GEOMETRY AND TOPOLOGY OF ANTI-QUASI-SASAKIAN MANIFOLDS

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Outline

Almost contact metric manifolds

Definition

An almost contact manifold is a smooth manifold M^{2n+1} endowed with a $(1,1)$-tensor field φ, a vector field ξ, and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

It follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and

$$TM = \mathcal{H} \oplus \langle \xi \rangle, \qquad J_{\mathcal{H}}^2 = -I,$$

where $\mathcal{H} := \operatorname{Ker} \eta = \operatorname{Im} \varphi$ is a distribution of rank $2n$ and $J_{\mathcal{H}} := \varphi|_{\mathcal{H}}$.

Given an almost contact manifold (M, φ, ξ, η), one can define an almost complex structure J on the product $M \times \mathbb{R}$ by setting

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

for every $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M \times \mathbb{R})$.

Normality

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for every $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M \times \mathbb{R})$.
Then,

$$J \text{ is integrable } \Leftrightarrow [J, J] = 0 \ \Leftrightarrow \ N_\varphi := [\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

In this case (M, φ, ξ, η) is called a normal almost contact manifold.

Definition

Let (M, φ, ξ, η) be an almost contact manifold. A Riemannian metric g is called compatible with the (φ, ξ, η)-structure if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

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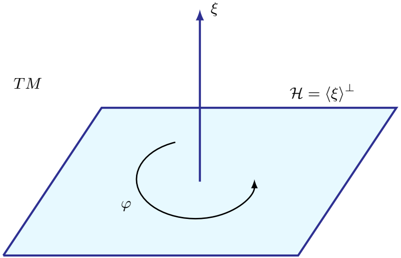
It follows that:

- $\|\xi\| = 1$, $\langle \xi \rangle = \mathcal{H}^{\perp}$;



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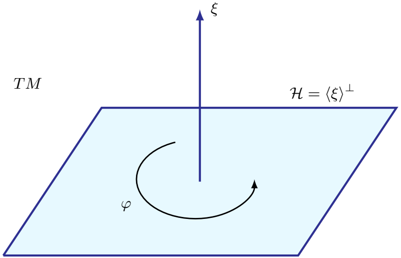

Compatible Riemannian metrics

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It follows that:

- $\|\xi\| = 1$, $\langle \xi \rangle = \mathcal{H}^{\perp}$;
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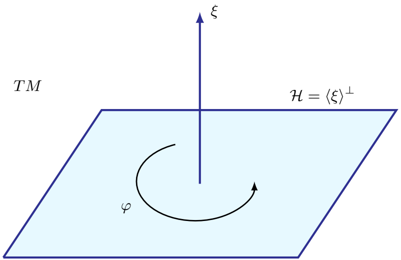
- $g(\varphi X, \varphi Y) = g(X, Y) \quad \forall X, Y \in \mathcal{H}$, i.e.
  $(J_{\mathcal{H}}, g)$ is an almost Hermitian structure on $\mathcal{H}$;



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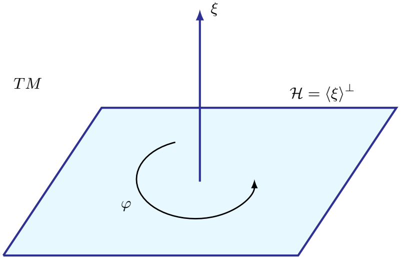
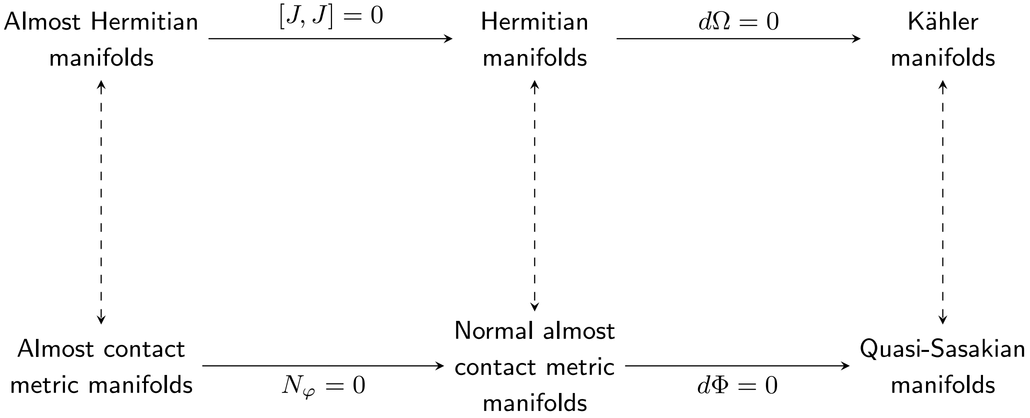
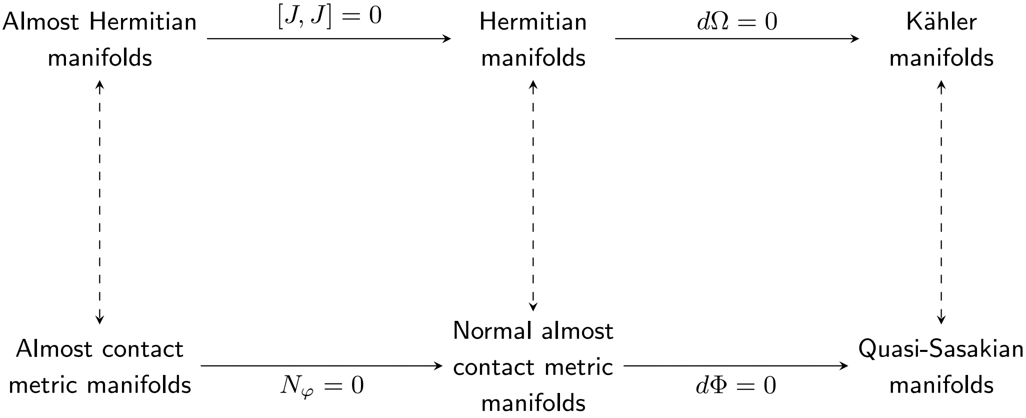
- φ is skew-symmetric w.r.t. $g \rightsquigarrow \Phi := g(\cdot, \varphi\cdot)$ is the fundamental 2-form of the structure.




Almost Hermitian manifolds
$[J, J] = 0$
Hermitian manifolds
$d\Omega = 0$
Kähler manifolds
Almost contact metric manifolds
$N_\varphi = 0$
Normal almost contact metric manifolds
$d\Phi = 0$
Quasi-Sasakian manifolds



Remark: Recall that (M, J, g) Kähler $\iff \nabla^g J = 0$.

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By Kanemaki, this turns out to be equivalent to

$$\begin{aligned}(\nabla^g_X \Phi)(Y, Z) &= g(Y, (\nabla^g_X \varphi)Z) \\ &= \eta(Z)g(Y, AX) - \eta(Y)g(X, AZ) \qquad \forall X, Y, Z \in \mathfrak{X}(M)\end{aligned}$$

where A is a $(1, 1)$-tensor field such that

$$A\varphi = \varphi A, \qquad g(AX, Y) = g(X, AY),$$

and it is uniquely determined, up to $f \in C^\infty(M)$, by $A = -\varphi \circ \nabla^g \xi + f\eta \otimes \xi$.

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A quasi-Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called

- Cokähler manifolds: $d\eta = 0$
  $\nabla^g \Phi = 0$ (or $\nabla^g \varphi = 0$)

- Sasakian manifolds: $d\eta = 2\Phi$
  $(\nabla^g_X \varphi)Y = g(X,Y)\xi - \eta(Y)X \quad \forall X, Y \in \mathfrak{X}(M)$;

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More generally, we say that a quasi-Sasakian structure has constant rank $2r+1$ if

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Remark: Tanno provided a 3-dim. example of quasi-Sasakian structure of maximal rank, which is not Sasakian.

For an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$

$$\begin{aligned} N_\varphi(\xi, X) &= -\varphi(\mathcal{L}_\xi \varphi)X + d\eta(\xi, X)\xi, \\ d\eta(\xi, X) &= \eta((\mathcal{L}_\xi \varphi)\varphi X) = (\mathcal{L}_\xi g)(\xi, X) = (\mathcal{L}_\xi \eta)X, \\ d\Phi(\xi, X, Y) &= (\mathcal{L}_\xi g)(X, \varphi Y) + g(X, (\mathcal{L}_\xi \varphi)Y). \end{aligned} \tag{1}$$

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Remark: In the Sasakian case $\omega = 2\Omega$.

Example (Cokähler)

For any Kähler manifold (N^{2n}, J, k), the Riemannian product $N \times \mathbb{R}$ has a cokähler structure.

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Example (Sasakian)

Consider $(\mathbb{S}^{2n+1}, g)$ as hypersurfaces of the complex space form $(\mathbb{C}^{n+1}, J, g_0)$, and let ν be a unit normal. For any $X \in \mathfrak{X}(\mathbb{S}^{2n+1})$, one can decompose

$$JX = \varphi X + \eta(X)\nu, \qquad \xi := -J\nu$$

Then, $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian manifold fibering on the complex space form $\mathbb{C}P^n$ via the Hopf fibration.

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Examples (Quasi-Sasakian)

- Orientable hypersurfaces of Kähler manifolds with $A_\nu \varphi = \varphi A_\nu$.
- Riemannian product of (quasi-)Sasakian and Kähler manifolds.
- Heisenberg Lie groups H^{2n+1}

Topological properties

- Cokähler manifolds
  - are formal $\rightsquigarrow$ compact nilmanifolds $\cong T^{2n+1}$
  - satisfy a version of the Hard Lefschetz property
  - all the Betti numbers are non-zero
  - $b_0 \leq b_1 \leq \cdots \leq b_n = b_{n+1} \geq b_{n+2} \geq \cdots \geq b_{2n+1}$
  - $b_{2p+1} - b_{2p}$ is even. In particular, b_1 is odd
  - ...
- Sasakian manifolds
  - compact nilmanifolds $\cong H^{2n+1}/\Gamma$
  - satisfy a version of the Hard Lefschetz property
  - b_p is even for odd $1 \leq p \leq n$ and even $n+1 \leq p \leq 2n+1$.
  - ...
- Quasi-Sasakian manifolds
  - have an almost formal model $\rightsquigarrow$ compact nilmanifolds $\cong (H^{2l+1} \times \mathbb{R}^{2(n-l)})/\Gamma$

Objective

Define a new class of almost contact metric manifolds such that the transverse geometry with respect to ξ is given by a Kähler structure endowed with a closed 2-form ω of type $(2,0)$.

Given an almost complex manifold (B, J), consider the canonical splitting

$$\Lambda^2_{\mathbb{C}}(B) = \Lambda^{2,0}(B) \oplus \Lambda^{1,1}(B) \oplus \Lambda^{0,2}(B).$$

Then, there is a one-to-one correspondence between complex $(2,0)$-forms and real J-anti-invariant 2-forms. In particular:

$$\{\omega_{\mathbb{C}} \in \Lambda^{2,0}(B),\ d\omega_{\mathbb{C}} = 0\} \overset{1:1}{\longleftrightarrow} \{\omega \in \Lambda^2(B),\ \omega(JX, JY) = -\omega(X,Y),\ d\omega = 0\}.$$

In the nondegenrate case, (B, J, ω) is a complex symplectic manifold and $\dim_{\mathbb{R}} B = 4n$.

Example

Hyperkähler manifolds are special complex symplectic manifolds.

Anti-quasi-Sasakian manifolds

Definition

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called anti-quasi-Sasakian (aqS) if

$$d\Phi = 0, \qquad N_\varphi = 2d\eta \otimes \xi.$$

We refer to $N_\varphi = 2d\eta \otimes \xi$ as the anti-normality condition.

Remark:

For an anti-quasi-Sasakian manifold

$$\text{normality} \;\Leftrightarrow\; d\eta = 0 \text{ (i.e. } \operatorname{rk}\eta = 1) \;\Leftrightarrow\; M \text{ is cokähler},$$

that is

$$\mathsf{qS} \cap \mathsf{aqS} = \{\text{cokähler}\}.$$

Applying (1), for every anti-quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, it turns out that:

$$d\eta(\xi, \cdot) = 0, \quad d\eta(\varphi X, \varphi Y) = -d\eta(X, Y).$$

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- If the φ-invariant distribution $\mathcal{E} = \mathcal{H} \cap \mathrm{Ker}(d\eta)$ has constant rank $2q$, then $\dim M = 2q + 4p + 1$, where $4p + 1 = \mathrm{rk}(\eta)$, i.e.
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$$\pi : M \to M/\xi,$$

- φ projects onto an almost complex structure J on M/ξ;
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- g projects onto a Kähler metric w.r.t. J;

Applying (1), for every anti-quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, it turns out that:

$$d\eta(\xi, \cdot) = 0, \quad d\eta(\varphi X, \varphi Y) = -d\eta(X, Y).$$

$$\mathcal{L}_\xi \varphi = 0, \quad \mathcal{L}_\xi g = 0, \quad \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi d\eta = 0.$$

- If the φ-invariant distribution $\mathcal{E} = \mathcal{H} \cap \mathrm{Ker}(d\eta)$ has constant rank $2q$, then $\dim M = 2q + 4p + 1$, where $4p + 1 = \mathrm{rk}(\eta)$, i.e.
$$\eta \wedge (d\eta)^{2p} \neq 0, \quad d\eta^{2p+1} = 0.$$

Considering the local submersion

$$\pi : M \to M/\xi,$$

- φ projects onto an almost complex structure J on M/ξ;
- $N_\varphi(X, Y, Z) := g(N_\varphi(X, Y), Z) = 0 \ \forall X, Y, Z \in \mathcal{H} \ \Rightarrow \ J$ is integrable;
- g projects onto a Kähler metric w.r.t. J;
- $d\eta$ projects onto a closed, J-anti-invariant 2-form ω (i.e., ω of type $(2, 0)$).

Theorem (Boothby-Wang type theorem)

Every anti-quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ *locally fibers onto a Kähler manifold* $(M/\xi, J, g)$ *endowed with a closed* $(2,0)$*-form* ω*.*

In particular, if ξ *is regular with compact leaves, then* M *is a principal* $\mathbb{S}^1$*-bundle over* M/ξ *and* η *is a connection form on* M*, whose curvature form is* $d\eta = \pi^*\omega$*.*

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Conversely:

Theorem

Let (B, J, k) be a Kähler manifold endowed with a closed (2,0)-form ω. If $[\omega] \in H^2(B, \mathbb{Z})$, then there exists a principal $\mathbb{S}^1$-bundle M over B endowed with an anti-quasi-Sasakian structure (φ, ξ, η, g) such that η is a connection form on M whose curvature form is $d\eta = \pi^*\omega$.

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Every anti-quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ locally fibers onto a Kähler manifold $(M/\xi, J, g)$ endowed with a closed $(2,0)$-form ω.
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This provides a method to construct examples of aqS manifolds.

- Let (B^{4n}, J, k) be a Kähler manifold, U a coordinate neighborhood such that $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$, $i = 1, \dots, 2n$. Consider

$$\omega = \sum_{i=1}^{p}(dx_i \wedge dx_{n+i} - dy_i \wedge dy_{n+i}), \qquad 1 \leq p \leq n,$$

  $\omega = d\beta$ is an exact 2-form of type $(2,0)$ and rank $4p$.

  The trivial bundle $U \times \mathbb{S}^1$ is endowed with an aqS structure (φ, ξ, η, g), where

$$\xi = \frac{d}{dt}, \quad \varphi\xi = 0, \quad \varphi X^* = (JX)^*,$$

$$\eta = dt + \pi^*\beta, \quad g = \pi^*k + \eta \otimes \eta.$$

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- Complex unit disc $D^{2n} \subset \mathbb{C}^{2n}$ endowed with the Kähler structure of constant holomorphic sectional curvature $c < 0$.
- Hermitian symmetric spaces of non-compact type of complex dimension $2n$.

- Hyperkähler manifolds $(B^{4n}, J_1, J_2, J_3, g)$:

$$J_i^2 = -I, \qquad g(J_iX, J_iY) = g(X, Y), \qquad \nabla J_i = 0,$$

$$J_1J_2 = J_3 = -J_2J_1$$

✓ All the fundamental 2-forms $\Omega_i = g(\cdot, J_i\cdot)$ are closed $(d\Omega_i = 0)$.

✓ For every even permutation of (1,2,3), Ω_j, Ω_k are J_i-anti-invariant.

- **IF** Ω_j (or Ω_k) is integral $\rightsquigarrow$ aqS structure on the principal $\mathbb{S}^1$-bundle M over (B, J_i, g, Ω_j).

V. Cortés, *A note on quaternionic Kähler manifolds with ends of finite volume*, Q. J. Math. **74** (2023), no. 4, 1489-1493.

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A remarkable class of examples

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This is equivalent to the existence of three almost contact metric structures $(\varphi_i, \xi, \eta, g)_{i=1,2,3}$ such that

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If $(M, \varphi_i, \xi, \eta, g)_{i=1,2,3}$ is an $Sp(n)$-almost contact metric manifold such that

$$d\Phi_1 = d\Phi_2 = 0, \quad d\eta = 2\Phi_3,$$

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Remark: For $n = 1$, i.e. $\dim M = 5$, then $Sp(1) = SU(2)$ and

$$\{\text{double aqS-Sasakian}\} = \{\text{contact Calabi-Yau}\} \subset \{K\text{-contact hypo } SU(2)\}.$$

Example (Weighted Heisenberg Lie group)

Let $n \geq 1$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $\mathfrak{h}_\lambda^{4n+1} := \mathrm{span}\{\xi, \tau_1, \ldots, \tau_{4n}\}$, such that

$$[\tau_r, \tau_{3n+r}] = [\tau_{n+r}, \tau_{2n+r}] = 2\lambda_r \xi, \qquad r = 1, \ldots, n.$$

Let H_λ^{4n+1} be the simply connected Lie group with Lie algebra $\mathfrak{h}_\lambda^{4n+1}$. We define three left-invariant almost contact metric structures $(\varphi_i, \xi, \eta, g)$ by setting $\varphi_i \xi = 0$ and

$$\varphi_i(\tau_r) = \tau_{in+r}, \ \varphi_i(\tau_{in+r}) = -\tau_r, \ \varphi_i(\tau_{jn+r}) = \tau_{kn+r}, \ \varphi_i(\tau_{kn+r}) = -\tau_{jn+r}.$$

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It turns out that:

- $(\varphi_1, \xi, \eta, g)$, $(\varphi_2, \xi, \eta, g)$ are anti-quasi-Sasakian, $(\varphi_3, \xi, \eta, g)$ is quasi-Sasakian and $\varphi_1\varphi_2 = \varphi_3 = -\varphi_2\varphi_1$.

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- If $\lambda_r \neq 0$ for all r, the structures have maximal rank.
  If $\lambda_r = 1$ for all r, then $(\varphi_3, \xi, \eta, g)$ is Sasakian.
- $\mathfrak{h}_\lambda^{4n+1}$ is a 2-step nilpotent Lie algebra.

Being 2-step nilpotent, if $\lambda_r \in \mathbb{Q}$ for every r, then H^{4n+1}_λ admits a cocompact discrete subgroup Γ (Malčev), so that an aqS strucure is induced on the compact nilmanifold H^{4n+1}_λ/Γ.

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Proposition

A $(4n+1)$-dimensional compact nilmanifold G/Γ admits an anti-quasi-Sasakian structure of maximal rank induced by a left-invariant aqS structure on G if and only $G \cong H_\lambda^{4n+1}$.

Theorem

Every nilpotent Lie algebra endowed with an anti-quasi-Sasakian structure of maximal rank is isomorphic to $\mathfrak{h}_\lambda^{4n+1}$, for some non-zero weights $\lambda_1, \ldots, \lambda_n$.

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This follows from the following

Proposition

Every nilpotent Lie algebras $\mathfrak{g}$ endowed with a transversely Kähler almost contact metric structure of maximal rank is the 1-dimensional central extensions of an abelian Lie algebra $\mathfrak{a}$, i.e.

$$\mathfrak{g} = \mathfrak{a} \oplus \mathbb{R}\xi, \quad [X, \xi] = 0, \quad [X, Y] = -\omega(X, Y)\xi$$

for every $X, Y \in \mathfrak{a}$ and for some 2-cocycle ω on $\mathfrak{a}$.

Geometric and topological obstructions

In the spirit of the Kanemaki's characterization of quasi-Sasakian manifold, we have the following:

Theorem

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ *is anti-quasi-Sasakian if and only if for every* $X, Y \in \mathfrak{X}(M)$

$$(\nabla^g_X \varphi)Y = 2\eta(X)AY + \eta(Y)AX + g(X, AY)\xi,$$

for some $(1, 1)$*-tensor field* A *such that* $g(AX, Y) = -g(X, AY)$ *and* $A\varphi = -\varphi A$*. In this case* A *is uniquely determined by* $A = -\varphi \circ \nabla^g \xi$*.*

Given an anti-quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, consider

$$A := -\varphi \circ \nabla^g \xi, \quad \psi := A\varphi = -\nabla^g \xi.$$

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Then $A\xi = \psi\xi = 0$, $\eta \circ A = \eta \circ \psi = 0$, A, ψ are skew-symmetric w.r.t. g and

$$A\varphi = \psi = -\varphi A, \quad \varphi\psi = A = -\psi\varphi, \quad \psi A = -\varphi A^2 = -A\psi$$

The associated 2-forms $\mathcal{A} := g(\cdot, A\cdot)$ and $\Psi := g(\cdot, \psi\cdot)$ satisfy

$$d\mathcal{A} = 0, \quad d\Phi = 0, \quad d\eta = 2\Psi.$$

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Remark: In general A, ψ are ***not*** almost contact structures, i.e. $A|^2_{\mathcal{H}} = \psi|^2_{\mathcal{H}} \neq -I$.

Proposition

Let $(M, \varphi, \xi, \eta, g)$ *be an anti-quasi-Sasakian manifold. Then*

1. $R(\xi, X)Y = (\nabla^g_X \psi)Y$;
2. $R(\xi, X)\xi = \psi^2 X = A^2 X$.

In particular M *has non-negative* ξ*-sectional curvatures, and* $K(\xi, X) = \lambda^2$*, for every unit* $X \in \mathcal{H}$ *such that* $\psi^2 X = -\lambda^2 X$.

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Theorem

Let $(M, \varphi, \xi, \eta, g)$ be anti-quasi-Sasakian manifold. Then the following are equivalent:

(a) $K(\xi, X) = 1$ *for every* $X \in \mathcal{H}$;

(b) $\psi^2 = A^2 = -I + \eta \otimes \xi$;

(c) $(A, \varphi, \psi, \xi, \eta, g)$ *is a double aqS-Sasakian structure.*

Furthermore, if one of the above condition holds, then M is transversely hyperkähler, hence transversely Ricci-flat.

Owing to $d\mathcal{A} = 0$, we have the following:

Proposition

Let $(M, \varphi, \xi, \eta, g)$ be an anti-quasi-Sasakian manifold. Then:

1. $\mathrm{Ric}(\xi, \xi) = \|\psi\|^2$;
2. $\mathrm{Ric}(\xi, X) = 0$;
3. $\mathrm{Ric}(X, Y) = \mathrm{Ric}^T(X', Y') - 2g(\psi X, \psi Y)$,

where Ric^T is the Ricci tensor field of the base space of the local Riemannian submersion $\pi : M \to M/\xi$, and $X, Y \in \mathcal{H}$ are basic vector fields projecting on X', Y'.

The scalar curvatures of M and M/ξ are related by

$$s = s^T - \|\nabla\xi\|^2 = s^T - \|\psi\|^2.$$

Theorem

Let $(M, \varphi, \xi, \eta, g)$ be a transversely Einstein, non coKähler, aqS manifold. Then:

$$\psi^2|_{\mathcal{H}} = -\lambda^2 I, \ \lambda \in \mathbb{R}^* \ \Leftrightarrow \ M \text{ is } \eta\text{-Einstein}.$$

In this case M turns out to be transversely Ricci-flat, $\dim M = 4n+1$, and

$$\mathrm{Ric} = -2\lambda^2 g + (4n+2)\lambda^2 \eta \otimes \eta, \quad s = -4n\lambda^2.$$

Proof ($\Rightarrow$): Apply the homothetic deformation:

$$\varphi' = \varphi, \quad \xi' = \frac{1}{\lambda}\xi, \quad \eta' = \lambda\eta, \quad g' = \lambda^2 g.$$

Then $(\varphi', \xi', \eta', g')$ is an aqS structure with associated operator $\psi' = \frac{1}{\lambda}\psi$, $A' = \frac{1}{\lambda}A$. In particular,

$$\psi'^2|_{\mathcal{H}} = -I \ \Rightarrow \ (A', \varphi', \psi', \xi', \eta', g') \text{ is double aqS-Sasakian}$$

Thus the metric g is transversely Ricci-flat ($\mathrm{Ric}^T = 0$), being homothetic to g', and by the previous theorem one gets the result. □

These results give obstructions to the existence of anti-quasi-Sasakian structures.

Theorem

If $(M, \varphi, \xi, \eta, g)$ is an anti-quasi-Sasakian manifold with constant sectional curvature, then it is flat and cokähler.

Proof: If (M, g) has constat sectional curvature κ, then M is Einstein and

$$R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y).$$

Hence,

$$\psi^2 = R(\xi, \cdot)\xi = \kappa(-I + \eta \otimes \xi),$$

that is

$$\psi^2|_{\mathcal{H}} = -\kappa I.$$

If $\kappa \neq 0$, M is η-Einstein, non Einstein. □

Theorem

There exist no compact regular, non-cokähler, aqS manifolds with $\mathrm{Ric} > 0$.
There exist no compact regular aqS manifolds of maximal rank with $\mathrm{Ric} \geq 0$.

Proof: If $(M, \varphi, \xi, \eta, g)$ is a compact regular, non cokähler, aqS manifold, then M/ξ is compact Kähler with a non-vanishing closed (hence holomorphic) $(2,0)$-form. Hence M/ξ cannot have positive definite Riemannian Ricci tensor.
On the other hand, for every $X' \in \mathfrak{X}(M/\xi)$ and $X \in \mathcal{H}$ basic vector field such that $\pi_* X = X'$, in both the cases of the statement one has:

$$\mathrm{Ric}^T(X', X') = \mathrm{Ric}(X, X) + 2\|\psi X\|^2 \;\Rightarrow\; \mathrm{Ric}^T > 0,$$

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Corollary

There exist no connected homogeneous aqS manifolds of maximal rank with $\mathrm{Ric} \geq 0$*.*

W.M. Boothby, H.C. Wang, *On contact manifolds*, Ann. of Math. **68** (1958), 721-734.

Theorem

There exist no compact homogeneous aqS manifolds of maximal rank.

Proof: Let $(M, \varphi, \xi, \eta, g)$ be a compact homogeneous aqS manifold of maximal rank, and let G a Lie group acting transitively on M.

Compact homogeneous manifolds

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Proof: Let $(M, \varphi, \xi, \eta, g)$ be a compact homogeneous aqS manifold of maximal rank, and let G a Lie group acting transitively on M. Then:

- η is an invariant contact form $\Rightarrow$ ξ is regular with compact orbits $\Rightarrow$ $\pi : M \to M/\xi$ is a principal circle bundle over a Kähler manifold endowed with a closed $(2,0)$-form ω such that $d\eta = \pi^*\omega$.

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- M/ξ is simply connected (Díaz-Miranda & Reventós).

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Theorem

There exist no compact homogeneous aqS manifolds of maximal rank.

Proof: Let $(M, \varphi, \xi, \eta, g)$ be a compact homogeneous aqS manifold of maximal rank, and let G a Lie group acting transitively on M. Then:

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Theorem

Every simply connected, compact, homogeneous Kähler manifold is a generalized flag manifold $G/Z_G(S)$.

$M/\xi \cong G/K$, with G compact semisimple and $K = Z_G(S)$. The Killing form $\mathrm{B} < 0$ gives rise to a reductive decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad \mathfrak{m} = \mathfrak{k}^\perp$$

Let T be a maximal torus in G. Then, $\mathfrak{h} := \mathfrak{t}^\mathbb{C} \subset \mathfrak{k}^\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$ and

$$\mathfrak{k}^\mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_K} \mathbb{C}E_\alpha, \quad \mathfrak{m}^\mathbb{C} = \bigoplus_{\alpha\in\Delta_M} \mathbb{C}E_\alpha \qquad (\Delta_M := \Delta \setminus \Delta_K).$$

- Invariant complex structures: $J : \mathfrak{m} \to \mathfrak{m}$, $JE_\alpha = i\varepsilon_\alpha E_\alpha$ $(\alpha \in \Delta_M)$.
- $\mathrm{B}(JX, JY) = \mathrm{B}(X, Y)$, $X, Y \in \mathfrak{m}$.
- Invariant closed 2-form: $\omega(X, Y) = \mathrm{B}([X, Y], Z_\omega)$, $X, Y \in \mathfrak{m}$, $Z_\omega \in \mathfrak{z}(\mathfrak{k})$.

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Proposition

Every closed, invariant 2-form on a generalized flag manifolds is of type $(1, 1)$.

Let $(M, \varphi_i, \xi, \eta, g)$ be a double aqS-Sasakian manifold. We define:

- Basic k-forms: $\Omega^*_B(M, \xi) := \{\alpha \in \Omega^k(M) \mid i_\xi \alpha = 0,\ \mathcal{L}_\xi \alpha = 0\}$

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  Denoting by d_B the restriction of d to basic forms, $(\Omega^*_B(M,\xi), d_B)$ is a subcomplex of the de Rham complex and $H^*_B(M,\xi) := \frac{\operatorname{Ker} d_B}{\operatorname{Im} d_B}$ is the basic cohomology of the Reeb foliation.

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Transverse Hodge theory

Let $(M, \varphi_i, \xi, \eta, g)$ be a double aqS-Sasakian manifold. We define:

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Theorem (A. El Kacimi-Alaoui, G. Hector)

If M is compact, then $\Omega^*_{\Delta_B}(M,\xi) \cong H^*_B(M,\xi)$.

Theorem

Let $(M^{4n+1}, \varphi_i, \xi, \eta, g)$ be a compact double aqS-Sasakian manifold. Then, the Betti numbers b_k, for odd $1 \leq k \leq 2n$, are divisible by 4.

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Proof: For any Sasakian manifold, the short exact sequence of complexes

$$0 \to \Omega^k_B(M) \xhookrightarrow{j} \Omega^k(M) \xrightarrow{i_\xi} \Omega^{k-1}_B(M) \to 0,$$

induces the long exact sequence in cohomology

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By the Transverse Hard Lefschetz property, the maps

$$H^k_B(M,\xi) \to H^{4n-k}_B(M,\xi) \qquad [\alpha]_B \mapsto [(d\eta)^{2n-k} \wedge \alpha]_B$$

are isomorphisms for all $0 \leq k \leq 2n$.

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are isomorphisms for all $0 \leq k \leq 2n$. In particular, the connection homomorphism is injective for $0 \leq k \leq 2n-1$, and the map $j_* : H^k_B(M,\xi) \to H^k(M)$ is surjective by the exactness, for all $1 \leq k \leq 2n$. It follows that for k in such range, $b_k = b^B_k - b^B_{k-2}$.

Claim: The odd basic Betti numbers b_{2p+1}^B are divisible by 4.

Proof: For $i = 1, 2, 3$ define the operators $K_i, \mathcal{I}_i : \Omega_B^k(M) \to \Omega_B^k(M)$ such that

$$(K_i\alpha)(X_1, \ldots, X_k) := \sum_{s=1}^{k} \alpha(X_1, \ldots \varphi_i X_s, \ldots, X_k)$$

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On every foliated chart $\mathcal{U} \subset M$, owing to $\pi_*\varphi_i = J_i\pi_*$, the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(\mathcal{U}/\xi) & \xrightarrow{\pi^*} & \Omega^k_B(\mathcal{U}) \\ {\scriptstyle \bar{K}_i}\downarrow & & \downarrow{\scriptstyle K_i} \\ \Omega^k(\mathcal{U}/\xi) & \xrightarrow[\pi^*]{} & \Omega^k_B(\mathcal{U}) \end{array}$$

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Since $[\Delta, \bar{K}_i] = 0$, $\bar{K}_i$ restricts to a map between harmonic forms:

$$\begin{array}{ccc} \Omega^k_{\Delta}(\mathcal{U}/\xi) & \xrightarrow{\pi^*} & \Omega^k_B(\mathcal{U}) \\ \bar{K}_i \downarrow & & \downarrow K_i \\ \Omega^k_{\Delta}(\mathcal{U}/\xi) & \xrightarrow[\pi^*]{} & \Omega^k_B(\mathcal{U}) \end{array}$$

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One can show that $\pi^* : \Omega^*_\Delta(\mathcal{U}/\xi) \to \Omega^*_{\Delta_B}(\mathcal{U}, \xi)$ is an isomorphism, so that K_i restricts to an operator between basic harmonic forms:

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Being $\mathcal{I}_i^2 = (-1)^k I$ and taking into account $\varphi_1\varphi_2 = \varphi_3 = -\varphi_2\varphi_1$, it follows that for k odd, the operators $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ behaves as the imaginary units of the quaternion algebra. Hence, for $k = 2p + 1$,

$$b^B_{2p+1} = \dim H^{2p+1}_B(M, \xi) = \dim \Omega^{2p+1}_{\Delta_B}(M) \text{ is divisible by } 4. \quad \square$$

Theorem

Let $(M^{4n+1}, \varphi_i, \xi, \eta, g)$ be a compact double aqS-Sasakian manifold. Then, for $1 \leq p \leq n$ we have that

$$\sum_{h=0}^{p} b_{2h} \geq \binom{p+2}{2}.$$

In particular, $b_2 \geq 2$.

Proof (sketch): The set

$$S_p := \{\Phi_1^{p_1} \wedge \Phi_2^{p_2} \wedge \Phi_3^{p_3} \mid p_1 + p_2 + p_3 = p\} \subset \Omega^{2p}_{\Delta_B}(M, \xi)$$

has $\binom{p+2}{2}$ linealrly independent elements. Therefore $b^B_{2p} \geq \binom{p+2}{2}$. Using $b^B_{2p} = b_{2p} + b^B_{2p-2}$ and iterating, one gets the statement.
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Corollary

The spheres $\mathbb{S}^{4n+1}$ cannot admit double aqS-Sasakian structures.

Theorem

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ a compact anti-quasi-Sasakian manifold. Then, the Betti numbers b_2 and b_{2n-1} are positive.

Proof:

$$\left.\begin{array}{r}\Phi \text{ is } \varphi\text{-invariant} \\ d\eta \text{ is } \varphi\text{-anti-invariant}\end{array}\right\} \Rightarrow \; d\eta \wedge \Phi^{n-1} = 0 \;\Rightarrow\; d(\eta \wedge \Phi^{n-1}) = 0.$$

If $b_2 = 0$, i.e. $H^2(M, \mathbb{R}) = \{0\}$, then $\Phi = d\alpha$ for some $\alpha \in \Lambda^1(M)$, and

$$0 \neq \int_M \eta \wedge \Phi^n = \int_M (\eta \wedge \Phi^{n-1}) \wedge d\alpha = -\int_M d(\eta \wedge \Phi^{n-1} \wedge \alpha) = 0 \; !! \quad \square$$

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Remark: Notice that Sasakian manifolds with positive curvature have $b_2 = 0$.

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Obrigado!

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