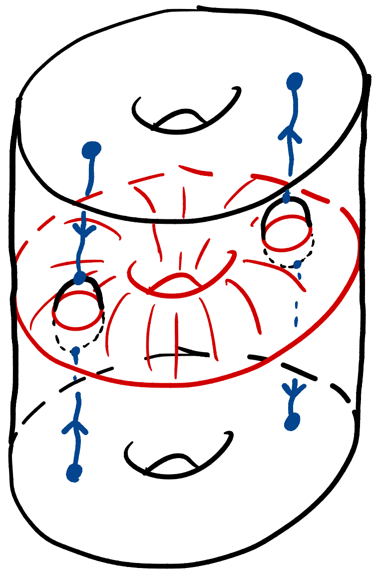
CFT/TFT correspondence beyond semisimplicity

Lisbon (online), 20.11.2024, Aaron Hofer

joint work with Ingo Runkel

based on [2405.18038] & work in progress



Outline: i) Motivation

ii) CFT's

iii) defect TFTs

iv) FRS-construction

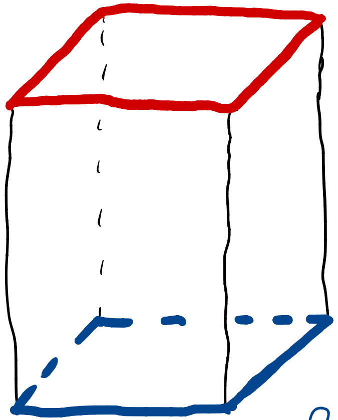
v) FRS-construction 2.0

i) Motivation

CFT–TFT correspondence:

top b.c.

$\mathcal{C} \boxtimes \mathcal{C}^{rev}$

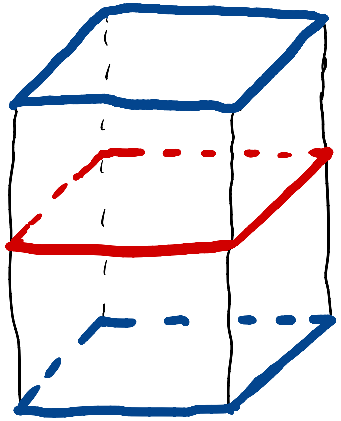


sym TFT

unfold $\rightsquigarrow$

$\leftsquigarrow$ fold

chiral CFT



$\mathcal{C}$

surface defect

$\mathcal{C}$

antichiral CFT

$Rep(VOA) = \mathcal{C}$

full 2d CFT

↑

What is this mathematically?

chiral conformal block space $\cong$ TFT state space

Working Hypothesis: vertex operator algebra $\longrightarrow$ sym. TFT

\+ modular functor

\+ field content

\+ correlators

conjecture

FRS-construction

\+ top. boundary condition / surface defect

ii) CFT's

Def A full modular functor is a symmetric monoidal 2-functor

$$\mathcal{B}l : \mathrm{Bord}^{oc}_{2+\varepsilon,2,1} \longrightarrow \mathrm{Prof}^{lex}_{\mathbb{K}}$$

0:	intervalls & circles	finite $\mathbb{K}$-linear ab. cats
1:	open-closed bordisms	left exact profunctors $\mathcal{A} \to\!\!\!\!\!\mapsto \mathcal{B} \Leftrightarrow \mathcal{A}^{op} \boxtimes \mathcal{B} \xrightarrow{lex} \mathrm{vect}_{\mathbb{K}}$
2:	diffeo's / isotopy	natural trafo's
$\diamond$:	gluing	convolution (coend $(\mathcal{G} \diamond \mathcal{F})(-,\sim) := \oint^{B \in \mathcal{B}} \mathcal{G}(B,\sim) \otimes_{\mathbb{K}} \mathcal{F}(-,B)$)
$\circ$:	composition	composition
$\sqcup$:	disjoint union	Deligne $\boxtimes$

Rem There is also a complex-analytic version of modular functors and it is conjectured to be "the same" as the topological one given above under certain conditions.

ii) CFT's

Def A full CFT for a full modular functor

$$Bl : \mathrm{Bord}^{oc}_{2+\varepsilon,2,1} \longrightarrow \mathrm{Prof}^{Lex}_{\mathbb{K}}$$

is a braided monoidal oplax natural transformation

$$\begin{tikzcd}
\mathrm{Bord}^{oc}_{2+\varepsilon,2,1} \arrow[r, bend left=50, "\Delta_{\mathbb{K}}"] \arrow[r, bend right=50, "Bl"'] \arrow[r, phantom, "\Downarrow \mathrm{Cor}"] & \mathrm{Prof}^{Lex}_{\mathbb{K}}
\end{tikzcd}$$

where $\Delta_{\mathbb{K}} : \mathrm{Bord}^{oc}_{2+\varepsilon,2,1} \longrightarrow \mathrm{Prof}^{Lex}_{\mathbb{K}}$ is the constant 2-functor to $\mathrm{vect}_{\mathbb{K}}$.

This definition encodes:

i) For every $\Gamma \in \mathrm{Bord}^{oc}_{2+\varepsilon,2,1}$ (1-manifold) a left exact profunctor:

$\mathrm{Cor}_{\Gamma}(-) : \mathrm{vect} \nrightarrow Bl(\Gamma) \cong \mathrm{Hom}_{Bl(\Gamma)}(\mathbb{F}_{\Gamma}, -)$ $\longrightarrow$ $\mathbb{F}_{\Gamma}$ state space on Γ (field content)

ii) For every 1-morphism (surface) $\Gamma \xrightarrow{\Sigma} \Gamma'$ in WS a natural transformation:

$\mathrm{cor}_{\Sigma} \in \mathrm{Nat}\left(\mathrm{cor}_{\Gamma'} \diamond \Delta_{\mathbb{K}}(\Sigma), Bl(\Sigma) \diamond \mathrm{cor}_{\Gamma}\right) \cong Bl(\Sigma)(\mathbb{F}_{\Gamma} ; \mathbb{F}_{\Gamma'})$ $\longrightarrow$ Correlators

space of conformal blocks

ii) CFT's

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This definition encodes:

iii) Naturality axioms encode mapping class group covariance (2-morphism naturality) and factorisation of correlators. (1-morphism naturality)

+ . . .

iii) defect TFT's

There is a notion of TFT with defects, where submanifolds of various codimension are decorated using extra data.

Thm [Reshetikhin-Turaev, Carqueville-Schaumann-Runkel]

Let $\mathcal{C}$ be a modular fusion cat. There exists a TFT with defects

$$Z_{\mathcal{C}} : \mathrm{Bord}^{\mathrm{def}}_{3,2}(\mathbb{D}_{\mathcal{C}}) \longrightarrow \mathrm{Vect}$$

constructed from $\mathcal{C}$.

Defect data

Rem i) The surface defects are obtained via orbifolding / condensation / gauging.

ii) For a rational VOA V, the category of representations $\mathrm{Rep}(V)$ is a modular fusion category.

iii) defect TFT's

There is a notion of TFT with defects, where submanifolds of various codimension are decorated using extra data.

Thm [Reshetikhin-Turaev, Carqueville-Schaumann-Runkel, De-Renzi-Gainutdinov-Geer-Patureau-Mirand-Runkel, H-Runkel]

Let $\mathcal{C}$ be a modular tensor cat. There exists a TFT with defects

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Defect data

Rem i) The surface defects are obtained via orbifolding / condensation / gauging.

ii) For a "finite log." VOA V, the category of representations $\mathrm{Rep}(V)$ is a modular tensor category.

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constructed from $\mathcal{C}$. Defect data

Thm [De-Renzi-Gainutdinov-Geer-Patureau-Mirand-Runkel, H-Runkel]

The above TFT induces a full modular functor

$$Bl_{\mathcal{C}} : \mathrm{Bord}^{oc}_{2+\varepsilon,2,1} \longrightarrow \mathrm{Prof}^{lex}_{\mathbb{K}}$$

$$I \longmapsto \mathcal{C} = \mathrm{Rep}(V)$$

$$S^1 \longmapsto \mathcal{C} \boxtimes \mathcal{C}^{rev}$$

$$\Sigma \longmapsto Z_{\mathcal{C}}(\Sigma \sqcup -\Sigma) \quad (\partial\Sigma = \emptyset)$$

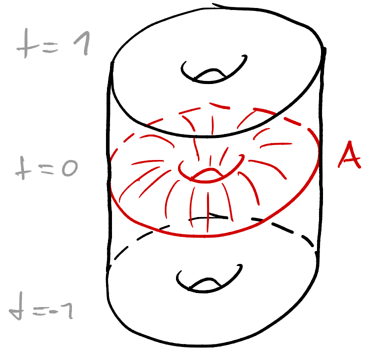
iv) FRS-construction

<u>Main idea:</u> $\mathrm{Cor}(\Sigma) \in Bl_{\mathcal{C}}(\Sigma) \cong Z_{\mathcal{C}}(\hat{\Sigma})$ with $\hat{\Sigma} = \Sigma \sqcup -\Sigma/\sim$ $\quad (p,+)\sim(p,-)$ if $p \in \partial^f \Sigma$

<u>Q</u>: Can we find a bordism $\emptyset \xrightarrow{M_\Sigma} \hat{\Sigma}$ such that $Z_{\mathcal{C}}(M_\Sigma)$ satisfies the conditions of a correlator?

Yes! But we need surface defect $A \in D^2_{\mathcal{C}} \subset D_{\mathcal{C}}$ as extra input.

e.g. $\Sigma =$ (torus)



$\xrightarrow{\text{Apply } Z_{\mathcal{C}}} \mathrm{Cor}_\Sigma \in Z_{\mathcal{C}}(\hat{\Sigma})$

connecting manifold

$M_\Sigma = \Sigma \times I/\sim$ with surface defect A at $\Sigma \times \{0\}$.

$(p,t)\sim(p,-t)$ if $p \in \partial^f(\Sigma)$

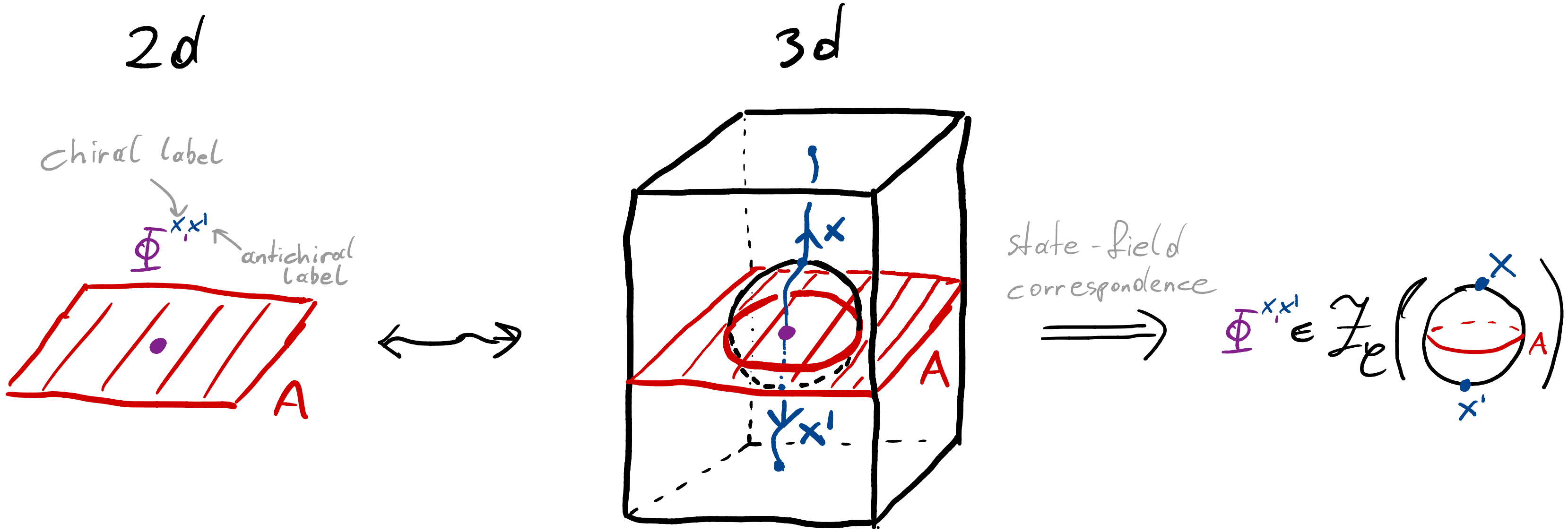
<u>Thm</u> [Felder-Fjelstad-Fröhlich-Fuchs-Runkel-Schweigert]

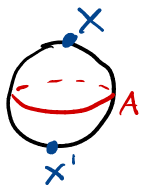
For $\mathcal{C}$ fusion, $\mathrm{Cor}^A_\Sigma = Z_{\mathcal{C}}(M^A_\Sigma)$ gives consistent correlators (only top level of $\Delta_{\mathbb{K}} \Rightarrow Bl_{\mathcal{C}}$) for any surface Σ.

<u>Q's</u> 1) In [F^4RS] the field content is determined algebraically can we get it topologically as well?

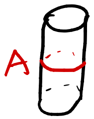
2) What about non-semisimple $\mathcal{C}$?

v) FRS-construction 2.0





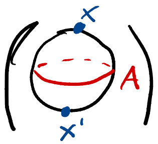
comes from the connecting manifold $M^A_{S^1}$ of S^1



$: \emptyset \longrightarrow \hat{S}^1 = S^1 \sqcup -S^1$

In analogy to construction of $Bl_{\mathcal{C}}$ on surfaces, we get a functor from $M^A_{S^1}$:

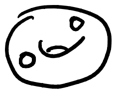
$$Cor^A_{S^1} : Bl_{\mathcal{C}}(S^1) = \mathcal{C} \boxtimes \mathcal{C}^{rev} \longrightarrow vect_{\mathbb{K}}$$

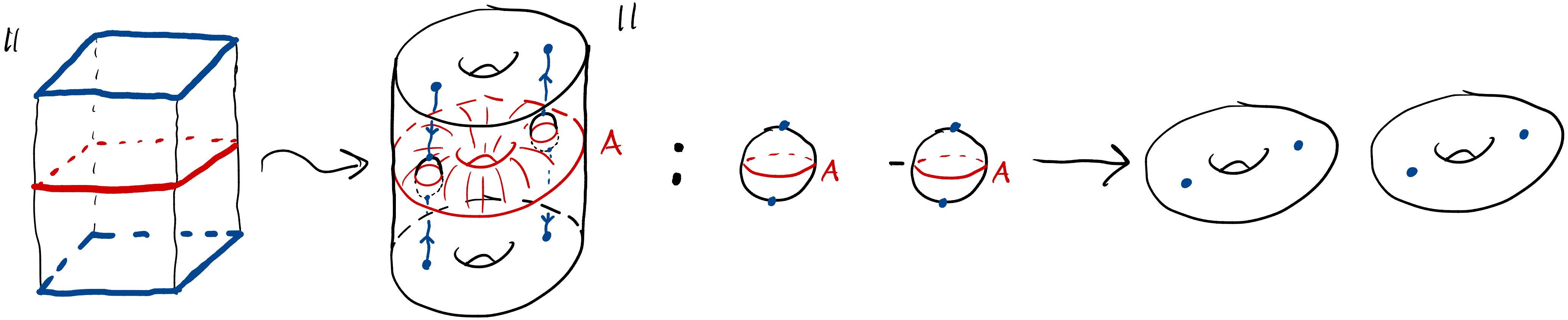
$x \boxtimes x' \longmapsto Z_{\mathcal{C}}($  $)$

$\mathbb{F}^A_{S^1} \in \mathcal{C} \boxtimes \mathcal{C}^{rev}$ is representing the space of CFT bulk fields!

$\simeq Rep(V \otimes \bar{V})$

v) FRS-construction 2.0

Back to correlators: $\Sigma =$  $: S^1 \to S^1$



$$Z_\mathcal{C}\left(M_\Sigma^{A\cdots}\right) : Z_\mathcal{C}\left(M_{S^1}^{A\cdots}\right) \otimes_{\mathbb{K}} Z_\mathcal{C}\left(-M_{S^1}^{A\cdots}\right) \longrightarrow Z_\mathcal{C}\left(\hat{\Sigma}^{\cdots}\right)$$

natural in
chiral & antichiral
labels

$$\Longrightarrow \qquad Cor_\Sigma^A : Cor_{S^1}^A \otimes_{\mathbb{K}} Cor_{S^1}^{A^+} \Longrightarrow Bl_\mathcal{C}(\Sigma)$$

$$\Longleftrightarrow \qquad Cor_{S^1}^A \diamond \Delta_{\mathbb{K}}(\Sigma) \Longrightarrow Bl_\mathcal{C}(\Sigma) \diamond Cor_{S^1}^A$$

v) FRS-construction 2.0

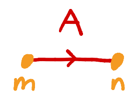
Thm [H-Runkel]
Let $\mathcal{C}$ be a modular tensor cat. Under one technical assumption on $Z_{\mathcal{C}}$, evaluation of the connecting manifold gives a full CFT

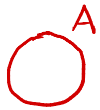
$$\begin{tikzcd} \mathrm{Bord}^{o.c.}_{2+\varepsilon,2,1} \arrow[r, bend left, "\Delta_{\mathbb{K}}"] \arrow[r, bend right, "Bl_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \mathrm{Cor}^{A}"] & \mathrm{Prof}^{\mathrm{dex}}_{\mathbb{K}} \end{tikzcd}.$$

for any $A \in D^2_{\mathcal{C}}$.

Rem We can also introduce boundary conditions & topological defects in the CFT and handle different A's at once.

Ex. Consider A the trivial surface defect: (Diagonal or Cardy case)
boundary conditions $\simeq \mathrm{ob}\,\mathcal{C}$

Field content: $I_{m,n}$ =  $\rightsquigarrow$ $\mathbb{F}_{I_{m,n}} \cong n \otimes m^* \in Bl_{\mathcal{C}}(I_{m,n}) = \mathcal{C}$

S^1 =  $\rightsquigarrow$ $\mathbb{F}_{S^1} \cong \int^{x \in \mathcal{C}} X \boxtimes X^* \in Bl_{\mathcal{C}}(S^1) = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$

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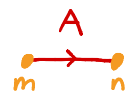
Field content: $I_{m,n}$ = 

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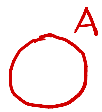
S^1 = 

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$\underset{\mathcal{C}\text{ fusion}}{\cong} \bigoplus_i i \boxtimes i^*$

Outlook

- Computations with A non-trivial?
- More general surface defects in $\mathcal{Z}_e$?
- Relation to other approaches?

Thanks for listening!