III) Discontinuous Solutions of Conservation Laws

III.1) Breakdown of continuity of solutions

Let $t \in [0,T) =: I$, $x \in \mathbb{R}$. Let $u \in C^1(I \times \mathbb{R}, \mathbb{R})$.

Ex: Assume $u = u(t,x)$ solves Burger's equ $\boxed{u_t + u u_x = 0}$ (BE)

Define "characteristic lines"

$$\begin{cases} \dot{\gamma}_a(t) = u(t, \gamma_a(t)) \\ \gamma_a(0) = 0 \end{cases}$$

Then $$\frac{d}{dt} u(t, \gamma_a(t)) = u_t + \overset{=u}{\dot{\gamma}_a} u_x = 0. \qquad (*)$$

(↳ Solution is constant along characteristic lines.)

$u_0(x) := u(0,x)$

$\Longrightarrow$ Solution of (BE) : $\underbrace{u(t,x) = u_0(a) \quad \text{for } \gamma_a(t) = x}_{\text{Defined as long that } \gamma_a\text{'s do not cross.}}$,

↳ Now, (*) implies $\gamma_a(t) = u_0(a) \cdot t + a$, a straight line.

Assume $u_0' < 0$.

Then first intersection of char. lines occurs at $t_b = \inf_{b>a} \frac{b-a}{u_0(a) - u_0(b)}$,

since for $b>a$, $\gamma_a(t) = \gamma_b(t)$

$$\Longleftrightarrow t = \frac{b-a}{u_0(a) - u_0(b)}.$$

Thus, continuity of solution u breaks down at $t_b = \inf_{b>a} \underbrace{\frac{b-a}{u_0(a)-u_0(b)}}_{}> 0$ ($u_0' < 0$).

Lemma:

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f'' > 0$, $I := [0,T)$.

Assume $u \in C^1(I \times \mathbb{R})$, $u(t,x) \in \mathbb{R}$, solves scalar conservation law

$$\boxed{u_t + f(u)_x = 0,}$$

then solution fails to be C^0 after time $T = \inf_{b>a} \dfrac{b-a}{f'(u_0(a)) - f'(u_0(b))}$.

proof:

Define $\begin{cases} \dot{\gamma}_a(t) = f'(u(t,\gamma_a(t))) \\ \gamma_a(0) = a \end{cases}$, "char. lines".

$$\Rightarrow \frac{d}{dt} u(t, \gamma_a(t)) = u_t + \underbrace{\dot{\gamma}_a}_{=f'(u)} u_x = u_t + f(u)_x = 0$$

$\Rightarrow u(t,x) = u_0(a)$ for $\gamma_a(t) = x$ and $\underbrace{\gamma_a(t) = f'(u_0(a))\, t + a}_{\text{straight line}}$

Assume $u_0' < 0$.

$$\Rightarrow \left(\gamma_a(t) = \gamma_b(t) \iff t = \frac{b-a}{f'(u_0(a)) - f(u_0(b))} > 0 \right)$$

$\blacksquare$

III.2) Weak solutions of Conservation Laws

Let $x \in \Omega \subset \mathbb{R}^n$ (open), $t \in [0,T]$. Let $u(t,x) \in \mathbb{R}^N$ and $f: \mathbb{R}^N \longrightarrow M(N,n)$ ($N \times n$-matrices) smooth.

Assume u solves $\begin{cases} u_t + \operatorname{div}_x f(u) = 0 \\ u(0,x) = u_0(x) \end{cases}$, $u_0 \in C^1(\mathbb{R}^n, \mathbb{R}^N)$

Then, for any $\phi \in C_0^\infty([-1,T] \times \Omega, \mathbb{R})$, we have by Gauss Thm

III.2) Weak solutions

Let $x \in \Omega \subset \mathbb{R}^n$, open, and $t \in [0,T) =: I$.

Let $u(t,x) \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to M(N,n)$ smooth. ($N \times n$-matrices)

$\left(f(u) = (f^1(u), \ldots, f^n(u)), \quad f^i(u) \in \mathbb{R}^N \ \forall i = 1, \ldots, n \right)$

Def

Assume $u \in L^1_{loc}(I \times \Omega, \mathbb{R}^N)$. We say u is a weak solution of

$$\begin{cases} u_t + \operatorname{div}_x (f(u)) = 0 \\ u(0,x) = u_0(x), \quad u_0 \in L^1_{loc}(\Omega, \mathbb{R}^N) \end{cases} \tag{CL}$$

if u satisfies $\forall \phi \in C_0^\infty((-T,T) \times \Omega, \mathbb{R})$

$$\int_{[0,T)\times\Omega} \left(u \cdot \phi_t + \sum_{i=1}^n f^i(u) \frac{\partial \phi}{\partial x^i} \right) dt\, d^n x = \int_{\mathbb{R}^n} u_0 \cdot \phi \Big|_{\{t=0\}} d^n x \tag{WCL}$$

Lemma

If $u \in C^1(I \times \Omega, \mathbb{R}^N)$ solves (CL), then u also solves (WCL).

proof:

$$\int_{[0,T)\times\Omega} \underbrace{\left(u \cdot \phi_t + \sum_{i=1}^n f^i(u) \frac{\partial \phi}{\partial x^i} \right)}_{= \operatorname{div}_{(t,x)} \left((u,f) \cdot \phi \right) - \underbrace{\left(u_t + \operatorname{div}_x f(u) \right)}_{=0, \text{ by (CL)}} \cdot \phi} dt\, d^n x = \int_{[0,T)\times\Omega} \operatorname{div}_{(t,x)} \left((u,f) \cdot \phi \right) dt\, d^n x$$

$$\overset{\text{Gauss}}{=} \int_{\Omega \times \{t=0\}} (u,f) \cdot \phi \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} d^n x$$

$$= \int_{\Omega} u_0 \, \phi \Big|_{\{t=0\}} d^n x$$

∎

III.3) Rankine Hugoniot conditions

Goal: Show that weak solution across a surface of discontinuity satisfies

$$\boxed{\underbrace{s}_{\substack{\text{shock}\\\text{speed}\\(n=1)}}\ \underbrace{[u]}_{\text{jump in } u} = \underbrace{[f]}_{\text{jump in } f(u)},}$$

"Rankine Hugoniot Condition" (RH)

Setting:
- Let $x \in \mathbb{R}$, i.e., one spatial dimension. Let $u(t,x) \in \mathbb{R}^N$.
- Let Σ be a smooth surface (submanifold) in $(0,T)\times\mathbb{R}$, parameterized by $\Sigma = \{(t, y(t)) \mid t \in (0,T)\}$.

  $\Rightarrow \quad s := \frac{dy}{dt}$ "shock speed"

  & $\begin{pmatrix} s \\ -1 \end{pmatrix}$ is normal to Σ
- Define $u_L(t) := \lim_{x \nearrow y(t)} u(t,x)$ & $u_R(t) := \lim_{x \searrow y(t)} u(t,x)$

  $\hookrightarrow [u] := u_L - u_R$ & $[f] := f(u_L) - f(u_R)$

Thm

Let $u \in L^1_{loc}((0,T)\times\mathbb{R}; \mathbb{R}^N) \cap C^1((0,T)\times\mathbb{R}\setminus\Sigma, \mathbb{R}^N)$.

Assume u solves $u_t + f(u)_x = 0$ point-wise in $(0,T)\times\mathbb{R}\setminus\Sigma$.

Then u is a weak solution in $(0,T)\times\mathbb{R}$ if and only if u satisfies the Rankine Hugoniot condition across Σ.

proof: Let $\phi \in C_0^\infty((0,T)\times\mathbb{R})$. Set $\mathcal{M} := (0,T)\times\mathbb{R}$

u weak solution $\iff 0 = \int \underbrace{(u\cdot\phi_t + f'\phi_x)}\, dt\,dx =$

$$= \operatorname{div}_{(t,x)}((u,f)\cdot\phi) - \cancelto{0}{(u_t + f(u)_x)}\cdot\phi \quad \text{on } \mathcal{M}\setminus\Sigma$$

$$= \int_{\mathcal{M}_L} + \int_{\mathcal{M}_R} \operatorname{div}_{(t,x)}((u,f)\cdot\phi)\, dt\,dx$$

for $\mathcal{M}_L := \{(t,x)\in\mathcal{M} \mid x < y(t)\}$ & $\mathcal{M}_R := \{(t,x)\in\mathcal{M} \mid x > y(t)\}$

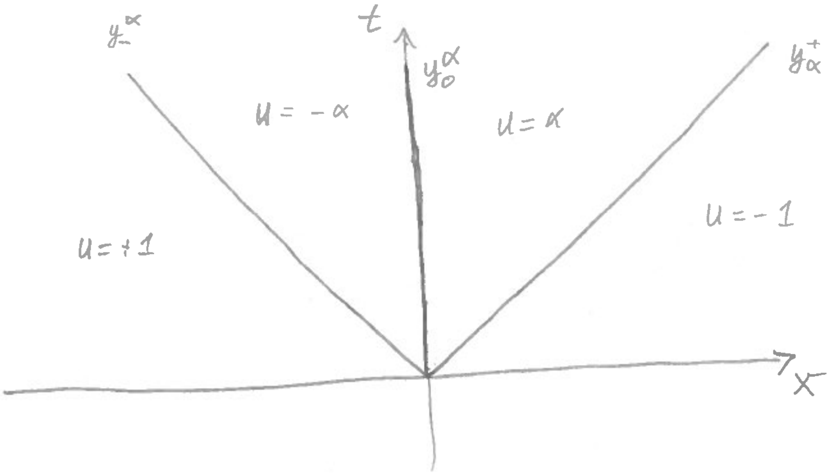
III.4) "Lax (entropy) conditions" for uniqueness

Example: Consider Burgers eqn $u_t + u u_x = 0$ (BE)

Let $u_0(x) := \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}$

Then for each $\alpha \geq 1$ we obtain a solution of (BE) by setting

$$u_\alpha(t,x) := \begin{cases} +1, & x < \frac{1-\alpha}{2} t =: y_-^\alpha(t) \\ -\alpha, & y_-^\alpha(t) < x < 0 =: y_0^\alpha(t) \\ \alpha, & 0 < x < \frac{\alpha-1}{2} t =: y_+^\alpha(t) \\ -1, & y_+^\alpha(t) < x \end{cases}$$



Clearly u_α solves (BE) away from $y_\pm^\alpha$, y_0^α and $u_\alpha(0,x) = u_0(x)$ $\forall \alpha \geq 1$.
Thus, to verify that RH-conditions hold across $y_\pm^\alpha$ & y_0^α:

- Across y_-^α: $[u] = u_L - u_R = 1 - (-\alpha)$,

  $[f] = f(u_L) - f(u_R) = \frac{1}{2}(+1)^2 - \frac{1}{2}(-\alpha)^2 = \frac{1-\alpha^2}{2}$, since $f(u) = \frac{1}{2}u^2$ for (BE)

  $\Rightarrow s = \frac{[f]}{[u]} = \frac{1-\alpha}{2} = \frac{dy_-^\alpha}{dt}$ ✓.

- Across y_+^α: $[u] = \alpha - (-1) = \alpha + 1$, $[f] = \frac{1}{2}(\alpha^2 - 1)$, $\Rightarrow s = \frac{\alpha-1}{2} = \frac{dy_+^\alpha}{dt}$ ✓

- Across y_0^α: $[u] = -2\alpha$, $[f] = \frac{1}{2}(\alpha^2 - \alpha^2) = 0$, $\Rightarrow s = 0 = \frac{dy_0^\alpha}{dt}$

Conclusion: By extending solution space to allow for weak solutions we loose uniqueness.

↳ To single out the unique "physical" condition, we need to impose the "Lax (entropy) condition".

$$\overset{\text{Gauss}}{=} \int_{\partial M_L} \phi \cdot (u, f) \cdot n_L \, d\sigma + \int_{\partial M_R} \phi \cdot (u, f) \cdot n_R \, d\sigma$$

unit normal of ∂M_L: $n_L = \frac{1}{s^2+1}\begin{pmatrix} s \\ -1 \end{pmatrix}$

unit normal of ∂M_R: $n_R = -\frac{1}{s^2+1}\begin{pmatrix} s \\ -1 \end{pmatrix}$

$$= \int_{\Sigma} \frac{\phi}{s^2+1} \Big(s \underbrace{(u_L - u_R)}_{=:[u]} - \underbrace{(f(u_L) - f(u_R))}_{=:[f]} \Big) \, d\sigma$$

$$\Longleftrightarrow \qquad s\,[u] = [f]$$

Rmk:

- In higher spatial dimensions Rankine Hugoniot conditions are $\big([u], [f^1(u)], \dots, [f^n(u)]\big) \cdot n = 0$, where n is normal to "space-time" surface Σ.
- For $T^{\mu\nu}$ energy-momentum-tensor in General Relativity, the RH-conditions are $[T^{\mu\nu}]\, n_\nu = 0$.

Rmk:

In one spatial dimension, the RH-conditions are the fundamental identity for constructing shock wave solutions by matching C^1-solution across surface of discontinuity ("shock surface"). But solution space is getting extended significantly, making uniqueness of solutions more delicate.