2-Representation Theory of $\mathfrak{gl}(1|1)$

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September 4, 2024

What algebra comes out of 4d topology?

<u>3d</u> Topology: braided monoidal cat., associators, KZ,
modular categories, fusion algebras...

Rep quantum $\mathfrak{g} \xrightarrow{\text{Forget}}$ Vect is monoidal, not braided

R-mat: $V_q \otimes W_q \xrightarrow{\sim} W_q \otimes V_q$

braid group $B_n \curvearrowright V_q^{\otimes n}$ $\dashrightarrow$ Knot invariants (Reshetikhin-Turaev)

<u>4d</u> 2-Rep $\mathfrak{g} \longrightarrow$ Cat not monoidal

$\mathcal{V} \circledcirc \mathcal{W} \;\cancel{\longmapsto}\; \mathcal{V} \otimes \mathcal{W}$

$\rightsquigarrow$ Complicated algebras from simple ones

Crane-Frenkel: $\exists$? monoidal cat $\mathcal{U}(\mathfrak{g})$, $K_0(\mathcal{U}(\mathfrak{g})) = U(\mathfrak{g})$,

$\mathcal{U}(\mathfrak{g})$-Mod gives invariants of 4d manifolds

"2-rep. of $\mathfrak{g}$"

Chuang-R, Khovanov-Lauda: construction of $\mathcal{U}(\mathfrak{g})$, $\mathfrak{g}$ Kac-Moody

R.: construction of $\otimes$, braiding.

4d Topological field theories

4d		3,4 d		2,3,4 d
Donaldson	~	Seiberg-Witten	~	Heegaard-Floer
Yang-Mills instantons		Kronheimer-Mrowka		Ozsváth-Szabó
ASD equations		monopole Floer		Lipshitz-Ozsváth-Thurston

HF: $\Sigma \mapsto \mathrm{Fuk}(S^g\Sigma)$

Douglas-Lipshitz-Manolescu, Manion-R: extension to dim 1:

$\bigcirc \mapsto$ 2-rep $\mathfrak{gl}(1|1)^+$

$gl(1|1)$

$gl(1|1) = \mathbb{C}\{e_+, e_-, h_1, h_2\}$ super Lie alg., $e_\pm$ odd

$U(gl(1|1)) = \mathbb{C}\langle e_+, e_-, h_1, h_2\rangle / e_+e_- + e_-e_+ = h_1 + h_2,\ e_\pm^2 = 0$ super-algebra

$h_i e_\pm - e_\pm h_i = \pm\varepsilon_i \cdot e_\pm,\ h_1h_2 = h_2h_1$ $\quad \varepsilon_i = \begin{cases} -1 & i=1 \\ 1 & i=2 \end{cases}$

$U_q(gl(1|1)) = \mathbb{C}(q)\langle e_+, e_-, k_1^{\pm 1}, k_2^{\pm 1}\rangle / e_+e_- + e_-e_+ = \dfrac{k_1k_2 - k_1^{-1}k_2^{-1}}{q - q^{-1}},\ e_\pm^2 = 0,$

$k_i e_\pm = q^{\pm\varepsilon_i} e_\pm k_i,\ k_1k_2 = k_2k_1$

$gl(1|1)^+ := \mathbb{C}e_+,\ U(gl(1|1)^+) = \mathbb{C}[e_+]/e_+^2,\ U_q(gl(1|1)^+) = \mathbb{C}(q)[e_+]/e_+^2$

Put $f = e_+,\ e = (q-q^{-1})k_1^{-1}k_2^{-1}e_-$. $\boxed{ef + fe = 1 - (k_1k_2)^{-2}}$ (Tian, contact geometry)

Irr. rep $U_q(gl(1|1))$: 1 or 2 dimensional.

$$\boxed{\mathcal{U}^+ = \mathcal{U}(\mathfrak{gl}(1|1)^+) := \text{monoidal cat with objects } \mathbb{1}, F,\ F^2 = F\otimes F = 0,\ \mathrm{End}(\mathbb{1}) = \mathrm{End}(F) = \Bbbk\cdot \mathrm{id}}$$

2-rep of $\mathfrak{gl}(1|1)^+$ on $\mathcal{V}$:= action of $\mathcal{U}^+$ on $\mathcal{V}$ = data of $F: \mathcal{V}\to\mathcal{V}$, $F^2=0$

Examples: $\cdot$ $\mathcal{V} = \Bbbk\text{-mod}$, $F=0$ (trivial rep.)

$\cdot$ $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, $F: \mathcal{V}_1 \to \mathcal{V}_2$.

$\cdot$ Vector 2-rep: take $\mathcal{V}_1 = \Bbbk[\varepsilon]/\varepsilon^2\text{-mod}$, $\mathcal{V}_2 = \Bbbk\text{-mod}$, $F = \mathrm{Res}$

$\mathcal{V}$ triangulated cat. Not rigid enough. Replace by A_∞-cat: $F \circlearrowright \mathcal{V}$, F^2 homotopic to 0.

$U_q(\mathfrak{gl}(1|1)^{\geq 0}) = \mathbb{Q}(q)\langle f, k_1^{\pm1}, k_2^{\pm1}\rangle \rightsquigarrow \mathcal{U}(\mathfrak{gl}(1|1)^{\geq 0})$: add $K_i^{\pm1}$ to $\mathcal{U}^+$

integral weight rep. $\quad V = \bigoplus_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}\\ \bar{\lambda}\in\mathbb{Z}/2 \text{ parity}}} V_{\lambda_1,\lambda_2,\bar{\lambda}}$ $\qquad$ $\mathcal{V} = \bigoplus_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}\\ d\in\mathbb{Z}}} \mathcal{V}_{\lambda_1,\lambda_2,d}$ graded

$$V_{\lambda_1,\lambda_2,\bar{\lambda}} \xrightarrow{f} V_{\lambda_1-1,\lambda_2+1,\bar{\lambda}+1} \qquad \mathcal{V}_{\lambda_1,\lambda_2,d} \xrightarrow{F} \mathcal{V}_{\lambda_1-1,\lambda_2+1,d+1}$$

$\mathrm{Rep}(\mathcal{U}^+)$ is a 2-category:

$\mathcal{V}_1, \mathcal{V}_2$ 2-rep $\quad \mathcal{H}om_{\mathcal{U}^+}(\mathcal{V}_1,\mathcal{V}_2)$: cat with obj $(\Phi : \mathcal{V}_1 \to \mathcal{V}_2, \alpha : \Phi F_1 \xrightarrow{\sim} F_2 \Phi)$

Internal Hom: $\quad \mathcal{H}om(\mathcal{V}_1,\mathcal{V}_2)$: cat with obj $(\Phi : \mathcal{V}_1 \to \mathcal{V}_2, \alpha : \Phi F_1 \to F_2 \Phi)$

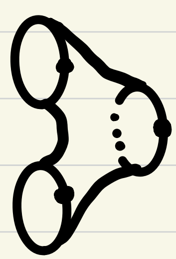
action of $\mathcal{U}^+$: $\quad F(\Phi,\alpha) := (\mathrm{cone}(\alpha), \mathrm{cone}(\alpha) F_1 \xrightarrow{\sim} F_2 \Phi F_1 \xrightarrow{\sim} F_2\, \mathrm{cone}(\alpha))$

(on $\mathrm{Hom}_{\mathbb{C}}(V_1,V_2)$, f acts by $f_2 - f_1$)

"Duals": $\mathcal{V}^\vee = \mathcal{V}^{opp} \subset \mathrm{Functors}(\mathcal{V}, \mathrm{Sets}) \rightsquigarrow \mathcal{V}_1 \otimes \mathcal{V}_2 = \mathcal{H}om(\mathcal{V}_1^\vee, \mathcal{V}_2)$ ($\mathcal{V}_1$ not too big)

Existence of left and right duals: require E, F biadjoint

Thm (Manion-R.) | Σ surface, $S^1 \hookrightarrow \partial\Sigma \rightsquigarrow$ 2-rep on $HF(\Sigma)$

 $\longleftrightarrow \otimes$

$\mathcal{V}_i = A_i$-mod, $F_i : (A_i, A_i)$-bimod. Assume $F_1 \otimes_{A_1} -$ has a left adjoint $E_1 \otimes_{A_1} -$

$\mathcal{V}_1 \otimes \mathcal{V}_2 = A$-mod, $A := A_1 \otimes A_2 \oplus E_1 \otimes F_2$

$\eta_1 \otimes \mathrm{id}$

left action of $E_1 \otimes F_2$ $\quad F := A_1 \otimes F_2 \oplus F_1 E_1 \otimes F_2 \oplus F_1 \otimes A_2 \quad$ right action of $E_1 \otimes F_2$

ε_1

Double construction of $\mathcal{U}(\mathfrak{gl}(1|1))$

Braiding

$$\begin{array}{ccccc} & \mathcal{V}_1 \otimes \mathcal{V}_2 & \xrightarrow{\sim} & \mathcal{V}_2 \otimes \mathcal{V}_1 & \\ & \| & c_{\mathcal{V}_1, \mathcal{V}_2} & \overset{\text{swap}}{\searrow\sim} & \\ A\text{-mod} & & \| & & A'\text{-mod} \\ & & (A_1 \otimes A_2) \otimes_A - & & \end{array}$$

$A' := A_1 \otimes A_2 \oplus F_1 \otimes E_2$

Map $\gamma_{\mathcal{V}_1, \mathcal{V}_2} : c_{\mathcal{V}_1, \mathcal{V}_2} \circ F \to F \circ c_{\mathcal{V}_1, \mathcal{V}_2}$

Def | $\mathcal{V}_2$ is a 2-rep of $\mathfrak{gl}(1|1)$ if $\gamma_{\mathcal{V}_1, \mathcal{V}_2}$ is an isomorphism for all $\mathcal{V}_1$

Same as a rep of a graded monoidal dg cat

$\mathcal{U}(\mathfrak{gl}(1|1)) = \langle E, F, K_i^{\pm 1}, \Sigma$, adjunctions (E,F) and $(F, \Sigma^2 K_1^2 K_2^2 E[1]) \rangle$

modulo relations

$E^2 = F^2 = 0$, K_1, K_2, Σ commute, $\Sigma F = F\Sigma[1]$, $K_1 F = q^{-1} F K_1$, $K_2 F = qFK_2$,

dist. tr. $\Sigma^{-2} K_1^{-2} K_2^{-2} \longrightarrow EF[1] \longrightarrow [1] \oplus FE \rightsquigarrow$ (Recall $ef + fe = 1 - (K_1K_2)^{-2}$)

$\mathcal{L} = \Bbbk \oplus \Bbbk[\varepsilon]/\varepsilon^2$ defining rep. Use tensor products of $\mathcal{L}, \mathcal{L}^\vee$

$\longrightarrow$ invariants of tangles and knots

Conj | This is Heegaard-Floer knot homology (Ozsváth-Szabó, Rasmussen)

Partial structure of braided monoidal $(\infty, 2)$-category

Rigidify: notion of 2-rep on an abelian cat.

F non exact functor, E only on derived cat

have $\otimes$ for abelian 2-rep

Next steps: . 3-manifolds invariants (recovering Heegaard-Floer homology)

· deformed versions ($HF^{\pm}$, necessary for 4d topology)