Skein Algebras of Surfaces – algebraic structure

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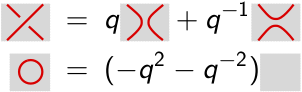
September, 2024

based on joint work with F. Costantino, J. Korinman, S. Sikora, T. Yu

Kauffman bracket polynomial

$D \subset \mathbb{R}^2$: unoriented link diagram $\longrightarrow \langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$



$$= q^6 + q^2 + q^{-2} + q^{-6}$$

• Invariant of framed unoriented link. Framed version of Jones polynomial.
• Extension (Reshetikhin-Turaev)

ribbon category	$\rightsquigarrow$	invariants of ribbon graph
simple Lie algebra $\mathfrak{g}$	$\rightsquigarrow$	ribbon category $\mathrm{Rep}(U_q(\mathfrak{g}))$
$\mathfrak{g} = sl_2$	$\rightsquigarrow$	(colored) Jones polynomial

Kauffman bracket skein module

Ground ring R: commutative domain, with fixed invertible q.
Example: $R = \mathbb{Z}[q, q^{-1}]$, $\mathbb{Q}(q)$, or $R = \mathbb{C} \ni q \neq 0$.
M oriented 3-manifold. Define (Przytycki, Turaev)

$$\mathscr{S}(M) = \frac{R\text{-span of framed unoriented links in } M}{\text{relations below}}$$

 $= q$  $+ q^{-1}$ ,  $= (-q^2 - q^{-2})$

Convention: $\emptyset$ is a framed link.
Orientation of M is used in the relations.
Kauffman's theorem: $\mathscr{S}(S^3) \cong R = \mathbb{Z}[q, q^{-1}], \ L \to \langle L \rangle$.
• Kauffman type invariants of links in M:

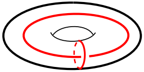
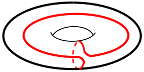
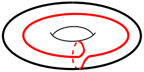
$$R\text{-linear} \quad f : \mathscr{S}(M) \to R = \mathbb{Z}[q, q^{-1}].$$

Surfaces

$\mathfrak{S}$: oriented surface. Thickened surface $\tilde{\mathfrak{S}} := \mathfrak{S} \times (-1, 1)$.

$$\mathscr{S}(\mathfrak{S}) := \mathscr{S}(\tilde{\mathfrak{S}}).$$

Links can be "drawn" on $\mathfrak{S}$ (diagrams).

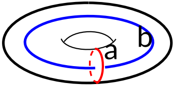
- Example:  $= q$  $+ q^{-1}$ 

- Przytycki: $\{$simple diagrams$\} = R$-basis
(simple=no crossings, no trivial knot).

Algebra structure

$\mathscr{S}(\mathfrak{S})$ is an **associative algebra with unit** (Turaev)

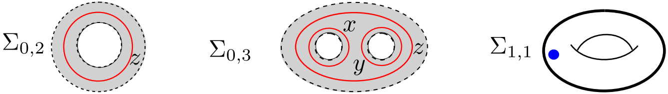
$$\alpha_1\alpha_2 = \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline\end{array}$$

example: $a.b =$ 

- unit $=$ empty link.
- $\mathscr{S}(\mathfrak{S})$ non-comutative in general.
- There are $\mathfrak{S} \neq \mathfrak{S}'$ with $\tilde{\mathfrak{S}} = \tilde{\mathfrak{S}}'$. But as algebra $\mathscr{S}(\mathfrak{S}) \neq \mathscr{S}(\mathfrak{S}')$.
- Related notion: skein categories (Walker, Johnson-Freyd)

Examples

$\mathscr{S}($$) \cong R, \qquad L \to \langle L \rangle$



$$\mathscr{S}(\Sigma_{0,2}) = R[z]$$
$$\mathscr{S}(\Sigma_{0,3}) = R[x,y,z]$$
$$\mathscr{S}(\Sigma_{1,1}) = R\langle x,y,z\rangle/Rel$$

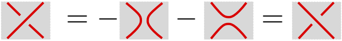
where *Rel* is

$$[x,y]_q = (q^2 - q^{-2})z, [y,z]_q = (q^2 - q^{-2})x, [z,x]_q = (q^2 - q^{-2})y$$

Here $[x,y]_q = qxy - q^{-1}yx$. (Bullock-Przytycki; $SO(3)$.)

Quantization of Character variety

If $R = \mathbb{C}, q = \pm 1$, then $\mathscr{S}_{\pm 1}(M)$ is a commutative algebra, where product is disjoint union:

$(q = -1)$ 

Turaev, Bullock, Przytycki-Sikora,
Bullock-Frohman-Kania-Bartoszynska:

• $\mathscr{S}_{-1}(M) \cong$ (canonically) the universal coordinate ring of the SL_2 character variety of M, an important classical object.
• For surfaces: quantization along the Atiyah-Bott-Goldman bracket.

• connects Jones polynomial and classical topology.
• Used in TQFT.
• Helps proving AJ conjecture for many knots.
• Closely related to (quantum) Teichmüller space, cluster algebras.

General ribbon category $\mathcal{C}$

$\mathcal{C}$ ribbon R-linear category, M oriented 3-manifold.

$$\mathscr{S}_{\mathcal{C}}(M) = \frac{R\langle\, \mathcal{C}\text{-ribbon graphs in } M \,\rangle}{\text{local RT operatior relations}}$$

Examples: $\mathfrak{g}$ semisimple Lie algebra.

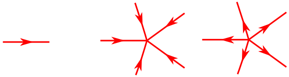
- $\mathcal{C}^{\mathrm{rat}} = \mathrm{Rep}(U_q(\mathfrak{g}))$, over $\mathbb{Q}(q^{1/D}) \rightsquigarrow \mathscr{S}_{\mathfrak{g}}^{\mathrm{rat}}(M)$
- $\mathcal{C}^{\mathbb{Z}} = \mathrm{Rep}^{\mathbb{Z}}(U_q^{\mathbb{Z}}(\mathfrak{g}))$, over $R = \mathbb{Z}[q^{\pm 1/D}] \rightsquigarrow \mathscr{S}_{\mathfrak{g}}(M)$
  $U_q^{\mathbb{Z}}(\mathfrak{g})$ Lusztig's integral version.
  $\mathrm{Rep}^{\mathbb{Z}}$ is generated by $V_\lambda^{\mathbb{Z}}, \lambda \in X_+$, tensor products.
  For $0 \neq \xi \in \mathbb{C}$ let $\quad \mathscr{S}_{\mathfrak{g},\xi}(M) = \mathscr{S}_{\mathfrak{g}}(M) \otimes_R \mathbb{C}$, $q \to \xi$.
- $R = \mathbb{C} \ni \xi$ root of 1, $\mathcal{C}^*_{\mathfrak{g},\xi} = \mathrm{Rep}^{1,fd}(U_\xi(\mathfrak{g})) \rightsquigarrow \mathscr{S}^*_{\mathfrak{g},\xi}(M)$
  $U_\xi(\mathfrak{g})$ Lusztig quantum group at ξ.

$$\mathcal{C}^{\mathbb{Z}}_{\mathfrak{g},\xi} \to \mathcal{C}^*_{\mathfrak{g},\xi} \rightsquigarrow F : \mathscr{S}_{\mathfrak{g},\xi}(M) \to \mathscr{S}^*_{\mathfrak{g},\xi}(M)$$

If F bijective?

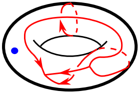
SL_n-skein algebra (Sikora 2003, twisted version)

• n-web-diagram on $\mathfrak{S}$: 1-dimensonal & oriented; locally either a smooth point, or an n-valent sink or source



Over/undercrossing like in link diagrams.

example of 3-web

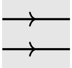


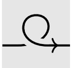
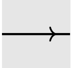
• n-web in $\tilde{\mathfrak{S}} = \mathfrak{S} \times (-1, 1)$, cyclic order at vertices, framing (normal vector field). These are special ribbon graphs, where strands are colored by the fundamental representation of $U_q(sl_n)$.
• considered up to isotopies in $\mathfrak{S} \times (-1, 1)$.

SL_n-skein algebra, definition

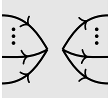
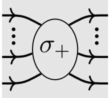
R any commutative domain, with $\hat{q}^{\pm 1}$. Usual $q = \hat{q}^{1/2n^2}$.

$$\mathscr{S}_{sl_n}(\Sigma) = \mathscr{S}_n(\Sigma) = R\text{-span}\,\{\text{isotopy classes of } n\text{-webs}\}/\text{Rel}$$

$q^{\frac{1}{n}}$  $-q^{-\frac{1}{n}}$  $= (q - q^{-1})$ ,

 $= (-1)^{n-1} q^{n-1/n}$ ,

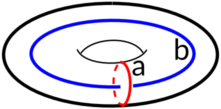
 $= (-1)^{n-1}[n]_q$, $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

 $= (-q)^{\binom{n}{2}} \cdot \sum_{\sigma \in S_n} (-q^{(1-n)/n})^{\ell(\sigma)}$ 

.

($\ell(\sigma)$ length, σ_+ positive braid)

$$\alpha_1\alpha_2 = \boxed{\begin{array}{c}\alpha_1\\ \hline \alpha_2\end{array}}$$

example: $a.b =$ 

SL_n-skein algebra

• Local n-webs: generating intertwiners of $U_q(sl_n)$ fundamental representation.
Defining Relations of skein algebra: (all) relations among elementary intertwiners.
• We don't use all ribbon graphs, all relations between ribbon graphs.
• $n = 2$, Kauffman bracket skein algebra.
• $n = 3$, Kuperberg skein algebra.
• other versions: MOY graphs, CKM graphs (restrictions on ground ring)
• Quantization of twisted character variety.

$$R = \mathbb{C}, \hat{q} = 1, \qquad \mathscr{S}_n(\Sigma) = \mathbb{C}[\text{ twisted } SL_n\text{-characters }]$$

Poisson structure: Atiyah-Bott-Goldman, Fock-Rosly.

Free? Domain?

• Understand $\mathscr{S}(\mathfrak{S})$ algebraically.

Questions.

(1) Is $\mathscr{S}(\mathfrak{S})$ free as an R-module? $R = \mathbb{Z}[\hat{q}^{\pm 1}]$.

(2) Is $\mathscr{S}(\mathfrak{S})$ a domain? $(xy = 0 \Rightarrow x = 0$ or $y = 0)$.

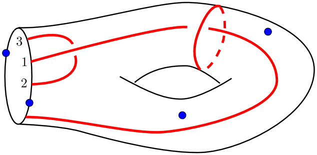
• sl_2 Yes to both. Przytycki-Sikora, Bonahon-Wong.
• Unknown answers: $R = \mathbb{Z}[\hat{q}^{\pm 1}]$,
$\mathfrak{S} = \Sigma_g, g \geq 1, sl_n, n \geq 3$. Even torus Σ_1.
$\mathfrak{S} = \Sigma_{g,p}, sl_n, n \geq 4$, except for $\Sigma_0, \Sigma_{0,1}, \Sigma_{0,2}$
$R = \mathbb{C} \ni \hat{q} \neq 0$ (domain question)
• (Costantino-Korinman-L.):
All sl_n, $\mathfrak{S} = \Sigma_{g,p}, p \geq 1$, and $R = \mathbb{Q}(\hat{q})$. Then $\mathscr{S}(\mathfrak{S})$ is a domain.
Proof use theory of stated skein algebra.
• difficulty: we don't know a (candidate for a) good basis for $\mathscr{S}_n(\mathfrak{S})$. We have basis $\mathfrak{g} = sl_2$, and $\mathfrak{g} = sl_3$ with $\Sigma_{g,p}, p \geq 1$.

Triangulation and coordinates

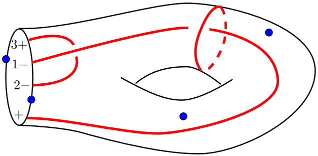
• Teichmüller space $\mathcal{T}(\mathfrak{S}) =$ set of all hyperbolic structures on $\mathfrak{S}$ up to isotopy.
$\mathcal{T}(\mathfrak{S})$ is a component of $\chi_{PSL_2(\mathbb{R})}(\mathfrak{S})$
• Given an ideal triangulation of $\mathfrak{S}$. Thurston and Penner coordinatize versions of $\mathcal{T}(\mathfrak{S})$.
• Can we do the same for $\mathscr{S}(\mathfrak{S})$?
$\rightsquigarrow$ skein algebra of *ideal triangle*.
Need to extend skein algebra to involve the boundary.

sl_2 case. Tangle diagrams

Goal: Extend $\mathscr{S}(\mathfrak{S})$ to involve boundary edges.
Links in $\widetilde{\mathfrak{S}} = \mathfrak{S} \times (-1)$ can "come" to boundary: Tangles in $\widetilde{\mathfrak{S}}$.



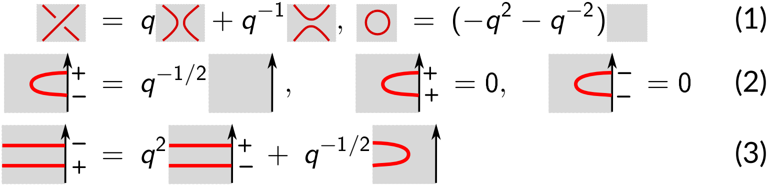
Tangle diagram α on $\mathfrak{S}$: closed curves and arcs with endpoints on $\partial\mathfrak{S}$.
In interior of $\mathfrak{S}$: like a link diagram
Each boundary edge b of $\mathfrak{S}$: a linear (height) order on $b \cap \partial\alpha$



State $s : \partial\alpha \to \{\pm\}$

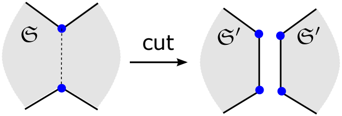
Stated skein algebra (L. 2016, $\mathfrak{g} = sl_2$)

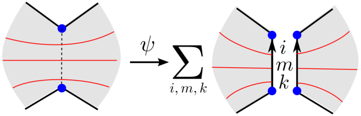
$$\mathscr{S}(\mathfrak{S}) := \frac{R\text{-span of stated tangles in } \widetilde{\mathfrak{S}}}{\text{(1), boundary rels (2) \& (3)}}$$



- RT operator invariant, (dual)canonical basis.
- product $\alpha\beta$: α is above β, higher on each boundary edge.

Cutting homomorphism





Theorem (L. 2016)

ψ is an algebra algebra homomorphism $\psi : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(\mathfrak{S}')$.
Injective (any ground ring).

The exact image is known. (Hochschild cohomology)

$$\text{triangulation } \lambda \rightsquigarrow \Psi : \mathscr{S}(\mathfrak{S}) \to \bigotimes_{\tau:\text{faces}} \mathscr{S}(\tau)$$

A presentation of $\mathscr{S}(\tau)$ is known.
$\rightsquigarrow$ many useful facts

sl_n, general ribbon category, half ribbon element

Stated skein algebra
$\mathfrak{g} = sl_3$: V. Higgins. Cutting homomorphism ψ is injective (using basis of $\mathscr{S}(\mathfrak{S})$).
sl_n: L.-Sikora. Cutting homomorphism ψ is injective if $\mathfrak{S}$ is connected and has boundary.

- Costantino-Korinman-L.: General Tanakian ribbon category. (half ribbon structure)

Arbitrary simple Lie algebra $\mathfrak{g}$. If ground ring is $\mathbb{Q}(\hat{q})$, then Cutting homomorphism ψ is injective, even for 3-manifolds.

- Costantino–L.: $\mathfrak{g} = sl_2, R = \mathbb{C}, \hat{q}$ root of 1. In general, the cutting homomorphism is not injective for 3-manifolds. (but injective for surfaces).
- over $\mathbb{Q}(\hat{q})$: related to factorization homology, lattice field theory of Alekseev-Grosse-Schomerus and Buffenoir-Roche, skein category (work of Ben-Zvi-Brochier-Jordan, Cooke, Haioun).

Quantum space, Quantum torus

- λ: an ideal triangulation of $\mathfrak{S}$. Each face $= \mathbb{P}_3$, triangle.

$$\mathscr{S}(\mathfrak{S}) \xrightarrow{\text{cut}} \bigotimes_{\tau:\text{faces}} \mathscr{S}(\tau)$$

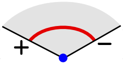
It turns out $\mathscr{S}(\mathbb{P}_3)$ is closely related to **quantum tori**.
Q: anti-symmetric $r \times r$ matrix, integer entries.

$$\mathbb{T}_+(Q) = R\langle x_i, i = 1, \ldots, r\rangle / (x_i x_j = q^{Q_{ij}} x_j x_i)$$
$$\mathbb{T}(Q) = R\langle x_i^{\pm 1}, i = 1, \ldots, r\rangle / (x_i x_j = q^{Q_{ij}} x_j x_i).$$

(Laurent) polynomials in q-commuting variables.
Simplest type of non-commutative algebra:
Noetherian domain. Gelfand-Kirilov dimension $= r$.
$q =$ root of 1: Azumaya algebra. Representation theory known.

Triangle $\mathbb{P}_3$, sl_2

- Costantino-L.: Reduced stated skein algebra

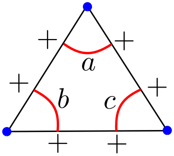
$$\overline{\mathscr{S}}(\mathfrak{S}) = \mathscr{S}(\mathfrak{S})/($$ 

$$)$$

(Relations comes from Bonahon-Wong work on quantum trace)

- $\overline{\mathscr{S}}(\mathbb{P}_3)$ is a quantum torus:

$$\overline{\mathscr{S}}(\mathbb{P}_3) = \frac{R\langle a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\rangle}{(ba = q^2 ab, cb = q^2 bc, ac = q^2 ca)}$$



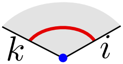
- λ: an ideal triangulation of $\mathfrak{S}$

$$\mathscr{S}(\mathfrak{S}) \to \overline{\mathscr{S}}(\mathfrak{S}) \xrightarrow{\text{cut}} \bigotimes_{\tau:\text{faces}} \overline{\mathscr{S}}(\tau) = \bigotimes_{\tau:\text{faces}} \mathbb{T}(\mathbb{P}_3)$$

$\rightsquigarrow$ quantum trace maps (L.-Yu).

Triangle $\mathbb{P}_3$, sl_n, $n \geq 3$

- L.-Yu Reduced stated skein algebra

$$\overline{\mathscr{S}}(\mathfrak{S}) = \mathscr{S}(\mathfrak{S})/($$ 

$$, i < k)$$

- $\overline{\mathscr{S}}(\mathbb{P}_3)$ is almost a quantum torus: $\exists$ antisymmetric matrix P,

$$\mathbb{T}_+(P) \subset \overline{\mathscr{S}}(\mathbb{P}_3) \subset \mathbb{T}(P)$$

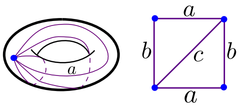
- λ: an ideal triangulation of $\mathfrak{S}$

$$\mathscr{S}(\mathfrak{S}) \to \overline{\mathscr{S}}(\mathfrak{S}) \xrightarrow{\text{cut}} \bigotimes_{\tau:\text{faces}} \overline{\mathscr{S}}(\tau) \to \bigotimes_{\tau:\text{faces}} \mathbb{T}(P)$$

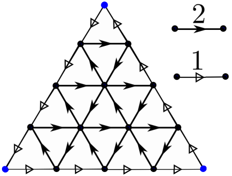
$\rightsquigarrow$ quantum trace maps (L.-Yu).

Fock Goncharov algebra (quantum higher Teichmüller theory, 2009)

λ: ideal triangulation of $\Sigma = \Sigma_g \setminus \mathcal{P}$,



- Subdivide each triangle into n^2 small triangles. Arrows on edges.



($n = 4$)
V = set of vertices (except $\mathcal{P}$).
$Q : V \times V \to \mathbb{Z}$ is the quiver matrix:

$$Q(x, y) = \#(x \to y) - \#(y \to x)$$

$$\overline{\mathcal{X}}_n(\Sigma, \lambda) := R\langle x^{\pm 1}, x \in V\rangle / (xy = \hat{q}^{2Q(x,y)} yx),$$ quantum torus

- Quantization of X-variety (if q replaces q^{1/n^2}.)

Sl_n quantum trace

Ideal triangulation λ of $\mathfrak{S}$

$$\mathscr{S}(\mathfrak{S}) \to \overline{\mathscr{S}}(\mathfrak{S}) \xrightarrow{\text{cut}} \bigotimes_{\tau:\text{faces}} \overline{\mathscr{S}}(\tau) \to \bigotimes_{\tau:\text{faces}} \mathbb{T}(P) \supset \overline{\mathcal{X}}_n(\Sigma, \lambda)$$

Theorem (L. & Yu)

$\exists$ algebra map, natural with respect to triangulation change

$$\text{tr}_\lambda^X : \overline{\mathscr{S}}(\mathfrak{S}) \to \overline{\mathcal{X}}(\mathfrak{S}; \lambda).$$

(i) $q = 1$ recovers classical map of Fock-Goncharov.
(ii) $n = 2$ Bonahon-Wong map.
(iii) $n \leq 3$ injective.

$n = 3$ independent work of H. Kim, partial result of D. Douglas.

- A-version for triangulable surfaces having no interior punctures.

THANK YOU!