Special Lagrangians in Calabi-Yau 3-Folds with a K3-Fibration

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Lisbon Geometry Webnar
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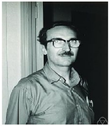
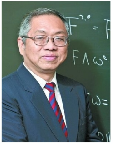
Outline of the Talk

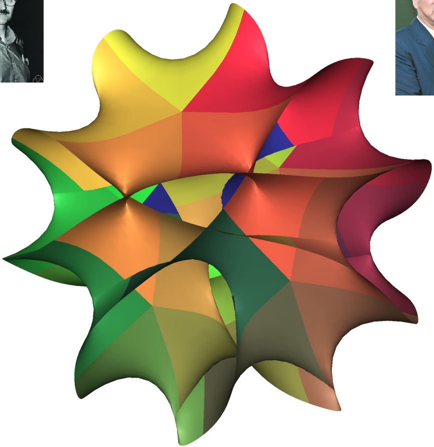
- Calabi-Yau Manifolds and Special Lagrangians
- Motivation from Tropical Geometry
- Geometric Setup and the Main Theorems
- "Smoothings" of Special Lagrangians
- Sketch of the Proof

Calabi-Yau Manifolds and Special Lagrangians

Calabi-Yau Manifolds







Calabi-Yau Manifolds

- A **Calabi-Yau** (CY) n-fold is an n-dim'l Kähler manifold w/
  1. a holomorphic volume form Ω, locally $f(z)dz_1 \wedge \cdots \wedge dz_n$.
  2. a Kähler form ω such that

  $$\omega^n = c\Omega \wedge \bar{\Omega},$$ complex Monge Ampere eq.

  where c is a constant only depends on n.

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- Ex: (Yau '76) Generic degree $n+1$ hypersurface in $\mathbb{P}^n$.

K3 Surfaces

- Compact simply connected CY surfaces called **K3 surfaces**.
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- K3 surfaces are hyperKähler (HK), i.e.
  $\exists$ integrable almost complex structure I, J, K satisfying
  $IJ = K = -JI, JK = I = -KJ, KI = J = -IK$.
  $\Rightarrow (aI + bJ + cK)^2 = -1$ if $a^2 + b^2 + c^2 = 1$.

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- Ex: Degree $(2, 4)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$
  Calabi-Yau 3-fold with a K3-fibration

Special Lagrangian Submanifolds

- L Lagrangian if $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} X$ and $\omega|_L = 0$.
- (Harvey-Lawson '82) A Lagrangian submanifold L in a CY is special Lagrangian (SLAG) if $\Omega|_L = e^{i\theta} vol_L$, for some constant $\theta \in S^1$.

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  If $[L] = [L']$, then

$$\int_L vol_L = \int_L \text{Re}\Omega|_L = \int_{L'} \text{Re}\Omega|_{L'} \leq \int_{L'} vol_{L'}$$

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- (Strominger-Yau-Zaslow conjecture) Calabi-Yau manifolds admits special Lagrangian torus fibration and the mirror is constructed by the dual fibration.
- SYZ conjecture is the guiding principle of mirror symmetry.
- "SLags conjecturally are stable objects in Fukaya category".

Examples of Special Lagrangians I, II

- Explicit Examples when the Calabi-Yau metric is explicit.

$$\text{Ex:} \qquad |x|^2 - |y|^2 = c_1, \text{Im}(xy) = c_2$$

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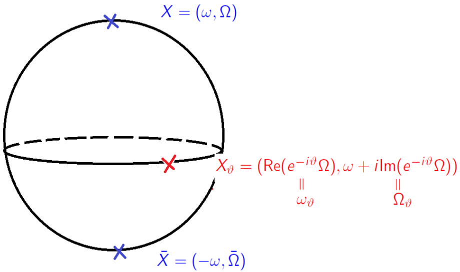
- Fix loci of an anti-holomorphic, anti-symplectic involution.

If $\iota^*\omega = -\omega, \iota^*\Omega = \bar{\Omega}$ and $\iota|_L = id$, then

$$\omega|_L = \iota^*\omega|_L = -\omega|_L \Rightarrow \omega|_L = 0.$$

Examples of SLAG III: HyperKähler Rotation

The hyperKähler triple (ω, Ω) induces an S^2-family of complex structures on the underlying space of X.



Then holomorphic curves in X $\Leftrightarrow$ special Lagrangians in X_ϑ.

A Useful Lemma

Lemma

Let $[L] \in H_2(K3, \mathbb{Z})$ *w/* $[\omega].[L] = 0$ *and* $[L]^2 = -2$*, then* $[L]$ *is represented by a special Lagrangian cycle.*

This is a consequence of HK rotation and Riemann-Roch theorem of surfaces.

Examples of SLAGs IV: LMCF

- Let L be a graded Lagrangian submanifold in X, i.e., $\exists$ the phase $\theta : L \to \mathbb{R}$ is the function such that

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  L is a special Lagrangian if θ is a constant.
- The mean curvature $\vec{H} = J\nabla\theta$ and the mean curvature flow is given by evolving family of immersions $F_t : L \to X$ with

$$\frac{\partial}{\partial t} F_t = \vec{H}.$$

- (Oh, Smoczyk) Lagrangian condition is preserved under mean curvature flow in Kähler–Einstein manifolds.
- Smooth Convergent Limit of LMCF gives Special Lagrangians.

LMCF

- (Wang '01) Conjectured that LMCF converges if the Lagrangian is almost calibrated.
- (Neves '10) ∃ Lagrangians arbitrary C^0-close to a special Lagrangian but LMCF develops finite time singularities.
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- (Joyce '14) Program of performing surgery before LMCF singularities arise to construct stability onf Fukaya categories.
- (CJL '19) First example of LMCF with smooth convergence w/o a priori knowing the limiting special Lagrangian exists.
- (CJL '24) Similar result for MCF.

Examples of SLAGs V: Unobstructed Deformation

Theorem (McLean '82)

Deformation of a special Lagrangian is unobstructed.

Given $\phi \in \Omega^1(L)$, define $V = \omega^{-1}\phi$ and $f_\phi : x \in L \mapsto exp_x(V(x))$.

$$\mathfrak{F} : \Omega^1(L) \mapsto \Omega^0(L) \oplus \Omega^2(L)$$
$$\phi \mapsto (*f_\phi^* \mathrm{Im}\Omega, f_\phi^* \omega)$$

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- The linearization is the Dirac operator $d + d^*$.
- Usually pair with known examples and the quantitative version of implicit function theorem.
- (Hein-Sun) CY near a conifold point admits a special Lagrangian vanishing cycle.

Motivation from Tropical Geometry

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- Mikhalkin considers the following self-diffeomorphism

$$(\mathbb{C}^*)^2 \xrightarrow{H_t} (\mathbb{C}^*)^2$$
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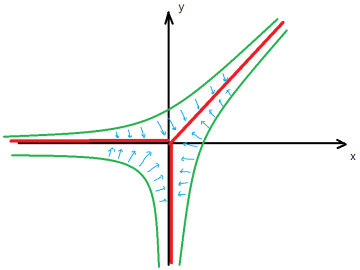
- This induces a new complex structure J_t.
- Metrically, this is the spirit of SYZ degeneration.

Tropical Curves as Collapsing Limits

- The image of $X + Y + 1 = 0$ under $Log \circ H_t$



converges (in the sense of Gromov-Hausdorff) to a tropical curve.

Characterization of Tropical Curves

Observation:

- Each edge is an affine line.
  They are gradient flow lines of certain area functionals.

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- Each edge is an affine line.
  They are gradient flow lines of certain area functionals.
- (balancing condition) At each vertex v,
  v_i: primitive integral vector tangent to the edge adjacent to v.

$$\sum_i w_i v_i = 0$$

New Special Lagrangians

Geometric Setting

- $\pi : X \to B$=Calabi-Yau 3-fold with K3 fibration and fibration is Lefschetz.
- $[\omega_X], [\omega_B]$: Kähler class of X, B
  ω_t unique CY metric $\in [\omega_X] + \frac{1}{t}\pi^*[\omega_B]$, $t \to 0$. adiabatic limit

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- (Tosatti '09) $b \notin \Delta$, then $\omega_t|_{X_b}$ converges to the unique Calabi-Yau metric in $[\omega_X|_{X_b}]$.
- (Li '18) Behavior of ω_t, $t \to 0$ with estimates.
- **Goal: Construction special Lagrangians in X when $t \to 0$.**

Quadratic Differential from the K3-Fibration

- $U \subseteq B$ open set and $[L] \in H_2(K3, \mathbb{Z})$ monodromy invariant within U (up to signs).
- $\alpha = \int_{[L]} \Omega$ a holomorphic 1-form on U.

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- $\alpha = \int_{[L]} \Omega$ a holomorphic 1-form on U.

  $\rightsquigarrow \phi = \alpha \otimes \alpha$ well-defined holomorphic quadratic differential.
- ex: $[L] =$ homology of the vanishing 2-spheres.
- Quadratic differential defines a flat metric s.t. dist $= \min_\gamma \int_\gamma |\alpha|$.

Admissible Paths and Admissible Loops

Input for the theorem:

- Admissible path: a path connecting two critical points of π such that
  1. vanishing cycles coincides up to sign via parallel transport.
  2. Geodesic respect to the vanishing cycle.
  3. Vanishing cycles represented by smooth special Lagrangian S^2 along the path.
- Admissible Loop: a loop in the base such that
  1. $\exists [L] \in H_2(X_y, \mathbb{Z})$ parallel invariant along the loop.
  2. Geodesic with respect to $[L]$.
  3. Smooth special Lagrangian representing $[L]$ along the loop.
  4. $\iota_v \omega_t$ is orthogonal to harmonic 1-forms in special Lagrangian lifting.

The Main Theorem

Theorem (Chiu-L '24)

Given an admissible path/loop in Y, $\exists$ smooth special Lagrangian L_t collapsing to the admissible path/loop.

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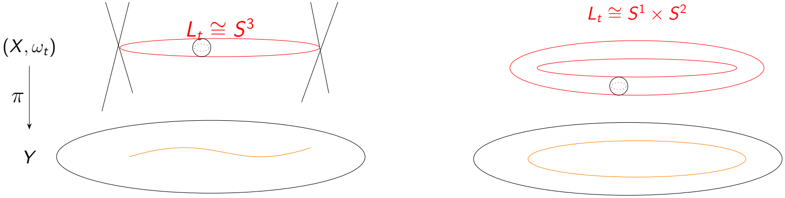

- We can engineer such admissible paths/loops.
- Near a conifold singularity, the shape of SLag S^3 is different from that of Hein-Sun.

The Donaldson-Scaduto Conjecture

- (Donaldson-Scaduto '19) Special Lagrangians in (X, ω_t) collapse to "gradient cycles".

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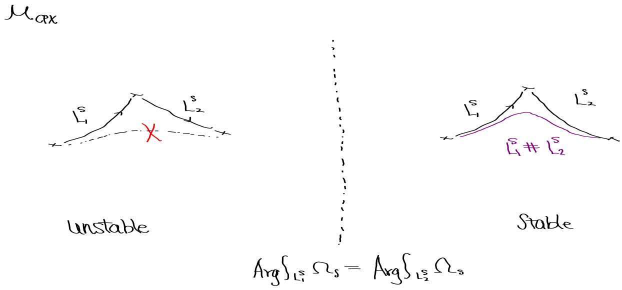
- (Donaldson-Scaduto '19) Special Lagrangians in (X, ω_t) collapse to "gradient cycles".
- Gradient cycles are union of geodesics of volume functional of certain 2-cycles in K3-fibres with "balancing conditions" at vertices.
- Admissible paths/loops are special cases of gradient cycles.

Application: Smoothing of Special Lagrangians



- This is related to the Thomas-Yau conjecture.
- This is the first example of the “smoothing” of two special Lagrangians in a compact Calabi-Yau manifolds.

Sketch of the Proof

Behavior of Collapsing CY Metrics

- Away from the singular fibres, ω_t is modeled by the semi-Ricci-flat metric.

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  $\omega_{SRF} = \omega_X + \frac{1}{t}\pi^*\tilde{\omega}_Y + i\partial\bar{\partial}\phi$
  w/ $\omega_X|_{X_b} + i\partial\bar{\partial}(\phi|_{X_b})$ CY metric of X_b.

- Near the singular fibre but away from the nodal point, ω_t is modeled by $\omega_{X_0} + \pi^*\omega_{\mathbb{C}}$, ω_{X_0} is the orbifold CY metric.

- Near the nodal point, ω_t is modeled by $\omega_{\mathbb{C}^3}$ after scaling.

The Non-Trivial Calabi-Yau Metric on $\mathbb{C}^3$

- (Li, Szekelyhidi, Colon-Rochon '17)
  Complete, full volume growth CY metric $\omega_{\mathbb{C}^3}$ but not flat.
- Opposite to $\mathbb{C}^2$ all complete, maximal volume growth CY metrics are flat.

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  Complete, full volume growth CY metric $\omega_{\mathbb{C}^3}$ but not flat.
- Opposite to $\mathbb{C}^2$ all complete, maximal volume growth CY metrics are flat.
- Tangent cone is $\mathbb{C}^2/\mathbb{Z}_2$.
- $t^{-1/3}\omega_t \to \omega_{\mathbb{C}^3}$ at the scale $r \lesssim O(t^{9/20})$ as $t \to 0$.
- $\omega_{\mathbb{C}^3}$ is asymptotic to fibrewise normalized Stenzel metrics.

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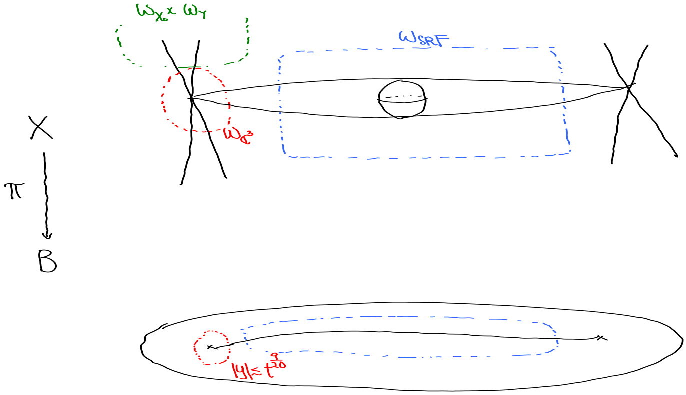
- The gluing region is at the scale $|y| \sim O(t^{9/20})$.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ω_SRF
ω_{C^3}
X
π
B
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Difficulties of the Perturbation

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    - Need to improve original Li's estimates.
2. The inverse of the linearized operator blows up as well.
    - The restriction of Li's weighted norm doesn't work.
3. The quadratic terms blows up.
    - This is because the diameter blows up when the geometry is scaled to be bounded.
    - Need to slightly change the implicit function theorem.

THANK YOU!