The twistor transform and the ADHM construction

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Summary

Spoilers

- Magnetic monopole as a prototype of a gauge theory with group $U(1)$, seen as the Hopf fibration.
- Generalize it to $SU(2)$ gauge theory, habitat of instantons.
- ADHM construction of instantons via twistor correspondence.

Electric charge

A point electric charge q located at the origin of an inertial frame determines an electric field **E** given by Coulomb's Law (written here in spherical coordinates ρ, ϕ, θ):

$$\mathbf{E} = \frac{q}{\rho^2}\mathbf{e}_\rho.$$

The magnetic field associated to the charge q, in this frame, is given by $\mathbf{B} = 0$.

Maxwell's equations

E and **B** satisfy the static, source-free Maxwell equations on $\mathbb{R}^3 \setminus \{0\}$:

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0 \\ \operatorname{div} \mathbf{B} &= 0 \\ \operatorname{curl} \mathbf{E} &= 0 \\ \operatorname{curl} \mathbf{B} &= 0. \end{aligned}$$

Magnetic charge

Suppose that Coloumb's Law also holds for a "magnetic point charge" (which has never been observed in nature). Thus we would have, by analogy,

$$\mathbf{E} = 0$$
$$\mathbf{B} = \frac{g}{\rho^2}\mathbf{e}_\rho,$$

where the constant g is the strength ("magnetic charge") of this magnetic monopole.

Maxwell's equations

- **E** and **B** clearly satisfy the static, source-free Maxwell equations on $\mathbb{R}^3 \setminus \{0\}$. In particular,

$$\operatorname{div} \mathbf{B} = 0 \text{ on } \mathbb{R}^3 \setminus \{0\},$$
$$\operatorname{curl} \mathbf{B} = 0 \text{ on } \mathbb{R}^3 \setminus \{0\}. \tag{1}$$

- (1) $+$ simple-connectedness of $\mathbb{R}^3 \setminus \{0\} \implies$ existence of *scalar* potential for **B**.
- Want: vector potential for **B**: vector field **A** such that $\mathbf{B} = \operatorname{curl} \mathbf{A}$ for physical reasons.

Potentials

- $\operatorname{div} \mathbf{B} = 0$ is necessary for the existence of a vector potential ($\operatorname{div} \operatorname{curl} = 0$), but not sufficient.
- Stokes' theorem tells us that there is no vector potential for $\mathbf{B}$ on $\mathbb{R}^3 \setminus \{0\}$.

Simple-connectedness

- Simple-connectedness + vanishing curl $\implies$ existence of *scalar* potential.
- Topological condition: vanishing of “fundamental group” $\pi_1(\mathbb{R}^3 \setminus \{0\}) = 0$.
- Fundamental group encodes classes of loops (maps from the circle S^1 to the space) deformable to one another.
- Also called 1-connectedness.

2-connectedness

- 2-connectedness: classes of maps from S^2 to a space U which form a group $\pi_2(U)$.
- Intuitively, if U is 1-connected, $\pi_2(U) = 0$ means that any copy of S^2 in U encloses only points of U.
- $\pi_2(\mathbb{R}^3 \setminus \{0\}) \neq 0$; take any sphere around the origin.

Existence of vector potential

Proposition

$U \subset \mathbb{R}^3$ open 1-connected, $\mathbf{F}$ smooth vector field on U. If $\operatorname{div} \mathbf{F} = 0$ and $\pi_2(U) = 0$, then there exists a smooth vector field $\mathbf{A}$ on U such that $\mathbf{F} = \operatorname{curl} \mathbf{A}$.

Example

- Let $Z_- = \{(0,0,z) \in \mathbb{R}^3 \mid z \leq 0\}$,
  $Z_+ = \{(0,0,z) \in \mathbb{R}^3 \mid z \geq 0\}$.
- The complements $U_\mp = \mathbb{R}^3 \setminus Z_\pm$ are simply-connected and 2-connected $\implies \exists$ vector potentials $\mathbf{A}_\pm$ for $\mathbf{B}$ on $U_\pm$.
- If $\mathbf{B} = (g/\rho^2)\mathbf{e}_\rho$, then we can calculate

$$\mathbf{A}_+ = \frac{g}{\rho \sin\phi}(1 - \cos\phi)\mathbf{e}_\theta, \qquad \mathbf{A}_- = -\frac{g}{\rho \sin\phi}(1 + \cos\phi)\mathbf{e}_\theta.$$

Example

- Note that $U_+ \cup U_- = \mathbb{R}^3 \setminus \{0\}$.
- On the overlap $U_+ \cap U_- = \mathbb{R}^3 \setminus \{z \text{ axis}\}$, we have

$$\mathbf{A}_+ - \mathbf{A}_- = \frac{2g}{\rho \sin\phi}\mathbf{e}_\theta = \nabla(2g\theta),$$

i.e. they differ by a gradient, which means

$$\operatorname{curl}\mathbf{A}_+ = \operatorname{curl}\mathbf{A}_-,$$

and so **B** is well-defined on the overlap.

Physical significance of potentials

Consider now that, apart from the magnetic monopole at the origin of an inertial frame, we have an electric charge q free to move in space.

- Classically, the dynamics of the charge are determined by the Lorentz Force Law and Newton's second law.
- In quantum mechanics, the dynamical variable is the wavefunction and it is determined by Schrödinger's equation.

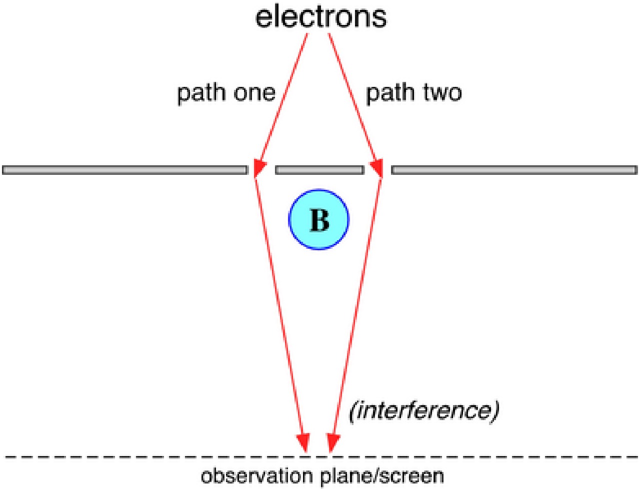
Physical significance of potentials

- In classical theory, the motion of the charge is unchanged by a transformation $\mathbf{A} \to \mathbf{A} + \nabla\Omega$ of the potential. Thus, it has no physical significance, only the field.
- In the quantum picture, Schrödinger's equation tells us that replacing $\mathbf{A} \to \mathbf{A} + \nabla\Omega$ transforms the wavefunction as

$$\psi \to e^{iq\Omega}\psi,$$

  changing only the amplitude (phase change) of the wavefunction.

The Aharonov-Bohm effect

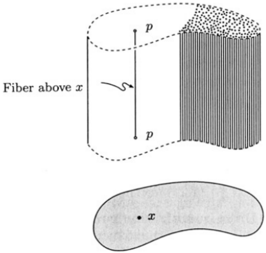


Bundles

- Lack of global potential introduces phase differences.
- How to keep track of the phase of a particle as it travels through space?

Bundles

- Associated to each point x in space, we have a circle (line segment with ends identified) of possible phase values.
- These circles (“fibers”) are “bundled” together.
- A phase change corresponds to an action of S^1 on each fiber.



Bundles

- Keeping track of the charge's phase is a "lifting problem": given the trajectory of the charge in space and a value of the phase at some given point, specify a curve through the total bundle space.
- The phase should vary continuously, so the fibers S^1 need to be bundled together topologically.

Differential forms

Let $U \subset \mathbb{R}^3$ be an open set.

- 0-forms on U are by definition real-valued functions on U.
- 1-forms on U: $f_1 dx + f_2 dy + f_3 dz$
- 2-forms on U: $g_1 dy \wedge dz + g_2 dz \wedge dx + g_3 dx \wedge dy$.
- Both correspond to a vector field on U with components (f_1, f_2, f_3) and (g_1, g_2, g_3), respectively.
- 3-forms on U: $f dx \wedge dy \wedge dz$.
- A p-form eats p vectors (skew-symmetric when exchange order) and spits out a number.

Differential forms

Let f be a 0-form, ω a 1-form and F a 2-form.

- Exterior differentiation takes a p-form to a $(p+1)$-form.
- $f \to df$ corresponds to ∇f.
- $\omega \to d\omega$ corresponds to $\operatorname{curl} \omega$
- $F \to dF$ corresponds to $\operatorname{div} F$.

Bundles over S^2

- The monopole field $\mathbf{B} = (g/\rho^2)\mathbf{e}_\rho$ has a corresponding 2-form F, and the vector potential can now be thought of as a 1-form A such that $F = dA$.
- The vector potentials $\mathbf{A}_+$ and $\mathbf{A}_-$ correspond to the 1-forms

$$A_+ = g(1 - \cos\phi)d\theta, \qquad A_- = -g(1 + \cos\phi)d\theta.$$

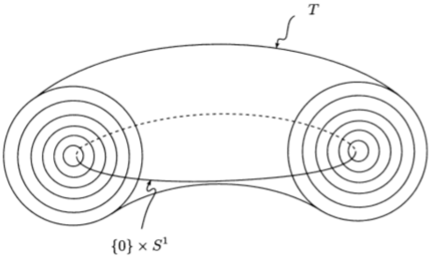
- Independence of $\rho \implies$ phase is constant in radial direction. Thus, only need to consider bundles over S^2.

The Hopf bundle

- Define S^3 as the set of $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$.
- Use polar coordinates in each entry: $z_1 = \cos(\phi/2)e^{i\xi_1}$, $z_2 = \sin(\phi/2)e^{i\xi_2}$ for $\phi \in [0, \pi]$ and $\xi_1, \xi_2 \in \mathbb{R}$.
- Let $T \subset S^3$ be given by $|z_1| = |z_2|$, i.e. all the points $(z_1, z_2) = (1/\sqrt{2})(e^{i\xi_1}, e^{i\xi_2})$, clearly a torus (product of two circles).

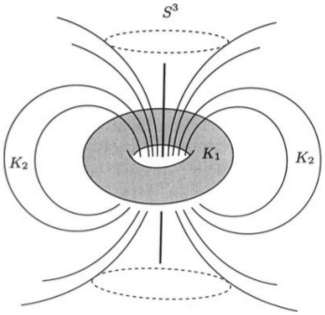
The Hopf bundle

- Let $K_1 \subset S^3$ be given by $|z_1| \leq |z_2|$, i.e. $\cos(\phi/2) \leq \sin(\phi/2) \implies \pi/4 \leq \phi/2 \leq \pi/2$.
- $\phi/2 = \pi/4$ gives us T; $\phi/2 = \pi/2$ means $z_1 = 0$ and so we have $\{0\} \times S^1$, a circle.



The Hopf bundle

- Let $K_2 \subset S^3$ be given by $|z_1| \geq |z_2|$. Similarly, we have $0 \leq \phi/2 \leq \pi/4$.
- Also a solid torus bounded by T, degenerating into a circle (here seen as a line through infinity).



The Hopf bundle

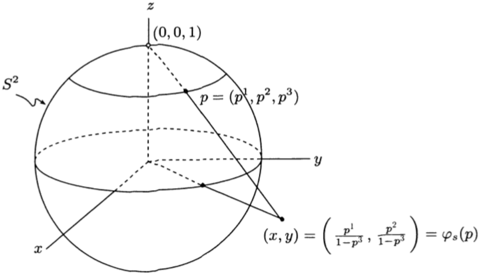
- Action of $U(1)$ (more traditional in gauge theory instead of S^1) on S^3:
$$(z_1, z_2) \cdot g = (z_1 g, z_2 g).$$
- Each orbit of the action corresponds to a point in S^2: if $(z_1, z_2) \sim (w_1, w_2)$, then $z_1/z_2 = w_1/w_2$, so it suffices to take one point of the orbit and take its ratio, which is a complex number (possibly ∞).

The Hopf bundle

Consider the map $\pi : S^3 \to S^2$ given by

$$\pi(z_1, z_2) = \varphi_S^{-1}\left(\frac{z_1}{z_2}\right),$$

where φ_S is stereographic projection from the north pole onto the extended complex plane.



The Hopf bundle

- In terms of the parameters ϕ, ξ_1, ξ_2 given by $(z_1, z_2) = (\cos(\phi/2)e^{i\xi_1}, \sin(\phi/2)e^{i\xi_2})$, we can write

$$\pi(\phi, \xi_1, \xi_2) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

  where $\theta = \xi_1 - \xi_2$.
- π maps S^3 onto S^2 and for any $x \in S^2$, the *fiber* $\pi^{-1}(x)$ is the orbit of any (z_1, z_2) above x, i.e. such that $\pi(z_1, z_2) = x$.

The Hopf bundle

Conclusion:

- $\pi : S^3 \to S^2$ identifies each orbit in S^3 (a copy of S^1) with a point in S^2.

Definition

Let M be a manifold (e.g. S^2) and G a Lie group (e.g. $U(1)$). A *principal bundle* over M (the *base space*) with *structure group* G consists of

- A manifold P (e.g. S^3), called *total space*,
- A map $\pi : P \to M$, called *projection*,
- An action of G on P.

Definition

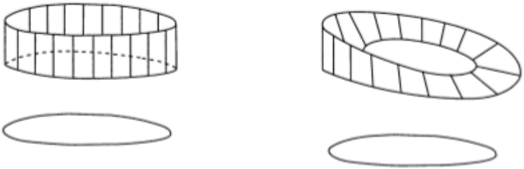
Moreover, we require that

1. The action of G on P leave the fibers invariant, i.e. $\pi(p \cdot g) = \pi(p)$.
2. *(Local triviality)* For each $x \in M$ there exists an open set $U \ni x$ and a diffeo $\Psi : \pi^{-1}(U) \to U \times G$ of the form $\Psi(p) = (\pi(p), \psi(p))$, where $\psi : \pi^{-1}(U) \to G$ satisfies

$$\psi(p \cdot g) = \psi(p)g \text{ for all } p \in \pi^{-1}(U) \text{ and } g \in G.$$

Local triviality

- The Hopf map $\pi : S^3 \to S^2$ is similar to the natural projection $S^2 \times U(1) \to S^2$.
- Both slice the total space into a disjoint union of circles hovering above each point of S^2.
- The fibers are glued together differently when viewed globally, but can "untwist" them locallly so as to be the same:



Local triviality of the Hopf bundle

- Cover the sphere with $U_S \cup U_N = S^2$.
- Can trivialize the Hopf bundle $\pi : S^3 \to S^2$ on each open set of the cover

$$
\begin{aligned}
\Psi_S : \pi^{-1}(U_S) &\to S^2 \times U(1) \\
(z_1, z_2) &\mapsto \left(\pi(z_1, z_2), \frac{z_2}{|z_2|} \right), \\
\Psi_N : \pi^{-1}(U_N) &\to S^2 \times U(1) \\
(z_1, z_2) &\mapsto \left(\pi(z_1, z_2), \frac{z_1}{|z_1|} \right).
\end{aligned}
$$

Transition functions for the Hopf bundle

- If $x \in U_S \cap U_N$, then Ψ_S and Ψ_N identify $\pi^{-1}(x)$ with $U(1)$ differently. Let $\psi_{S,x}, \psi_{N,x} : \pi^{-1}(x) \to U(1)$ be the restriction to the fiber. We have

$$\psi_{S,x} \circ \psi_{N,x}^{-1}(g) = \left(\frac{z_2/|z_2|}{z_1/|z_1|}\right) g,$$
$$\psi_{N,x} \circ \psi_{S,x}^{-1}(g) = \left(\frac{z_1/|z_1|}{z_2/|z_2|}\right) g,$$

for any $(z_1, z_2) \in \pi^{-1}(x)$.

Transition functions for the Hopf bundle

- Get functions

$$g_{SN} : U_S \cap U_N \to U(1), \qquad g_{NS} : U_S \cap U_N \to U(1)$$

called *transition functions*, defined by

$$\psi_{S,x} \circ \psi_{N,x}^{-1}(g) = g_{SN}(x)g, \qquad \psi_{N,x} \circ \psi_{S,x}^{-1}(g) = g_{NS}(x)g.$$

- In terms of the parameters ϕ, ξ_1, ξ_2, with $\theta = \xi_1 - \xi_2$, we have

$$g_{SN}(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = e^{-i\theta},$$
$$g_{NS}(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = e^{i\theta}.$$

Connections on principal bundles

Consider again $A_+ = g(1 - \cos\phi)d\theta$, $A_- = -g(1 + \cos\phi)d\theta$ for $g = 1/2$ (physical reasons).

- Correspond to potential 1-forms on $U_S, U_N \subset S^2$.

$$A_N = \frac{1}{2}(1 - \cos\phi)d\theta, \qquad A_S = \frac{1}{2}(1 + \cos\phi)d\theta.$$

- $A_N = A_S + d\theta$ on $U_S \cap U_N$.
- Replace $A_N \to -iA_N$ and $A_S \to -iA_S$, can write above equation as

$$\begin{aligned} A_N &= e^{i\theta} A_S e^{-i\theta} + e^{i\theta} d e^{-i\theta} \\ &= g_{SN}^{-1} A_S g_{SN} + g_{SN}^{-1} d g_{SN}. \end{aligned}$$

Lie algebra-valued 1-forms

- Locally defined 1-forms A_1, A_2 cannot be spliced together into a globally-defined form unless they agree on the intersection.
- Lie algebra-valued 1-forms *can*, if they satisfy the consistency condition

$$A_2 = g_{12}^{-1} A_1 g_{12} + g_{12}^{-1} dg_{12},$$

be spliced together into a globally-defined Lie algebra-valued form *on the principal bundle*.

Wrapping up

- Lie algebra of $U(1)$ is $T_1 U(1) \cong \operatorname{Im} \mathbb{C}$. Rotating the circle identifies $T_\theta U(1) \cong T_1 U(1) \cong \operatorname{Im} \mathbb{C}$.
- Each $p \in S^3$ has through it a $U(1)$ fiber $\implies$ each $T_p S^3$ has $T_1 U(1) \cong \operatorname{Im} \mathbb{C}$ as subspace.
- A Lie algebra-valued 1-form ω on S^3 is a correspondence $p \mapsto \omega_p : T_p S^3 \to \operatorname{Im} \mathbb{C} \subset T_p S^3$ (like a projection).
- The collection $\ker \omega_p$ as p varies in S^3 determine a 2-dim. distribution and the derivative of $\pi : S^3 \to S^2$ restricts to an isomorphism $\ker \omega_p \cong T_{\pi(p)} S^2$ for each p.

Wrapping up

- Each velocity vector along a curve in S^2 lifts to a unique vector in the tangent space of S^3 via the isomorphisms.
- Given an initial condition, i.e. an initial phase, the lifted vectors can be fitted with a unique integral curve lifting the original curve in S^2.

Nomenclature

- These Lie algebra-valued 1-forms ω are called *connections* on the principal bundle.
- The exterior derivative $\Omega = d\omega$ is called the *curvature* of the connection.
- In the Hopf bundle, the connection replaces the potential and its curvature corresponds to the field of the magnetic monopole.

Instantons

- Analogous Hopf bundle replacing $\mathbb{C}$ by *quaternions* $\mathbb{H} \cong \mathbb{R}^4$.
- $S^7 \subset \mathbb{H}^2$ as pairs of quaternions (q_1, q_2) with $|q_1|^2 + |q_2|^2 = 1$.
- S^3 can be identified with unit quaternions, so there is action on S^7.
- Identify orbits of this action with points in S^4 via stereographic projection.
- Principal bundle over S^4 with structure group $SU(2) \cong S^3$.

Instantons

- Models particles with *isotopic spin*.
- Interesting connections satisfy the *Yang-Mills* equations (restricts to Maxwell's equations for $G = U(1)$):

$$d_A F_A = 0$$
$$d_A * F_A = 0,$$

with the first being an identity (Bianchi's identity; holds for every curvature 2-form).

(Anti-)Self-duality

- Hodge star: operator on oriented Riemannian manifolds. Spits out the remaining form for the volume form with sign respecting orientation:

$$*(dx_1 \wedge dx_2) = dx_3 \wedge dx_4$$
$$*(dx_1 \wedge dx_4 \wedge dx_3) = -dx_2.$$

- In dimension 4, $*$ takes 2-forms into 2-forms and $*^2 = \mathrm{Id}$ $\implies$ eigenvalues ± 1. We can then decompose the space of 2-forms as

$$\Omega^2 = \Omega_+ \oplus \Omega_-,$$

self-dual and *anti-self-dual* parts.

(Anti-)Self-duality

- If a curvature 2-form F is SD or ASD, i.e. $*F = \pm F$, then Yang-Mills equation $d_A * F_A = 0$ follows trivially from Bianchi's identity $d_A F_A = 0$.
- (Anti-)-self-dual solutions to the Yang-Mills equations are called *instantons*.
- Basic instanton is given by

$$A(x) = \operatorname{Im}\left(\frac{x d\bar{x}}{1 + |x|^2}\right)$$

(Lie algebra of $SU(2)$ can be identified with imaginary quaternions).

Vector bundles

- $\pi : E \to M$ projection from total space to base space.
- Each fiber $E_x = \pi^{-1}(x)$ is now a vector space.
- Trivializations are linear when restricted to each fiber.
- Operations on vector spaces carry on to bundles (fiberwise): E^*, $E \oplus F$, $E \otimes F$, $\Lambda^r E$, E/F, etc.
- Arise from principal bundles by choosing a representation of the structure group.

Idea

- Any conformal transformation of S^4 will give a new instanton, since the Hodge star is conformally invariant.

Problem: exhibit all instantons.

- The twistor correspondence gives a 1-1 correspondence between instanton bundles over S^4 and holomorphic bundles over $\mathbb{P}^3$, called the twistor space.
- The ADHM construction is a recipe to build such holomorphic bundles over $\mathbb{P}^3$ and it can be proven that it exhausts all instantons.

Complex manifolds

- Transition functions between open sets of $\mathbb{C}^n$ are holomorphic.
- Introduce *complex structure* on TM, i.e. $J : TM \to TM$ with $J^2 = -1$.
- Action of $\mathbb{C}$ on each tangent space: $i \cdot v := J(v)$.
- Complexify each tangent space $\implies$ J has eigenvalues $\pm i$.
- $TM = T^{1,0}M \oplus T^{0,1}M$ splits into holomorphic and anti-holomorphic parts.
- Decomposition carries over to T^*M and its exterior powers: (p, q) forms.

Holomorphic bundles

- Defined over complex manifolds; admits holomorphic trivialization maps.
- Complex vector spaces as fibers.

Proposition

Given a hermitian metric on each fiber, there is a unique connection such that its Lie algebra-valued 1-form A satisfies

1. $A^* = -A$ under unitary trivializations,
2. A is of type $(1, 0)$ under holomorphic trivializations,

called the *Chern connection*.

Twistor fibration

- In complex projective space $\mathbb{P}^3$, every $\ell \in \mathbb{P}^3$ is a (complex) line passing through the origin in $\mathbb{C}^4 \cong \mathbb{H}^2$.
- Associate to each $\ell \in \mathbb{P}^3$ the *quaternionic* line $\mathbb{H}\ell$ passing through the origin in $\mathbb{H}^2$, which gives us a map $\pi : \mathbb{P}^3 \to \mathbb{HP}^1$.
- Each quaternionic line $L \in \mathbb{HP}^1$ is a copy of $\mathbb{C}^2$, and thus the fiber $\pi^{-1}(L)$ is the set of lines through the origin in this $\mathbb{C}^2$, i.e. $\pi^{-1}(L) \cong \mathbb{P}^1$.
- Identifying $S^4 \cong \mathbb{HP}^1$, we have a map $\pi : \mathbb{P}^3 \to S^4$ whose fibers are $\pi^{-1}(L) \cong \mathbb{P}^1$. This map is called the *twistor fibration*.

Twistor fibration

- Can identify $\mathbb{C}^4 \cong \mathbb{H}^2$ via

$$(z_1, z_2, z_3, z_4) \mapsto (z_1 + z_2 j, z_3 + z_4 j).$$

- $(z_1 + z_2 j, z_3 + z_4 j) \in \mathbb{H}^2$ left multiplied by j corresponds to $(-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3) \in \mathbb{C}^4$.
- Have map $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$(z_1, z_2, z_3, z_4) \mapsto (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)$$

  in homogeneous coordinates.
- σ has invariant lines $\mathbb{P}^1$ which are precisely the fibers of the twistor fibration.

ASD and complex structures

Lemma

A 2-form on S^4 is ASD if and only if its lift (pullback via π) to twistor space is of type $(1, 1)$.

Applying this result to the curvature 2-form:

Proposition

A $U(n)$-bundle with a metric-compatible connection on S^4 has ASD curvature iff the lifted bundle on twistor space has curvature of type $(1, 1)$.

Instantons and holomorphic bundles

- The lifted bundle has a natural holomorphic structure: let $E \to X$ be a Hermitian vector bundle over a complex manifold, equipped with a connection ∇ such that $F_\nabla \in \Omega^{1,1}(E)$.

Proposition

E has a natural holomorphic structure such that ∇ is the Chern connection of E.

- As a pullback bundle, each fiber $\tilde{E}_z = E_{\pi(z)} \implies$ topologically trivial on the $\mathbb{P}^1$ fibers, can show they are holomorphically trivial on the fibers.

Instantons and holomorphic bundles

Theorem

Let $E \to S^4$ be an Hermitian vector bundle with an ASD connection and let $F = \pi^* E$ be the lifted bundle, where $\pi : \mathbb{P}^3 \to S^4$ is the twistor fibration. Then

1. F is holomorphic.
2. F restricts to a holomorphic trivial bundle over each fiber $P_x := \pi^{-1}(x)$.
3. There is a holomorphic isomorphism $\tau : \sigma^* \overline{F} \to F^*$ such that τ induces an Hermitian inner product on $H^0(P_x, F)$.

Conversely, every such bundle $F \to \mathbb{P}^3$ is given by $F = \pi^* E$ for some bundle $E \to S^4$ with ASD connection.

ADHM construction

- There is a non-degenerate skew-form on $\mathbb{C}^4$ defined by $\omega(u, jv) = \langle u, v \rangle$.
- Let $L_z \subset \mathbb{C}^4$ be the line corresponding to the point $[z] \in \mathbb{P}^3$.
- Consider the complement wrt this skew-form L_z^ω which has dimension 3.
- The collection of $E_z = L_z^\omega / L_z$ with $[z]$ varying over $\mathbb{P}^3$ defines a vector bundle E over $\mathbb{P}^3$ with fiber $\mathbb{C}^2$.

ADHM construction

- E is holomorphic because it is the quotient of two holomorphic vector bundles.
- The definition of the skew-form ω implies that $L_z^\omega = L_{jz}^\perp$. Have orthogonal decomposition

$$\mathbb{C}^4 = L_z \oplus R_x \oplus L_{jz},$$

with $R_x = L_z^\omega \cap L_{jz}^\omega$ depending only on the fiber, i.e. on the point of $S^4 \implies E$ is trivial on the fibers of the twistor fibration.
- Corresponds to $SU(2)$-instanton bundle via twistor correspondence.

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