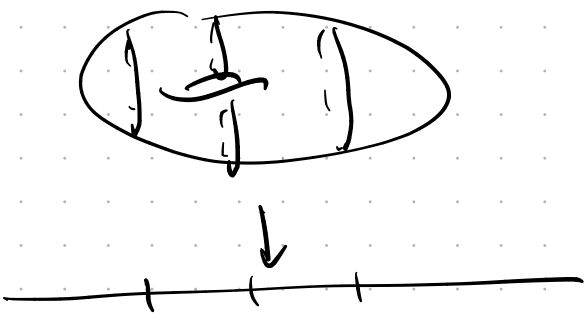
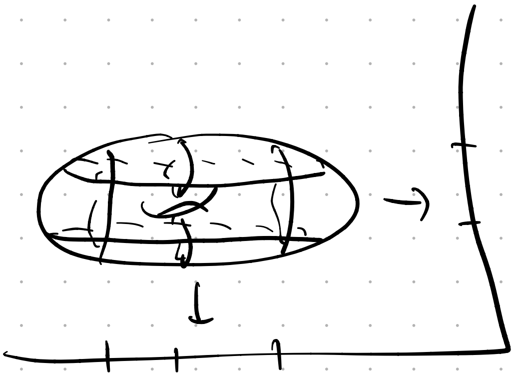
Domain walls & oplax natural transformations

Atiyah-Segal framework: a TQFT is a symmetric monoidal functor

$$F : \mathrm{Bord}_{n,\dots,n-k} \longrightarrow \mathcal{C}$$

Picture: $n=2, k=1$ $\qquad$ $n=2, k=2$





Feature: e.g.

Question: Given $F_A, F_B : \mathrm{Bord}_n \longrightarrow \mathcal{C}$, what is a map from F_A to F_B?

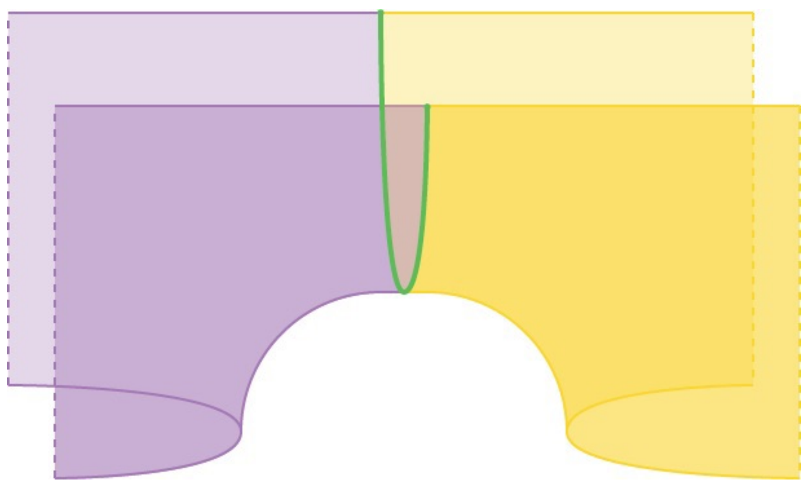
$$\tau F_A : \mathrm{Bord}_{n-1} \longrightarrow \mathrm{Bord}_n \xrightarrow{F_A} \mathcal{C}$$

$$
\begin{tikzcd}
\text{Domain walls} \arrow[d, "\simeq"', "\text{dom wall cob hyp.}"] \arrow[r, shift left, "\simeq"] & \text{(Op)lax Natural transformations } \tau F_A \text{ to } \tau F_B \arrow[l, shift left, "\text{Thm/Cor}"'] \arrow[d, "\simeq"', "\text{using cob. hypothesis}"] \\
F_A(pt^+) \to F_B(pt^+) \text{ ``fully dualizable''} \arrow[r, "\simeq \text{Thm}"] & F_A(pt^+) \to F_B(pt^+) \text{ ``partial dualizability''}
\end{tikzcd}
$$

framed Fully extended theories

Note: $F_A(pt^+)$ & $F_B(pt^+)$ are "fully dualizable".

Domain walls



$$i_A, i_B : \mathrm{Bord}_n \longrightarrow \mathrm{Bord}_n^{dom}$$

Domain wall from F_A to F_B is

$$Z : \mathrm{Bord}_n^{dom} \longrightarrow \mathcal{C}$$

s.t. $Z \circ i_A = F_A$, $Z \circ i_B = F_B$.

Oplax natural transformation

Let $F, G : \mathcal{B} \to \mathcal{C}$ be functors of bicategories. An oplax natural transformation $\beta : F \Rightarrow G$ consists of

$$X \in \mathcal{B} \quad \longmapsto \quad F(X) \xrightarrow{\beta(X)} G(X)$$

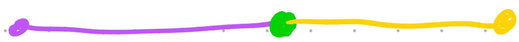
$$f : X \to Y \quad \longmapsto \quad \begin{tikzcd} F(X) \arrow[r, "Ff"] \arrow[d, "\beta_X"'] & F(Y) \arrow[d, "\beta_Y"] \arrow[dl, phantom, "\swArrow \beta_f"] \\ G(X) \arrow[r, "Gf"'] & G(Y) \end{tikzcd}$$

Defining oplax natural transformation

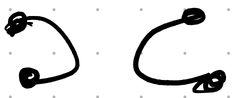
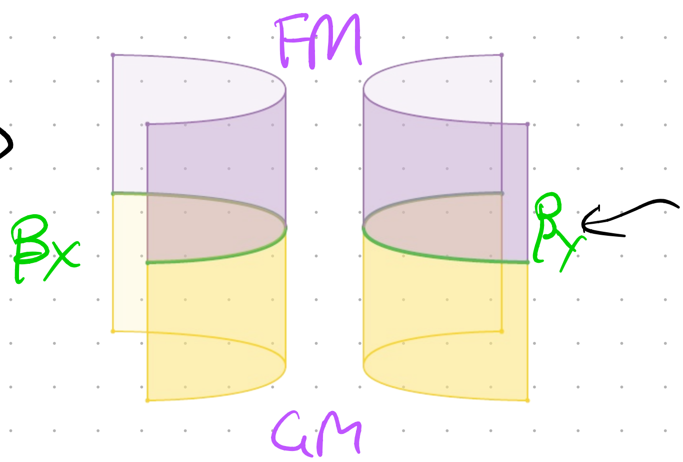
Given $Z: \mathrm{Bord}_n^{\mathrm{dom}} \to \mathcal{C}$ construct $\beta: \tau i_A \Rightarrow \tau i_B$

$$\beta_Z := \begin{tikzcd} \mathrm{Bord}_{n-1} \arrow[r, bend left, "\tau i_A"] \arrow[r, bend right, "\tau i_B"'] \arrow[r, phantom, "\Downarrow \beta_{\mathrm{Id}}"] & \mathrm{Bord}_n^{\mathrm{dom}} \arrow[r, "Z"] & \mathcal{C} \end{tikzcd}$$

Morally: β_{Id} is "dimensional reduction" along

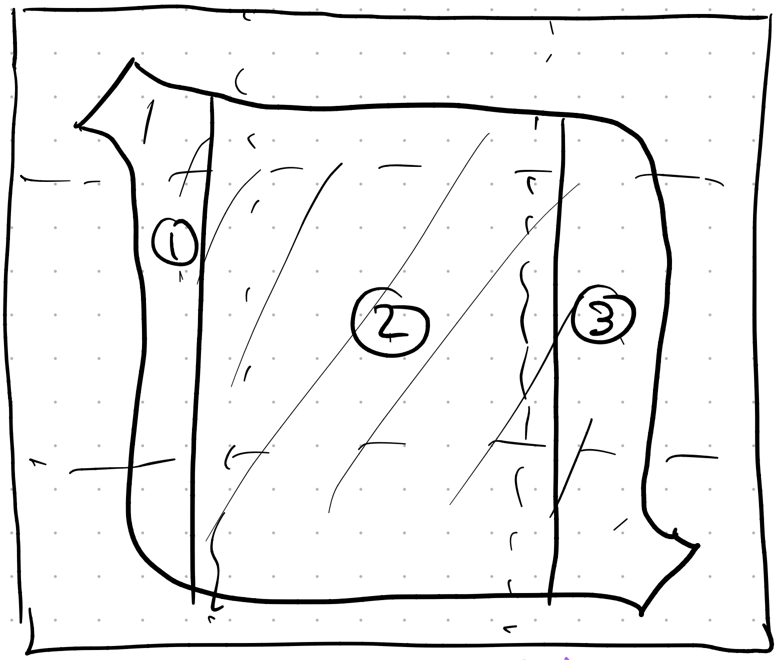
$Z($  $) =: \tilde{I}$

e.g. $O \longmapsto O \times \tilde{I} \cong Z($  $)$

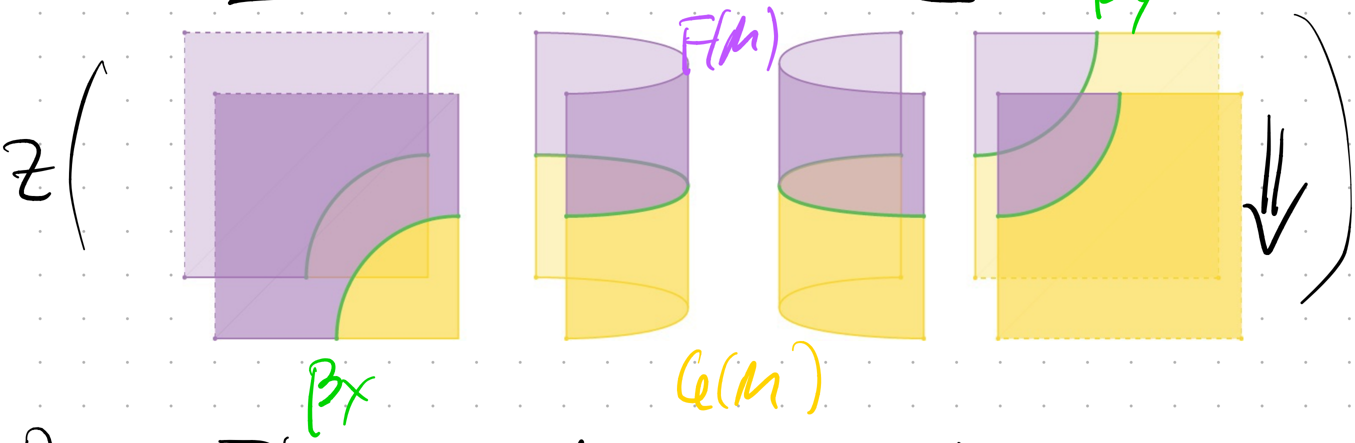
Issue:  $\longmapsto$ 

$\parallel$

M

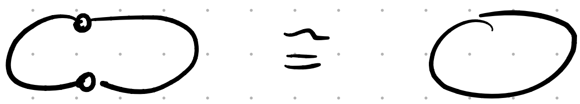


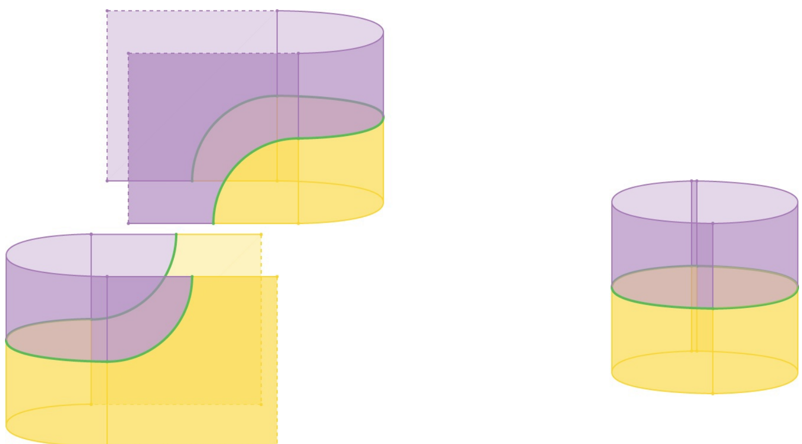
$$\begin{tikzcd} \cdot \arrow[r, "F(M)"] \arrow[d, "\beta_x"'] & \cdot \arrow[d, "\beta_y"] \arrow[dl, phantom, "\swarrow"] \\ \cdot \arrow[r, "G(M)"'] & \cdot \end{tikzcd}$$



Prop: This produces an oplax trans. Bld.

Functorality:





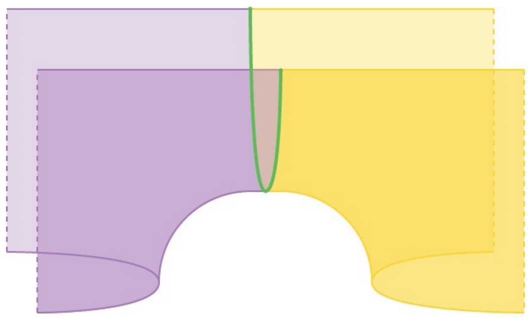
Recovering the domain wall

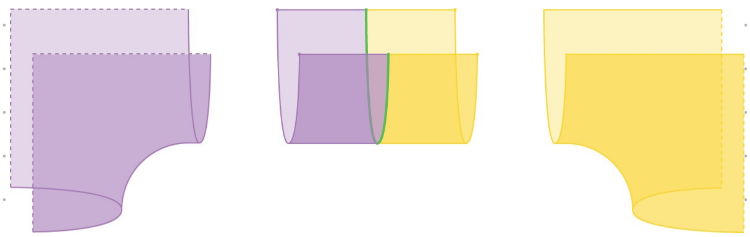
$$\beta : \tau F_A \Rightarrow \tau F_B \quad \rightsquigarrow \quad Z : \mathrm{Bord}_n^{dom} \to \mathcal{C}$$

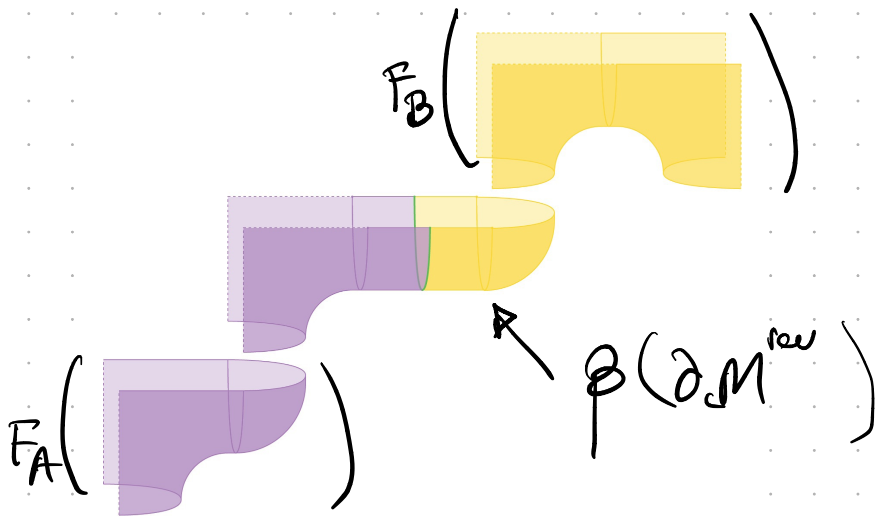
Thm (Collar Nbhd)

$$M \cong M_A \circ \left(\tilde{I} \times \underline{\partial_c M} \right) \circ M_B$$

E.g.







Thm: Let $X, Y \in \mathcal{C}$ are n-dualizable.
Let $f: X \to Y$ be a 1-mor.

TFAE

- f is "n-dualizable" $(f: X \to Y) \in \mathcal{C}^{\mathrm{nd}}$

- f is n-times right adjunctible

$$f \dashv_{u, v} f^R$$

$\mathcal{C}$ is a 2-category

f has right adjoint $\iff$ f has left.

$$\underbrace{f^L = S_X^{-1} \circ f^R \circ S_Y}$$

$f^L: Y \to X$