On the Donaldson-Scaduto conjecture

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Geometria em Lisboa Seminar

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Introduction

- Topic: Calabi-Yau 3-folds and special Lagrangians.

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- Motivation: differential geometry, Ricci flat.
- Algebraic geometry + symplectic topology: Mirror Symmetry.
- String theory: $10 = (3 + 1) + 6$.

- Calabi's conjecture/Yau's theorem: the existence of compact Calabi-Yau manifolds.

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- There are non-compact explicit examples.

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$$\omega^3 = \text{const.}\Omega \wedge \bar{\Omega}.$$

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- $L \subset Z$ is a special Lagrangian if

$$\text{Lagrangian:} \qquad \omega_L \equiv 0,$$

$$\text{special:} \qquad \mathrm{Im}(\Omega)_L \equiv 0.$$

I. The model Calabi-Yau 3-fold

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- Kodaira: Any compact hyperkähler 4-manifold is either a K3 surface or a torus $\mathbb{T}^4$.

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- Special interests:

$$Z = \text{K3} \times \mathbb{R}^2.$$

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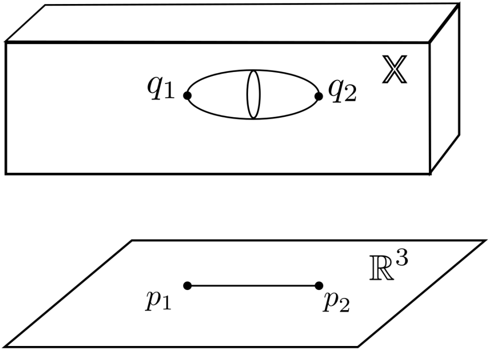

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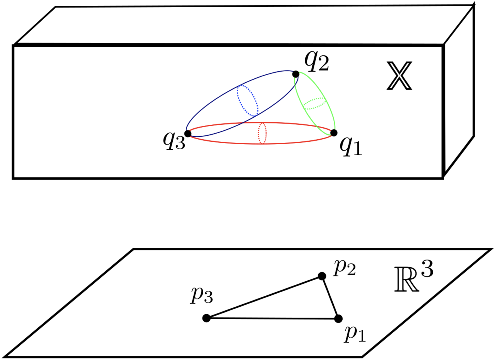

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  $V(u_1, u_2, u_3) = A + \sum_{i=1}^{n} \frac{1}{2|u-p_i|}, \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3, \quad A = \text{constant} \geq 0.$

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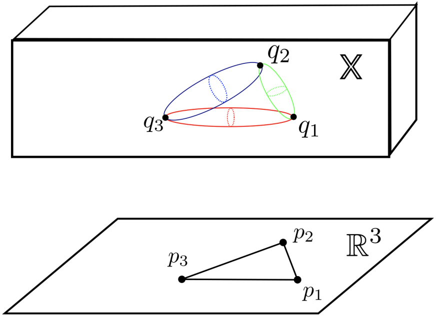
- The metric is given by $g_X = V^{-1}\theta^2 + V \sum_{i=1}^{3} du_i^2$.

Gibbons-Hawking

- A_2 type ALE hyperkähler manifold X_{A_2}: let $n = 3$ and $V = \sum_{i=1}^{3} \frac{1}{2|u-p_i|}$.
- Three 2-sphere $\Sigma_i := \pi^{-1}[p_i, p_{i+1}] \subset X$ is holomorphic.



II. Donaldson-Scaduto conjecture

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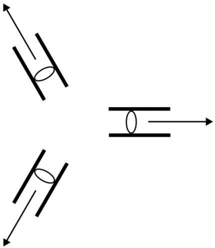
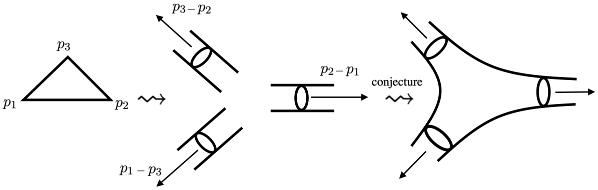


Donaldson-Scaduto conjectures

Conjecture (K3 Donaldson-Scaduto)

[...For suitable choices of $\Sigma_1, \Sigma_2, \Sigma_3$...], there is an associative submanifold homeomorphic to a three-holed 3-sphere $P \subset \mathrm{K3} \times \mathbb{R}^2$ with three ends asymptotic to cylinders P_1, P_2, and P_3.



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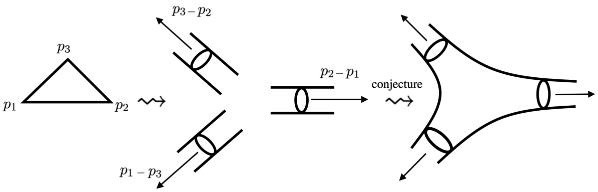
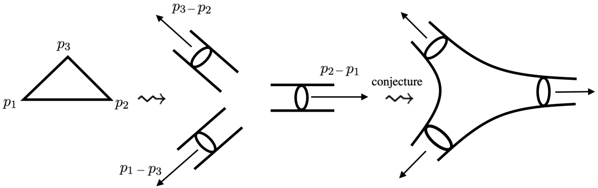


Conjecture 1 *Let* $\alpha_1, \alpha_2, \alpha_3$ *be* -2 *classes on the K3 manifold* X *with* $\alpha_1 + \alpha_2 + \alpha_3 = 0$. *Let* $\mathbf{R}^3 = H \subset H^2(X)$ *be a maximal positive subspace corresponding to a hyperkähler structure and* v_i *be the projection of* α_i *to* H. *Assume that the* (α_i, H) *are irreducible. Then there is an associative submanifold* $\Pi \subset X \times \mathbf{R}^3$ *with three ends asymptotic to* $\Sigma_i \times \mathbf{R}^+ v_i$ *where* Σ_i *is the complex curve representing* α_i, *for the complex structure defined by* v_i, *and* Π *is unique up to the translations of* $\mathbf{R}^3$.

- Expectation: many Calabi-Yau 3-folds/G_2-manifolds admit K3-fibrations.



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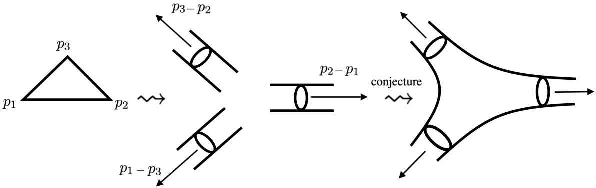

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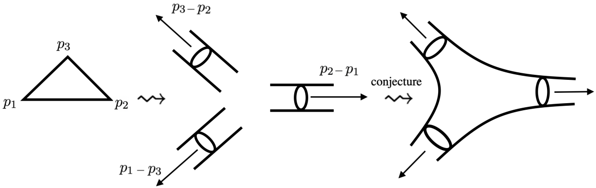
- Plumbing joints: building blocks in the gluing construction of special Lagrangians.



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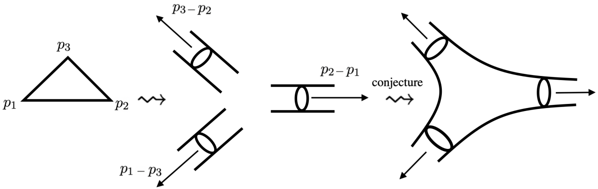
Theorem (E-Li)

There exists an $(n-3)$-dimensional family of special Lagrangians $P \subset X \times \mathbb{R}^2$ homeomorphic to a n-holed 3-sphere, with n ends asymptotic to the half cylinders $P_1, \ldots, P_n$.



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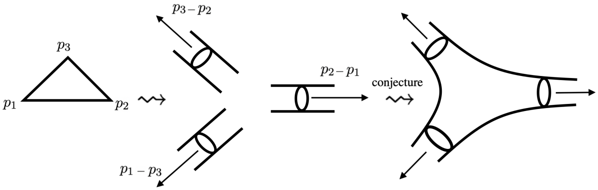

- The case $n = 3$, $A = 0$ implies the A_2 Donaldson-Scaduto conjecture.

III. Proof

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- Step 4. Show the special Lagrangians satisfy the conjecture: asymptotics + topology.

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  2. This action is Hamiltonian with moment maps $u_3 : Z \to \mathbb{R}$.
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- The dimensionally reduced Lagrangian is

$$L_{\text{red}} := L/U(1) \subset Z_{\text{red}} := u_3^{-1}(0)/U(1).$$

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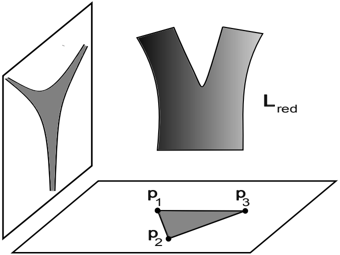

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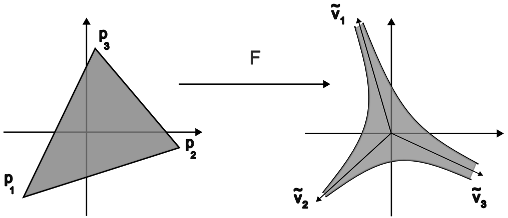

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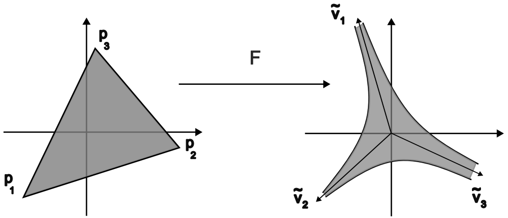

- The (special) condition $du_1 \wedge dy_1 + du_2 \wedge dy_2 = 0$ implies $F = (F_1, F_2)$ satisfies

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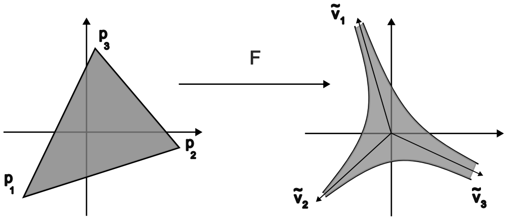

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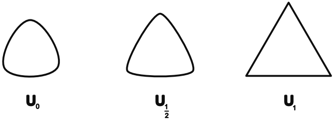
- The second condition (Lag) implies a degenerate Monge–Ampère equation equation:

$$\det D^2\varphi = V = A + \sum_{i=1}^{n} \frac{1}{2|u - p_i|}.$$

Solve the PDE

- Solving the Dirichlet problem: an approximation method and a compactness argument.



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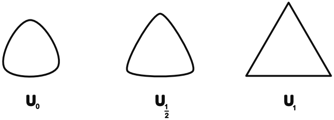

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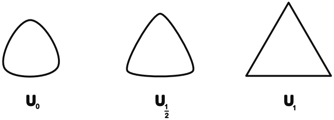

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  1. Caffarelli: The singular set must propagate along some line segment to the boundary.
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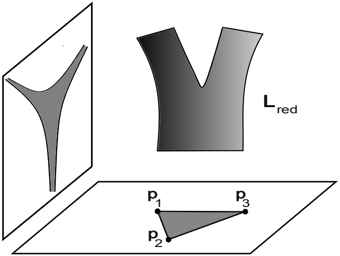
- This proves the existence of φ and therefore, the cruve.

Smoothness

- $L^\circ = \pi^{-1}(\mathrm{Graph}(F)_U)$.



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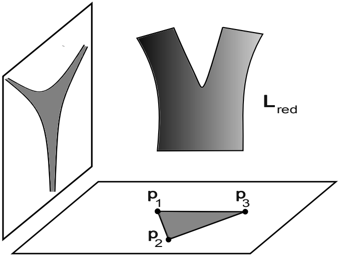

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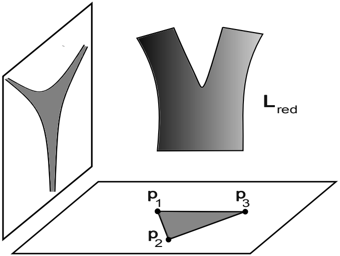

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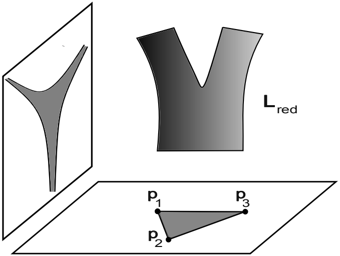

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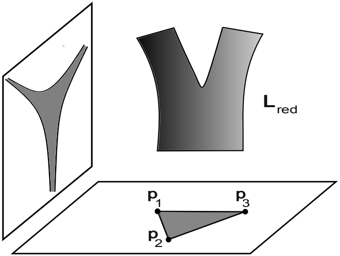
- $\pi_{(u_1,u_2)}(x)$ cannot be an interior point of U or any point over an open edge.
- The only possibility $\pi_{(u_1,u_2)}(x) \in \{p_1, \dots, p_n\}$.



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- Proposition: A point $x \in \text{supp}(L)$ is a smooth point if and only if every tangent cone $N \subset \mathbb{C}^3$ at x is a 3-plane with multiplicity one.
- There is a classification of $U(1)$-invariant special Lagrangian cones in $\mathbb{C}^3$ due to Joyce/Haskins.

Smoothness

- Proposition (Joyce/Haskins): Let N be a $U(1)$-invariant SL cone in $\mathbb{C}^3$. Then, exactly one of the following holds:
  1. N is a $\mathbb{T}^2$-cone.
  2. N is the singular union of two flat 3-planes.
  3. N is SL cone described in terms of Jacobi elliptic functions.
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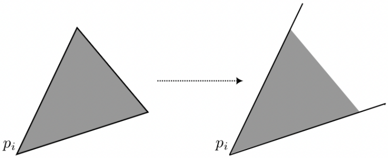
  4. N is a flat 3-plane with multiplicity $m \in \mathbb{Z}$.
- Using properties of the Monge–Ampère equation, we rule out every case but a flat 3-plane with $m = 1$.

Tangent cone analysis

- The projection of $\mathbb{T}^2$-cone is surjective.



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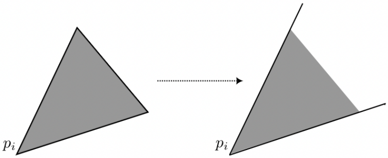

Tangent cone analysis

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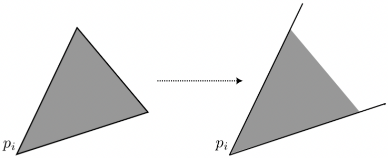
- Rulling out the Jacobi elliptic case and flat 3-plane with $|m| \geq 1$ follows from a variation of Joyce graphicallity argument + some GMT ingrediants.



Special Lagrangian cones

- Asymptotically cylindrical:

  1. Using Legendre transform + quasi-Elliptic property: C^0 convergence.
  2. Allard's regularity: $C^{1,\alpha}$-decay, and then C^k-decay.
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- This completes the proof.

IV. Epilogue

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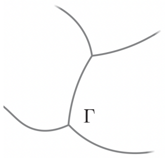
- Donaldson initiated a program to study G_2-manifolds through coassociative K3 fibrations, in the adiabatic limit where the diameters of the K3 fibers shrink to zero.
- Subsequent work of Donaldson and Scaduto provided a conjectural limiting description of associative submanifolds in the adiabatic setting.



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  $A2$ conjecture $\Rightarrow$ K3 conjecture for open subset of moduli of hyperkähler K3 surfaces.
- Therefore, the K3 Donaldson-Scaduto conjecture is proven for infinitely many K3 surfaces, but not all, which is the topic of current research topic.

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Thank you for your attention!