Dark soliton solutions to the Nonlinear Schrödinger Equation

Salvador López Martínez

in collaboration with

André de Laire

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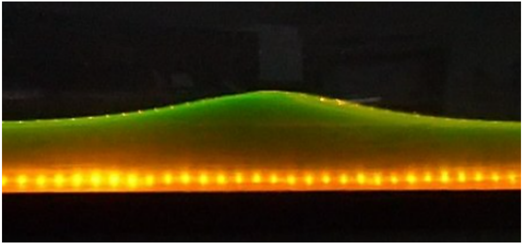
What is a soliton?

It is a physical object that **travels with constant velocity** in some direction **without changing shape**, even after mutual collisions.



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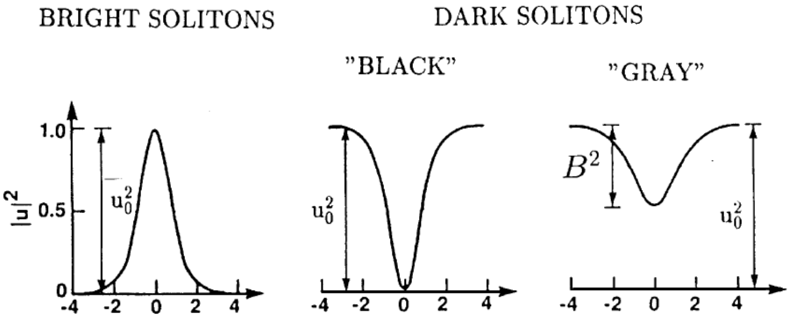
What is a soliton?

The soliton wave was recreated on the Scott Russell Aqueduct in 1995
Image Credit: Department of Mathematics, Heriot-Watt University

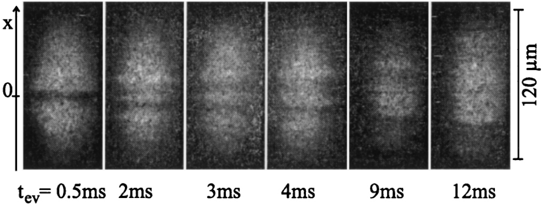
Dark vs Bright solitons



Y.S. Kivshar and B. Luther-Davies (1998)

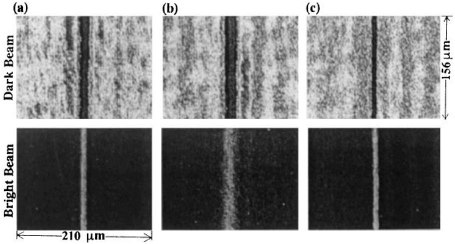
Are there dark solitons in nature?

In Bose-Einstein condensates



S. Burger et al. (1999)

In optics



Z. Chen et al. (1996)

Defocusing NLS (or Gross-Pitaevskii) equation

$$\boxed{i\partial_t\Psi = \Delta\Psi + \Psi\big(1 - |\Psi|^2\big) \quad \text{in } \mathbb{R}\times\mathbb{R}^N}$$

It models

- Bose-Einstein condensation with repulsive interactions between bosons.
- Evolution of optical pulses in nonlinear self-defocusing media.

Mathematically, a dark soliton is a traveling wave...

$$c > 0,\ \Psi(t,x) = u(x_1 - ct, x') \implies \boxed{ic\partial_{x_1}u + \Delta u + u(1 - |u|^2) = 0 \quad \text{in } \mathbb{R}^N}$$

...with finite Ginzburg-Landau energy:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N}(1-|u|^2)^2 dx < \infty \implies \boxed{|u(x)| \to 1, \text{ as } |x| \to \infty}$$

Some basics

$$\boxed{ic\partial_{x_1} u + \Delta u + u(1 - |u|^2) = 0 \quad \text{in } \mathbb{R}^N}$$

- The energy space is **not** a vector space...

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2 dx$$

$$\boxed{\mathcal{E}(\mathbb{R}^N) = \{u \in H^1_{\text{loc}}(\mathbb{R}^N;\mathbb{C}) : \ \nabla u \in L^2(\mathbb{R}^N),\ 1 - |u|^2 \in L^2(\mathbb{R}^N)\}}$$

  ... and contains oscillating functions (for $N = 1$): $e^{i\log(x^2+1)} \in \mathcal{E}(\mathbb{R})$.

- Infinite energy solutions: $u(x) = e^{-cx_1 i}, \quad |u| = 1$.
- Trivial solutions: $u \equiv \text{cst}, \quad |u| = 1$.
- Invariances:
  - $u(x) \rightsquigarrow u(x + x_0),\ x_0 \in \mathbb{R}^N$
  - $u \rightsquigarrow e^{i\theta_0} u,\ \theta_0 \in \mathbb{R}$

The 1d case

$$\boxed{icu' + u'' + u(1 - |u|^2) = 0 \quad \text{in } \mathbb{R}}$$

Theorem (Béthuel, Gravejat, Saut (2008))

1. *If* $c \geq \sqrt{2}$, *we have* **nonexistence**.
2. *If* $c \in [0, \sqrt{2})$, *then the* **unique (u.t.i.) nontrivial finite-energy solution** *is*

$$u_c(x) = \sqrt{\frac{2 - c^2}{2}} \tanh\left(\frac{\sqrt{2 - c^2}}{2} x\right) - i\frac{c}{\sqrt{2}}.$$

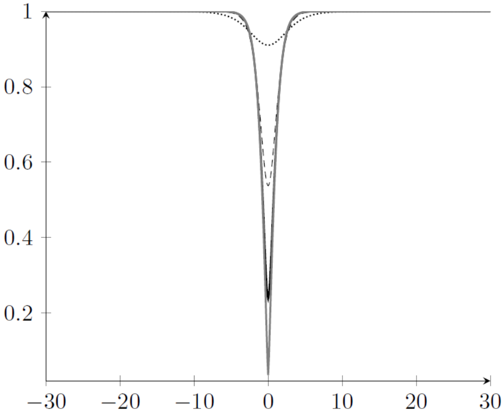
Moreover, u_c *is* **orbitally stable.**

Properties:

1. $1 - |u_c|^2$ is **analytic, even** and **radially monotone,** and **decays exponentially**.
2. u_c has **different limit values** at $-\infty$ and $+\infty$.
3. u_c has a **vortex** (i.e. $|u_c| = 0$ at some point) iff $c = 0$.

The 1d case

$$icu' + u'' + u(1 - |u|^2) = 0 \quad \text{in } \mathbb{R}$$

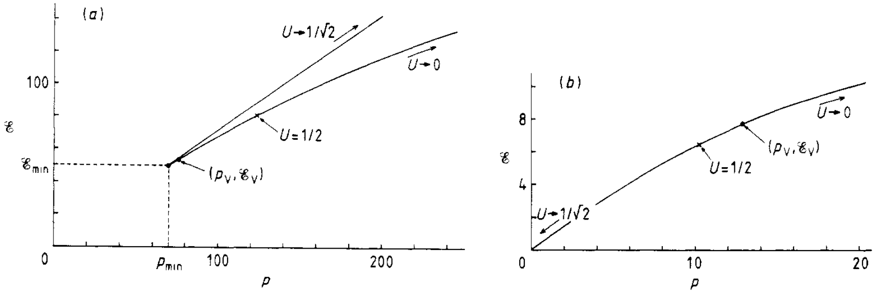


The Jones, Putterman and Roberts program ($N = 2, 3$)

$$ic\partial_{x_1} u + \Delta u + u(1 - |u|^2) = 0 \quad \text{in } \mathbb{R}^N$$

Jones, Putterman and Roberts conjectures for $N = 2, 3$ (1986)

Existence, asymptotics and stability $\forall c \in (0, \sqrt{2})$. **Nonexistence** $\forall c > \sqrt{2}$.



Nonexistence and asymptotics

Theorem (P. Gravejat (2003, 2004))

If $N \geq 2$*, nonexistence for every* $c > \sqrt{2}$*. If* $N = 2$*, nonexistence for* $c = \sqrt{2}$*.*

Theorem (P. Gravejat (2004-2006))

Precise asymptotic behavior of the dark solitons u *at infinity. In particular,*

1. $u(x) \to 1$ *as* $|x| \to \infty$.
2. $(u(x) - 1)|x|^{N-1}$ *is bounded.*

Existence of orbitally stable solutions
The momentum

$$P(u) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_{x_1} u, u-1\rangle dx$$

Not well-defined in $\mathcal{E}(\mathbb{R}^N)$

Alternatives:

1. Work in a smaller space.
2. Consider renormalized momentums, etc.

Theorem (Béthuel, Gravejat, Saut (2009), Chiron, Mariş (2013))

If $N = 2$, for every $p > 0$ there exists a dark soliton $u_p \in \mathcal{E}(\mathbb{R}^N)$ with speed $c_p \in (0, \sqrt{2})$ such that

$$E(u_p) = \min\{E(v) : \ v \in \tilde{\mathcal{E}}(\mathbb{R}^N), \ P(v) = p\}.$$

Moreover, the set of minimizers is orbitally stable.
If $N = 3$, the same result holds for every $p \geq p_0$, for some $p_0 > 0$.
Remark: *The speed c_p is a Lagrange multiplier and cannot be prescribed.*

Existence for prescribed subsonic speed

1. **By the Mountain Pass Lemma**: existence for $c \gtrapprox 0$ in dimensions $N = 2$ [Béthuel, Saut (1999)] and $N = 3$ [Chiron (2004)].

2. **By minimizing the energy under a Pohozaev constraint:** existence for all $c \in (0, \sqrt{2})$ in dimension $N = 3$ [Mariş (2013)].

3. **By pertubative methods (Lyapunov-Schmidt reduction)**: existence in dimension $N = 2$ for $c \gtrapprox 0$ [Liu, Wei (2020), Chiron, Pacherie (2021)] and for $c \lessapprox \sqrt{2}$ [Liu, Wang, Wei, Yang (2021)].

4. **By the MPL + Monotonicity trick:** existence in dimension $N = 2$ for *almost every* $c \in (0, \sqrt{2})$ [Bellazzini, Ruiz (2023)].

The Jones, Putterman and Roberts program is *almost* complete. □

The nonlocal model

$$\boxed{i\partial_t\Psi = \Delta\Psi + \Psi\big(\mathcal{W} * (1-|\Psi|^2)\big) \quad \text{in } \mathbb{R}\times\mathbb{R}^N}$$

- It was originally proposed by [Gross (1963)] and [Pitaevskii (1961)].
- $\mathcal{W}$ is an even tempered distribution that captures **nonlocal interactions** between bosons or with the optical medium.

Example

$$\mathcal{W} = \delta_0, \quad \mathcal{W} \approx e^{-x^2}, \quad \mathcal{W} \approx e^{-|x|}, \quad \mathcal{W} \approx \delta_0 - e^{-|x|}.$$

We still look for traveling waves with finite energy:

$$\Psi(t,x) = u(x_1 - ct, x') \implies \boxed{ic\partial_{x_1}u + \Delta u + u\big(\mathcal{W} * (1-|u|^2)\big) = 0 \quad \text{in } \mathbb{R}^N}$$

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} (1-|u|^2)\big(\mathcal{W} * (1-|u|^2)\big)\, dx$$

$$\widehat{\mathcal{W}} \in L^\infty(\mathbb{R}^N) \implies E(u) < \infty \, \forall u \in \mathcal{E}(\mathbb{R}^N)$$

The (nonlocal) Jones, Putterman and Roberts program

So far, only for $N = 1$ (except nonexistence for $N \geq 2$, $c > \sqrt{2}$ [de Laire (2012)]).

1. Existence of energy minimizers under (renormalized) momentum constraint [de Laire, Mennuni (2020)].
2. Existence for almost every $c \in (0, \sqrt{2})$, and for all $c \in (0, \sqrt{2})$ provided a priori estimates hold [de Laire, L.-M. (2022)].
3. Nonexistence for $c = \sqrt{2}$ [de Laire, L.-M. (2022)].
4. Existence of symmetric solution for $c \in (\varepsilon, \sqrt{2} - \varepsilon)$ and for $\mathcal{W} \approx \delta_0$ [de Laire, L.-M. (2024)].
5. Qualitative properties for relevant examples [de Laire, L.-M. (2022, 2024)]:
   - A priori estimates.
   - Exponential convergence to different limits at $\pm\infty$.
   - Analytic regularity.
   - Monotonicity breaking.
   - Nonlocal-to-local limit.
6. Study of the case $c = 0$ [de Laire, L.-M. (2024)].

Proof of existence for a.e. $c \in (0, \sqrt{2})$

Simplifying the framework

(TW$_c$) $$\boxed{icu' + u'' + u\big(\mathcal{W} * (1 - |u|^2)\big) = 0 \quad \text{in } \mathbb{R}}$$

Lemma

Every solution $u \in \mathcal{E}(\mathbb{R})$ to (TW$_c$) is smooth and vortexless, i.e. $|u| > 0$. In particular, there is a lifting $u = \rho e^{i\theta}$ with $1 - \rho \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$, $\theta \in C^\infty(\mathbb{R})$. Moreover, ρ satisfies (P_c).

(P$_c$) $$\boxed{-\rho'' + \frac{c^2}{4}\frac{1-\rho^2}{\rho^3} = \rho\big(\mathcal{W} * (1 - \rho^2)\big) \quad \text{in } \mathbb{R},} \qquad \rho : \mathbb{R} \to (0, +\infty)$$

Lemma

Let $\rho \in 1 + H^1(\mathbb{R})$ be a **positive solution** *to (P_c). We define $u = \rho e^{i\theta}$, with $\theta(x) = \frac{c}{2}\int_a^x \frac{1-\rho^2}{\rho^2}$. Then, $u \in \mathcal{E}(\mathbb{R})$ is a* **nontrivial solution** *to (TW$_c$).*

Proof of existence for a.e. $c \in (0, \sqrt{2})$

Variational structure

$$J_c(1-\rho) = \frac{1}{2}\int_{\mathbb{R}} (\rho')^2 + \frac{1}{4}\int_{\mathbb{R}} (\mathcal{W} * (1-\rho^2))(1-\rho^2) - \frac{c^2}{8}\int_{\mathbb{R}} \frac{(1-\rho^2)^2}{\rho^2}$$

$$J_c(1-\rho) \in \mathbb{R} \cup \{-\infty\} \quad \forall 1-\rho \in H^1(\mathbb{R})$$

Lemma

$J_c \in \mathcal{C}^2(\Omega)$, *where* $\Omega = \{1-\rho \in H^1(\mathbb{R}) : \ \rho > 0 \text{ in } \mathbb{R}\}$. *Moreover, given* $1-\rho \in \Omega$,

$$J_c'(1-\rho) = 0 \iff \rho \text{ is a positive solution to } (P_c).$$

Proof of existence for a.e. $c \in (0, \sqrt{2})$

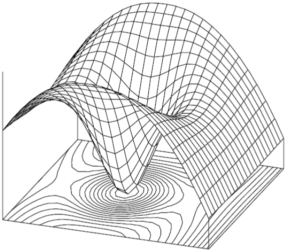
Mountain Pass geometry

(H) $\widehat{\mathcal{W}}(\xi) \geq 1 - \kappa\xi^2$ in $\mathbb{R}$ for some $\kappa \in [0, \frac{1}{2})$.

Lemma

Let $c_0 \in (0, \sqrt{2})$. *Then, there exists* $\phi \in \Omega$ *such that, for all* $c \in (c_0, \sqrt{2})$,

1. $J_c(\phi) < 0$, *and*
2. J_c *achieves a strict local minimum at* 0, *with* $J_c(0) = 0$.



Jabri (2003)

Proof of existence for a.e. $c \in (0, \sqrt{2})$

Palais-Smale sequences

$$1 - \rho_n = v_n \in \Omega : \quad \boxed{J_c(v_n) \to \alpha} \quad \boxed{\|J_c'(v_n)\|_{H^{-1}(\mathbb{R})} \to 0}$$

Theorem (Classical Mountain Pass Lemma)

$\Omega = H^1(\mathbb{R})$ and MP structure $\implies$ $\exists$ PS sequence $\{v_n\}$ at level $\alpha = \gamma(c)$, where

$$\boxed{\gamma(c) := \inf_{g \in \Gamma(c_0)} \max_{t \in [0,1]} J_c(g(t)) > 0} \quad \forall c \in (c_0, \sqrt{2}),$$

$$\Gamma(c_0) := \{g \in \mathcal{C}([0,1], \Omega) : \; g(0) = 0, \; g(1) = \phi\}.$$

Moreover, if $v_n \to v$, then $J_c(v) = \gamma(c)$, $J_c'(v) = 0$.

Questions

1. Do PS sequences **exist** in Ω?
2. Are they **bounded** in $H^1(\mathbb{R})$?
3. Do they remain **away from the boundary** of Ω?

Proof of existence for a.e. $c \in (0, \sqrt{2})$

Monotoniciy trick of Struwe (and Jeanjean)

$$\boxed{J_c(1-\rho) = \frac{1}{2}\int_{\mathbb{R}} (\rho')^2 + \frac{1}{4}\int_{\mathbb{R}} (\mathcal{W} * (1-\rho^2))(1-\rho^2) - \frac{c^2}{8}\int_{\mathbb{R}} \frac{(1-\rho^2)^2}{\rho^2}}$$

$$\boxed{\gamma(c) := \inf_{g\in\Gamma(c_0)} \max_{t\in[0,1]} J_c(g(t))} \quad \forall c \in (c_0, \sqrt{2})$$

Monotonicity trick (Struwe (1988), Jeanjean (1999))

$\gamma : (c_0, \sqrt{2}) \to (0, +\infty)$ is non increasing $\implies \exists \gamma'(c)$ for a.e. $c \in (c_0, \sqrt{2})$

$\implies \exists$ **bounded** PS sequence

Proof of existence for a.e. $c \in (0, \sqrt{2})$

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$\implies$ $\exists$ **bounded** PS sequence

Theorem (de Laire, L.-M. (2022))

Under (H), there exists a nontrivial solution $u \in \mathcal{E}(\mathbb{R})$ for a.e. $c \in (0, \sqrt{2})$.

Proof of existence for every $c \in (0, \sqrt{2})$

$$(\mathrm{TW}_{c_n}) \qquad \boxed{ic_n u_n' + u_n'' + u_n\big(\mathcal{W} * (1 - |u_n|^2)\big) = 0 \quad \text{in } \mathbb{R}}$$

$c_n \to c \in (0, \sqrt{2})$ fixed

Proof of existence for every $c \in (0, \sqrt{2})$

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$c_n \to c \in (0, \sqrt{2})$ fixed

Question 1: $u_n \to u$ solution to (TW_c)?

$\boxed{\|u_n\|_{W^{k,\infty}(\mathbb{R})} \leq C}$ (holds for many relevant potentials $\mathcal{W}$)

Proof of existence for every $c \in (0, \sqrt{2})$

(TW$_{c_n}$) $$\boxed{ic_n u_n' + u_n'' + u_n\big(\mathcal{W} * (1 - |u_n|^2)\big) = 0 \quad \text{in } \mathbb{R}}$$

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Question 1: $u_n \to u$ solution to (TW$_c$)?

$\boxed{\|u_n\|_{W^{k,\infty}(\mathbb{R})} \leq C}$ (holds for many relevant potentials $\mathcal{W}$)

Question 2: Is u nontrivial?

$$\boxed{\|1 - |u_n|^2\|_{L^\infty(\mathbb{R})} \geq C_{\mathcal{W}}(2 - c_n^2).}$$

Proof of existence for every $c \in (0, \sqrt{2})$

$$(\mathrm{TW}_{c_n}) \qquad \boxed{ic_n u_n' + u_n'' + u_n\big(\mathcal{W} * (1 - |u_n|^2)\big) = 0 \quad \text{in } \mathbb{R}}$$

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$$\boxed{\|1 - |u_n|^2\|_{L^\infty(\mathbb{R})} \geq C_{\mathcal{W}}(2 - c_n^2).}$$

Question 3: $u \in \mathcal{E}(\mathbb{R})$?

Profile decomposition of Palais-Smale sequences, versions of Brezis-Lieb lemma, Pohozaev identity...

Proof of existence for every $c \in (0, \sqrt{2})$

Theorem (de Laire, L.-M. (2021))

Let $\mathcal{W}$ be an even signed Borel measure with $\widehat{\mathcal{W}} \in W^{2,\infty}(\mathbb{R})$, $\widehat{\mathcal{W}} \geq 0$, $\widehat{\mathcal{W}}(0) = 1$. Assume that there exist $m \in [0,1)$ and $k_c > 0$ such that $mk_c < 1$ and

$$\big(\widehat{\mathcal{W}}\big)'(\xi) \geq -m\xi \ \forall \xi \geq 0, \quad \|u\|^2_{L^\infty(\mathbb{R})} \leq k_c \ \forall u \in \mathcal{E}(\mathbb{R}) \text{ solution.}$$

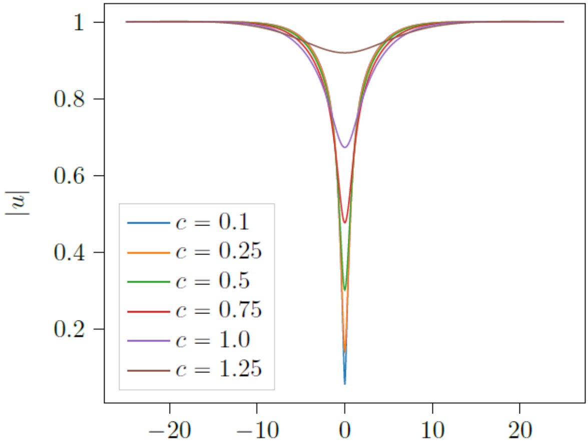
Then there exists a nontrivial solution $u \in \mathcal{E}(\mathbb{R})$ to (TW$_c$) for every $c \in (0, \sqrt{2})$.

Example

$$\mathcal{W} = \delta_0, \quad \mathcal{W} \approx e^{-x^2}, \quad \mathcal{W} \approx e^{-|x|}, \quad \mathcal{W} \approx \delta_0 - e^{-|x|}.$$

Numerical plots

$\mathcal{W} \approx \delta_0 - e^{-|x|}$



de Laire, Dujardin, L.-M. (2023)

Numerical plots

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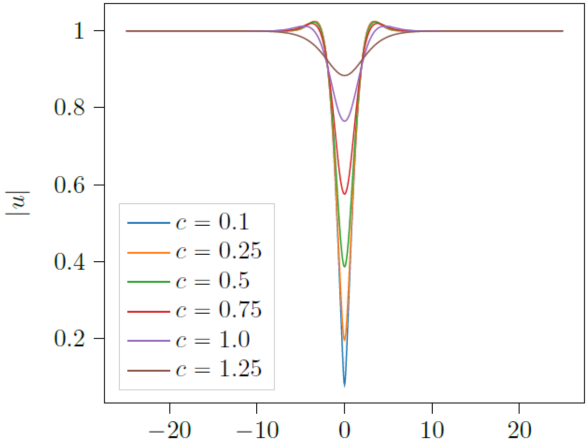

de Laire, Dujardin, L.-M. (2023)

Open problems

In the local case

1. Existence of dark solitons **for every** $c \in (0, \sqrt{2})$ in dimension $\boxed{N = 2}$.
2. Existence of **2-dimensional** dark solitons for every $c \in (0, \sqrt{2})$ in **a strip of** $\mathbb{R}^2$.
3. Existence of **dark-bright solitons** for systems such as

$$\begin{cases} i\partial_t \Psi = \Delta \Psi + \Psi(1 - |\Psi|^2 - \alpha|\Phi|^2), & |\Psi| \to 1 \text{ as } |x| \to \infty, \\ i\partial_t \Phi = \Delta \Phi + \Phi(\Lambda - \alpha|\Psi|^2 - \beta|\Phi|^2), & |\Phi| \to 0 \text{ as } |x| \to \infty. \end{cases}$$

In the nonlocal case

1. Essentially everything for $N \geq 2$.

In dimension $N = 1$

2. Nonexistence for $c > \sqrt{2}$.
3. Uniqueness (or multiplicity).
4. Stability of Mountain Pass solutions.
5. Existence of energy minimizers with fixed momentum for more general potentials.

Thanks for your attention!

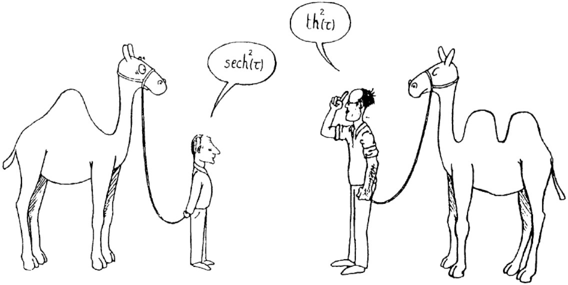

Fig. 1. Do these 'animals' belong to the same soliton family? (the drawing made by Marc Haelterman in 1989).