surface area.

The $V_i : \mathcal{K}^n \mapsto \mathbb{R}$ are $\overline{O(n)}$-invariant and i-homogeneous. Furthermore, their value is independent of the ambient space, and $V_k \equiv \lambda_k$ when restricted to k-dimensional convex bodies. Hence, they are called *intrinsic volumes*.

Example

Let K be the n-dimensional rectangle with side lengths $x_1, \ldots, x_n$. Then $V_i(K)$ is the i-th elementary symmetric polynomial on the variables $x_1, \ldots, x_n$:

- $V_0(K) = 1$;
- $V_1(K) = x_1 + x_2 + \cdots + x_n$;
- $V_2(K) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$;
- $V_i(K) = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} x_{j_1} \cdots x_{j_i}$;
- $V_n(K) = x_1x_2 \ldots x_n$.

More generally, a functional $\varphi : \mathcal{K}^n \to \mathbb{R}$ is called a *valuation* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$. Let Val denote the space of translation-invariant continuous valuations and

$$\mathrm{Val}^G := \{\varphi \in \mathrm{Val} \mid \varphi(gK) = \varphi(K) \quad \forall g \in G\}.$$

Hadwiger's Characterization Theorem

Let $\varphi \in Val^{SO(n)}$. Then there are constants $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$\varphi(K) = \sum_{i=0}^{n} c_i V_i(K)$$

for all $K \in \mathcal{K}^n$, that is, $\mathrm{Val}^{SO(n)}$ is a finite dimensional real vector space.

Invariant measures

$SO(n)$ has a Haar probability measure ν.

On $\overline{SO(n)}$, there is a Haar measure μ with $\mu(\gamma([0,1]^n \times SO_n)) = 1$.

Let Gr_q the set of all q-dimensional linear subspaces of $\mathbb{R}^n$, and $\overline{Gr}_q$ the set of all q-dimensional affine subspaces of $\mathbb{R}^n$.

There exists a $SO(n)$-invariant probability measure ν_q on Gr_q.

There exists a $\overline{SO(n)}$-invariant measure μ_q on $\overline{Gr}_q$. It is unique up to a constant factor.

Hadwiger's General Integral Geometric Theorem

If $\varphi \in \mathrm{Val}^{SO(n)}$, then

$$\int_{\overline{SO(n)}} \varphi(K \cap \overline{g}M) d\mu(\overline{g}) = \sum_{k=0}^{n} \varphi_{n-k}(K) V_k(M)$$

for $K, M \in \mathcal{K}^n$, where the coefficients $\varphi_{n-k}(K)$ are given by

$$\varphi_{n-k}(K) = \int_{\overline{Gr}_k} \varphi(K \cap E) d\mu_k(E)$$

Proof idea: show that $M \mapsto \int_{\overline{SO(n)}} \varphi(K \cap \overline{g}M) d\mu(\overline{g}) \in Val^{SO(n)}$ and apply Hadwiger's Characterization Theorem.

If $\varphi = V_j$, one can also show that $\varphi_{n-k} \in \mathrm{Val}^{SO(n)}$, so we obtain

Principal Kinematic Formula

$$\int_{\overline{SO(n)}} V_j(K \cap \overline{g}M)d\mu(\overline{g}) = \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K)V_k(M)$$

Crofton's Formula

$$\int_{\overline{Gr}_k} V_j(K \cap E)d\mu_k(E) = \alpha_{njk} V_{n+j-k}(K)$$

Here, $\alpha_{njk} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n+j-k+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}$

Applications

Let $K, K_0 \in \mathcal{K}^n$ such that $K \subset K_0$ and consider the $X_k \in \overline{Gr}_k$ such that $K_0 \cap X_k \neq \emptyset$

$$\mathbb{E}V_i(K \cap X_k) = \frac{\int_{\overline{Gr}_k} V_i(K \cap E) d\mu_k(E)}{\int_{\overline{Gr}_k} V_0(K_0 \cap E) d\mu_k(E)} = \frac{\alpha_{nik} V_{n+i-k}(K)}{\alpha_{n0k} V_{n-k}(K_0)}.$$

If K_0 is assumed to be known and $V_i(K \cap X_k)$ is observable, then

$$\frac{\alpha_{n0k} V_{n-k}(K_0)}{\alpha_{nik}} V_i(K \cap X_k)$$

is an unbiased estimator of $V_{n+i-k}(K)$.

What about for $GL(n, \mathbb{R})$? Trying to follow the proof of $SO(n)$ verbatim wouldn't work, since there is no Hadwiger's Characterization Theorem for $GL(n, \mathbb{R})$. Solution: use polar decomposition!

Polar Decomposition

Every $g \in GL(n, \mathbb{R})$ can be written uniquely as

$$g = \vartheta S,$$

where $\vartheta \in O(n)$ and $S \in SPD(n)$. Furthermore, this decomposition is unique.

The exponential map

$$\exp : \mathrm{Sym}(n) \to SPD(n)$$

is a diffeomorphism.

In Sym(n), we define the Gaussian measure

$$\gamma_n(A) = \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{n(n+1)}{2}} \int_A e^{-\frac{1}{2}\|X\|_F^2} d\rho(X),$$

where $\rho(X)$ is a Lebesgue measure in Sym(n) with respect to the Frobenius inner product $\langle X, Y\rangle_F = \mathrm{tr}(X^T Y)$. It is $O(n)$-conjugation invariant.

Hadwiger's General Integral Geometric Theorem for $GL(n, \mathbb{R})$

Let $\overline{GL(n,\mathbb{R})} := GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$ and $\mu_{\overline{GL(n,\mathbb{R})}} := \nu \times \gamma_n \times \lambda_n$. Then

$$\int_{\overline{GL(n,\mathbb{R})}} \varphi(K \cap gM) d\mu_{\overline{GL(n,\mathbb{R})}}(g) = 2 \sum_{k=0}^{n} c_k \varphi_{n-k}(K) V_k(M),$$

where φ_{n-k} is the same as above and

$$c_k = \frac{\int_{\mathrm{Sym}(n)} V_k(e^X B^n) d\gamma_n(X)}{\binom{n}{k} \frac{\kappa_n}{\kappa_{n-k}}}.$$

Remark

It is possible to compute $\int_{\text{Sym}(n)} V_k(e^X B^n) d\gamma_n(X)$ explicitely: $X = \vartheta D(X)\vartheta^T \in \text{Sym}(n)$ is diagonalizable, so

$$V_k(e^X B^n) = V_k(e^{\vartheta D(X)\vartheta^T} B^n) = V_k(\vartheta e^{D(X)}\vartheta^T B^n) = V_k(e^{D(X)} B^n).$$

In other words, $X \mapsto V_k(e^X B^n)$ is $O(n)$-conjugation invariant. Using a Weyl Integration Formula, it is enough to know the intrinsic volumes of ellipsoids. The latter was done recently by Gusakova, Spodarev and Zaporozhetsby in 2022.

In the 2000's, breakthroughs were made in regards to kinematic formulas in $\mathbb{C}^n$.

McMullen's Decomposition, 1977

Let V be a real n-dimensional vector space and

$$\mathrm{Val}_k(V) = \{\varphi \in \mathrm{Val}(V) \mid \varphi(tK) = t^k \varphi(K) \quad t \geq 0\}.$$

Then

$$\mathrm{Val}(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V).$$

Alesker, 2001

$$\dim \mathrm{Val}_k^{\mathrm{U}(n)} = \min\left\{ \left\lfloor \frac{k}{2} \right\rfloor, \left\lfloor \frac{2n-k}{2} \right\rfloor \right\} + 1.$$

Alesker, 2002

For non-negative integers p and k such that $2p \leq k \leq 2n$, let

$$U_{k,p}(K) = \int_{\overline{Gr}^{\mathbb{C}}_{n-p}} V_{k-2p}(K \cap E) d\mu^{\mathbb{C}}_{n-p}(E).$$

Then the $U_{k,p}$, for $0 \leq p \leq \min\left\{\left\lfloor \frac{k}{2} \right\rfloor, \left\lfloor \frac{2n-k}{2} \right\rfloor\right\} + 1$, form a basis of $\mathrm{Val}_k^{U(n)}(\mathbb{C}^n)$.

Alesker, 2002

For $K, M \in \mathcal{K}(\mathbb{C}^n)$, we have

$$\int_{\overline{U(n)}} U_{k,p}\left(K \cap gM\right) d\mu^{\mathbb{C}}(g) =$$

$$\sum_{k_1+k_2=2n} \sum_{p_1=0}^{\frac{\min\{k_1,k_2\}}{2}} \sum_{p_2=0}^{\frac{\min\{k_1,k_2\}}{2}} \gamma_{k_1,k_2,p_1,p_2}^{k,p} U_{k_1,p_1}\left(K\right) U_{k_2,p_2}\left(M\right),$$

for some constants $\gamma_{k_1,k_2,p_1,p_2}^{k,p}$.

We have a symplectic polar decomposition

$$Sp(2n, \mathbb{R}) \simeq U(n) \times Sp(2n, \mathbb{R}) \cap SPD(2n),$$

and the exponential map is a diffeomorphism between $\mathfrak{sp}(2n) \cap \mathrm{Sym}(2n)$ and $Sp(2n, \mathbb{R}) \cap SPD(2n)$. The former is a real $n(n+1)$-dimensional vector space, and we can define a Gaussian measure

$$\gamma_n^{\mathbb{C}}(A) = \left(\frac{1}{\sqrt{2\pi}}\right)^{n(n+1)} \int_A e^{-\frac{1}{2}\|X\|_F^2} d\rho(X),$$

Let $K, M \in \mathcal{K}(\mathbb{C}^n)$ and $\mu_{\overline{Sp(2n,\mathbb{R})}} := \lambda_{2n} \times \gamma^{\mathbb{C}} \times \nu^{\mathbb{C}}$. We have that

$$\int_{\overline{Sp(2n,\mathbb{R})}} U_{k,p}(K \cap gM) d\mu_{\overline{Sp(2n,\mathbb{R})}} =$$

$$\sum_{k_1+k_2=2n} \sum_{p_1=0}^{\frac{\min\{k_1,k_2\}}{2}} \sum_{p_2=0}^{\frac{\min\{k_1,k_2\}}{2}} \sum_{p_3=0}^{\min\left\{\left\lfloor\frac{k_2}{2}\right\rfloor, \left\lfloor\frac{2n-k_2}{2}\right\rfloor\right\}+1}$$

$$\gamma^{k,p}_{k_1,k_2,p_1,p_2} \beta_{k_2,p_2,p_3} U_{k_1,p_1}(K)\, U_{k_2,p_3}(M),$$

for some constants $\gamma^{k,p}_{k_1,k_2,p_1,p_2}, \beta_{k_2,p_2,p_3}$.

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