2-Vector Bundles

Peter Kristel
with Konrad Waldorf & Matthias Ludewig

October 09, 2024

Overview

Context

- String Geometry
- Extended TQFT
- Categorification

Transgression/Regression

$$LM = C^\infty(S^1, M)$$

$$\begin{array}{c}\{\text{degree } n \text{ geometric object on } LM\} \\ \text{transgression} \uparrow \quad \downarrow \text{regression} \\ \{\text{degree } n+1 \text{ geometric object on } M\}\end{array}$$

Some examples:

LM	M	
$\Omega^n(LM)$	$\Omega^{n+1}(M)$	
fusive function ($n = 0$)	$\mathrm{U}(1)$-bundle ($n = 1$)	[Waldorf; '09]
fusive $\mathrm{U}(1)$-bundle ($n = 1$)	bundle gerbe ($n = 2$)	[Waldorf; '10]
fusive vector bundle ($n = 1$)	2-vector bundle ($n = 2$)	[this talk]

String Geometry

$$\{\text{fusive vector bundles}\} \rightleftarrows \{\text{2-vector bundles}\}$$

Theorem: K. & Waldorf [2020]

A string structure on a manifold M leads to a *fusive spinor bundle* on LM.

According to the transgression/regression principle, there should be a corresponding 2-Vector Bundle.
This 2-Vector Bundle exists[1], we call it the *Stringor Bundle*.

[1] K., Ludewig & Waldorf, arXiv:2206.09797

Extended TQFT

Theorem: Berwick-Evans & Pavlov [2023]

Smooth one-dimensional topological field theories over a manifold M are vector bundles with connection on M.

1-d TFT	Vector bundle & connection
value at a point	fibre over a point
value at a path	parallel transport

Might 2-Vector bundles with connection be related to (extended) two-dimensional field theories?

Outline

Definition

The bicategory of 2-Vector Spaces is the (framed) bicategory with

- **Objects** Algebras
- **1-Morphisms** Bimodules
- **2-Morphisms** Intertwiners of bimodules

and composition of

- **1-morphisms** Relative tensor product of bimodules
- **2-morphisms** Composition of maps

Are these 2-Vector Spaces?

- This bicategory is symmetric monoidal.
- The unit object is the base field.
- The morphism category of the unit object is the category of vector spaces.

Indeed, a $\mathbb{K}$-$\mathbb{K}$-bimodule is just a vector space.
An intertwiner of $\mathbb{K}$-$\mathbb{K}$-bimodules is just a linear map.

Flavours

One may consider different flavours of this bicategory:

Finite-dimensional Finite-dimensional algebras and bimodules.

Super Everything is $\mathbb{Z}_2$-graded.

von Neumann Algebras are von Neumann algebras, Bimodules are Hilbert spaces, relative tensor product is the Connes Fusion product.

An open problem is to find an infinite-dimensional setting that plays nicely with smoothness. Hilbert spaces are generally too rigid for this.

Outline

Complex associated to an open cover

Let M be a manifold. Let $\{U_i\}_{i\in I}$ be a (good) open cover of M. Set

$$Y = \coprod_{i\in I} U_i, \qquad \text{and} \qquad p : Y \to M.$$

and

$$Y^{[2]} = Y \times_M Y = \{(y_1, y_2) \in Y^2 \mid p(y_1) = p(y_2) \in M\}.$$

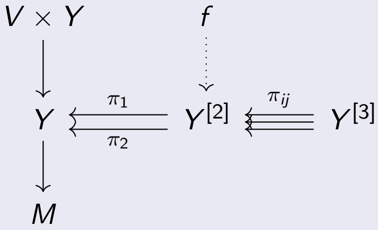
An element of $Y^{[2]}$ is essentially an element of $U_i \cap U_j$.
Consider the diagram

$$\begin{array}{ccccc} Y & \underset{\pi_2}{\overset{\pi_1}{\leftleftarrows}} & Y^{[2]} & \overset{\pi_{ij}}{\Lleftarrow} & Y^{[3]} \\ \downarrow & & & & \\ M & & & & \end{array}$$

Vector Bundles

Definition

A Vector Bundle with typical fibre V over M is a diagram



Where f is a (smooth) family of linear automorphisms of V parametrized by $Y^{[2]}$, i.e. $f_{(y_1,y_2)} : V \to V$ satisfying a compatibility condition over $Y^{[3]}$.

Somewhat tautologically, f is an isomorphism of vector bundles: $f : \pi_1^*(V \times Y) \to \pi_2^*(V \times Y)$.

Comparing Vect to 2-Vect

Let $Y \to M$ be a surjective submersion, e.g. $Y = \coprod_i U_i$.

	Y	$Y^{[2]}$	$Y^{[3]}$
Vect	vector space	linear maps	consistency condition
2-Vect	algebra	bimodules	intertwiners
C	Objects	1-morphisms	2-morphisms

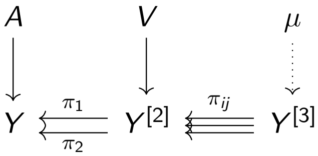
For 2-Vect, we will require a consistency condition on $Y^{[4]}$.

Replacing Vect by 2-Vect

- If $y \in Y$, then A_y is an algebra.
- If $(y_1, y_2) \in Y^{[2]}$, then $V_{(y_1,y_2)}$ is an A_{y_2}-A_{y_1}-bimodule.
- If $(y_1, y_2, y_3) \in Y^{[3]}$, then

$$\mu_{(y_1,y_2,y_3)} : V_{(y_2,y_3)} \otimes_{A_{y_2}} V_{(y_1,y_2)} \to V_{(y_1,y_3)}$$

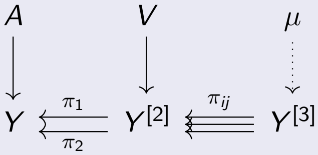
  is an isomorphism.



2-vector bundles

Definition [M. Ludewig, K. Waldorf, PK '21]

A *2-vector bundle* over M is a diagram



- $Y \to M$ is a surjective submersion
- $A \to Y$ is an algebra bundle
- V is a $\pi_2^* A$-$\pi_1^* A$-bimodule bundle
- $\mu : \pi_{23}^* V \otimes_{A_2} \pi_{12}^* V \to \pi_{13}^* V$ is an associative isomorphism

1-Morphisms of 2-Vector Bundles

Let $\mathcal{V}_1 = (Y, A_1, V_1, \mu_1)$ and $\mathcal{V}_2 = (Y, A_2, V_2, \mu_2)$ be 2-Vector Bundles.

A *1-morphism* from $\mathcal{V}_1$ to $\mathcal{V}_2$ consists of

- an A_1-A_2-bimodule bundle $P \to Y$
- an intertwiner of bimodule bundles over $Y^{[2]}$

$$\phi : \pi_2^* P \otimes_{\pi_2^* A_1} V_1 \to V_2 \otimes_{\pi_1^* A_2} \pi_1^* P.$$

- such that ϕ is compatible with μ_1 and μ_2.

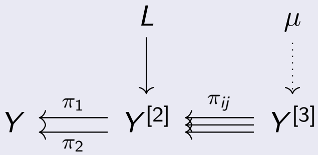
If $\mathcal{V}_1$ and $\mathcal{V}_2$ have different surjective submersions $Y_1 \to M$ and $Y_2 \to M$, first pull back to the common refinement $Z = Y_1 \times_M Y_2$.

Bundle gerbes

Line bundles are geometric realizations of elements of $H^2(M,\mathbb{Z})$.
Bundle gerbes are geometric realizations of elements of $H^3(M,\mathbb{Z})$.

Definition [Murray, 1994]

A *Bundle gerbe* on M is a diagram



where L is a line bundle, and μ an isomorphism
$\mu : \pi_{23}^* L \otimes \pi_{12}^* L \to \pi_{13}^* L$.

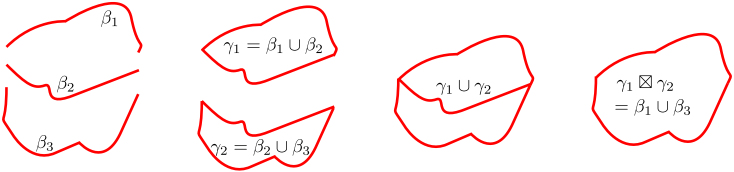
Observation: Bundle gerbes are 2-Line Bundles. Moreover,
Morphisms of bundle gerbes are 1-morphisms of 2-Line Bundles.

Outline

Smooth loop space of a manifold

$$LM := C^\infty(S^1, M), \qquad PM := C^\infty([0,\pi], M),$$
$$L_xM := \{\gamma \in LM, \gamma(0) = x\}, \quad P_xM := \{\beta \in PM, \beta(0) = x\}.$$

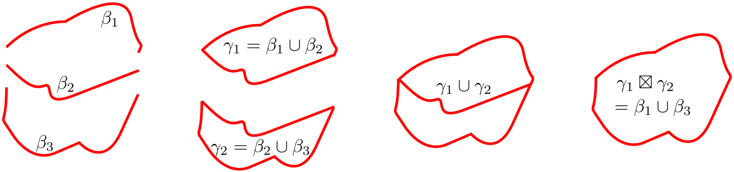
The space LM "remembers" M through the operation of fusion:



$$P_xM^{[3]} \overset{\pi_{ij}}{\Rrightarrow} P_xM^{[2]}$$

Loops/paths that can be fused together are called *compatible*.

Fusion products: fibrewise



Let $L \to LM$ be a line bundle. A fusion product on L is an operation that covers the operation of fusing compatible paths:

$$\mu_{321} : L_{\beta_2 \cup \beta_3} \otimes L_{\beta_1 \cup \beta_2} \to L_{\beta_1 \cup \beta_3},$$

which satisfies the natural associativity condition when given four compatible paths.

Fusion products: global

Consider $p : P_x M \to M, \beta \mapsto \beta(\pi)$. We identify $P_x M^{[2]}$ with $L_x M$. This is technically incorrect, and the solution is non-trivial, but morally it suffices.
Let $L \to LM$ be a line bundle.

$$LM \overset{\pi_{ij}}{\Lleftarrow} P_x^{[3]} M$$

Definition

A *fusion product* on a line bundle $L \to LM$ is an associative isomorphism

$$\mu : \pi_{23}^* L \otimes \pi_{12}^* L \to \pi_{13}^* L.$$

$$\mu_{321} : L_{\beta_2 \cup \beta_3} \otimes L_{\beta_1 \cup \beta_2} \to L_{\beta_1 \cup \beta_3},$$

Regression

We can piece everything together:

$$\begin{array}{ccccc} & & L & & \mu \\ & & \downarrow & & \vdots \\ P_xM & \overset{\pi_1}{\underset{\pi_2}{\leftleftarrows}} & P_xM^{[2]} & \overset{\pi_{ij}}{\Lleftarrow} & P_xM^{[3]} \\ \downarrow & & & & \\ M & & & & \end{array}$$

We see that a fusion product is exactly the data required to obtain a bundle gerbe with $Y = P_xM$.

Transgression/Regression of Line Bundles

$$\begin{array}{ccc} \{\text{Bundle Gerbes on } M\} & \underset{\text{Regression}}{\overset{\text{Transgression}}{\rightleftarrows}} & \{\text{Fusive Line Bundles on } LM\} \\ {\scriptstyle\text{Dixmier-Douady}}\downarrow & & \downarrow{\scriptstyle\text{Chern}} \\ H^3(M,\mathbb{Z}) & \longrightarrow & H^2(LM,\mathbb{Z}) \end{array}$$

(To invert the bottom arrow, look to the differential cohomology diagram.)

Outline

Clifford algebras

Let Cl be the Clifford algebra generated by $L^2(S^1, \mathbb{C}^d)$, i.e. subject to the relation

$$fg + gf = \langle f, \overline{g} \rangle \mathbb{1} \qquad f, g \in L^2(S^1, \mathbb{C}^d).$$

Let $L \subset L^2(S^1, \mathbb{C}^d)$ be the subspace consisting of functions with vanishing negative fourier coefficients; set

$$F = \Lambda L = \mathbb{C} \oplus L \oplus (L \wedge L) \oplus \ldots .$$

By splitting elements of Cl into creation and annihilation operators, it acts on F. That is, we have a homomorphism from Cl to the algebra of bounded operators on F:

$$\mathrm{Cl} \to B(F).$$

The fibre

Denote by A the Clifford algebra of $L^2(I, \mathbb{C}^d)$. By including I into S^1 as the upper (lower) semi-circle, we get an (anti)-inclusion of A into Cl.
This turns F into an invertible A-A-bimodule, i.e. we have algebra homomorphisms:

$$A \to \mathrm{Cl} \to B(F)$$
$$A^{\mathrm{op}} \to \mathrm{Cl} \to B(F)$$

with commuting images.

Spin structures on loop spaces

Let M be a spin manifold with spin frame bundle $\mathrm{Spin}(M)$. Then $L\mathrm{Spin}(M)$ has structure group $L\mathrm{Spin}(d)$. The group $L\mathrm{Spin}(d)$ has a "basic" central extension:

$$\mathrm{U}(1) \to \widetilde{L\mathrm{Spin}}(d) \to L\mathrm{Spin}(d).$$

Definition (T.P. Killingback; '87)

A *spin structure on the loop space LM* is a lift

$$\begin{array}{ccc} \widetilde{L\mathrm{Spin}}(M) & \longrightarrow & L\mathrm{Spin}(M) \\ & \searrow & \downarrow \\ & & LM \end{array}$$

The loop spinor representation

The loop group $L\mathrm{SO}(d)$ acts on Cl by pointwise action on the functions $f \in L^2(S^1, \mathbb{C}^d)$. This induces an action of $\widetilde{L\mathrm{Spin}}(d)$ on Cl, simply through the projection $\widetilde{L\mathrm{Spin}}(d) \to L\mathrm{SO}(d)$.

Theorem

The group $\widetilde{L\mathrm{Spin}}(d)$ admits a representation on F by intertwiners for the Cl action on F:

$$\varphi(a_1 \triangleright \psi \triangleleft a_2) = \varphi(a_1) \triangleright \varphi(\psi) \triangleleft \varphi(a_2),$$

for $\varphi \in \widetilde{L\mathrm{Spin}}(d)$, $a_1, a_2 \in A$, $\psi \in F$.

The spinor bundle

Spinor bundle on loop space:

$$F(LM) = \widetilde{L\mathrm{Spin}}(M) \times_{\widetilde{L\mathrm{Spin}(d)}} F.$$

Clifford algebra bundle on loop space:

$$\mathrm{Cl}(LM) = L\mathrm{SO}(M) \times_{L\mathrm{SO}(d)} \mathrm{Cl}(d).$$

The spinor bundle is a module bundle:

$$\mathrm{Cl}(LM) \times_{LM} \mathcal{F}(LM) \to \mathcal{F}(LM).$$

Outline

Clifford bundles over path space

The group $P\mathrm{SO}(d)$ acts on A through the pointwise action of $P\mathrm{SO}(d)$ on $L^2(I, \mathbb{C}^d)$. Set

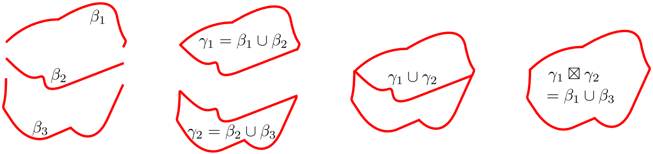
$$A(PM) = P\mathrm{SO}(M) \times_{P\mathrm{SO}(d)} A.$$

If $(\beta_1, \beta_2) \in P_x M^{[2]}$ then we have an algebra homomorphism

$$A(PM)_{\beta_1} \times A(PM)^{\mathrm{op}}_{\beta_2} \to \mathrm{Cl}(LM)_{\beta_1 \cup \beta_2}.$$

So that $F(LM)_{\beta_1 \cup \beta_2}$ becomes an $A(PM)_{\beta_1}$-$A(PM)_{\beta_2}$-bimodule.

Fusion



Assume that the spin structure $\widetilde{L\mathrm{Spin}(d)} \to LM$ is fusive.

Theorem: Fusion of Fock spaces [K. Waldorf, PK '20]

For each triple $(\beta_1, \beta_2, \beta_3)$, there exists an isomorphism of $A(PM)_{\beta_1}$-$A(PM)_{\beta_3}$ bimodules

$$\mu_{1,2,3} : F(LM)_{\gamma_1} \boxtimes_{A(PM)_{\beta_2}} F(LM)_{\gamma_2} \xrightarrow{\simeq} F(LM)_{\gamma_1 \boxtimes \gamma_2}.$$

such that these isomorphisms are associative, i.e. have a commutative square for each quadruple $(\beta_1, \beta_2, \beta_3, \beta_4)$.

The stringor bundle

Fix a basepoint $\{*\} \in M$. Let $A(P_*M)$ and $F(L_*M)$ be the restriction of $A(PM)$ and $F(LM)$ to P_*M and L_*M, respectively.

The stringor 2-vector bundle