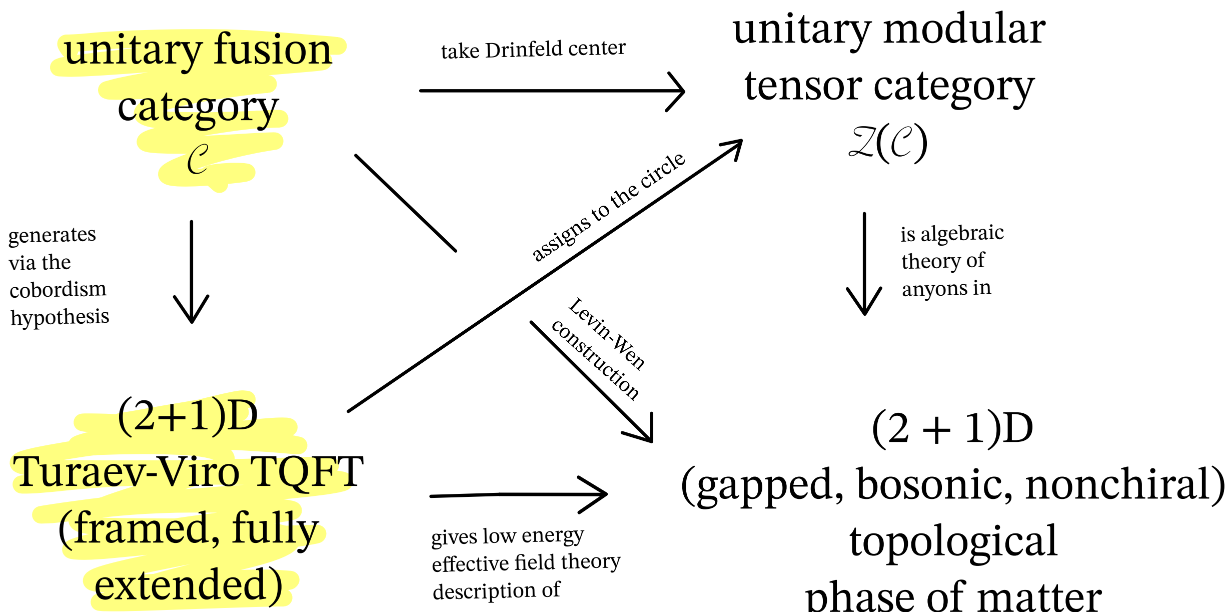
An efficient* classical algorithm for some quantum 3-manifold invariants

Based on arXiv:2311.08514 and work in progress
[with Clement Maria and Eric Samperton]

TQFT Club Talk
August 28, 2024

Motivating questions

Complexity-theoretic classification of 3D (semisimple) TQFTs/TOs



Motivating questions

Complexity-theoretic classification of 3D (semisimple) TQFTs/TOs

↳ Can look at braid/mapping class group images, framed link and **3-manifold invariants**

How quantum vs. how classical is a given TQFT/TO?

What is the contribution to the complexity from algebra vs. topology?

Background: review of fusion categories and TVBW state-sum invariants of 3-manifolds

Computing state-sum invariants of 3-manifolds

[Turaev-Viro, Barrett-Westbury]

simplicial triangulation of 3-manifold M

unitary skeletal fusion category $\mathcal{C}$

state-sum formula

topological invariant $|M|_{\mathcal{C}} \in \mathbb{C}$

Data type of a spherical (multiplicity-free) fusion category

$$\{N_c^{ab}, [F_d^{abc}]_{e,f}, t_a : a,b,c,d,e,f \in L\}$$

Fusion rules

$$a \otimes b = \bigoplus_{c \in L} N_c^{ab}\, c$$

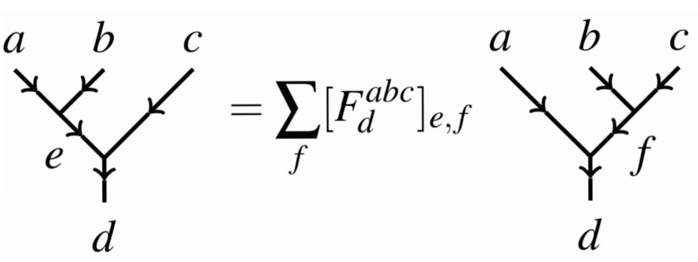
Label set

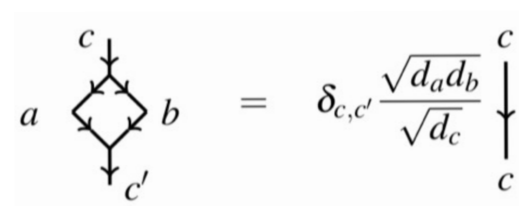
$$L = \{1, a, a^*, b, b^*, \ldots\}$$

Pivotal coefficents

$$d_a = a\bigcirc = t_a^{-1}[F_a^{aa^*a}]_{11}^{-1} \;;\; D = \sum_{a \in L} d_a^2$$

F-symbols





Data type of a spherical (multiplicity-free) fusion category

Defining equations of a fusion ring

$$\sum_x N_x^{ab} N_d^{xc} = \sum_x N_x^{bc} N_d^{ax}$$
$$N_b^{1a} = N_b^{a1} = \delta_{ab}$$
$$N_1^{a^*b} = N_1^{ba^*} = \delta_{ab}$$
$$N_c^{ab} = N_b^{a^*c} = N_a^{cb^*}$$

Pivotal equations

...

Pentagon equations

$$[F_d^{1bc}]_{b,d} = 1 \qquad [F_d^{a1c}]_{a,c} = 1 \qquad [F_d^{ab1}]_{d,b} = 1$$

$$\sum_h [F_g^{abc}]_{f,h} [F_e^{ahd}]_{g,k} [F_k^{bcd}]_{h,l} = [F_e^{fcd}]_{g,l} [F_e^{abl}]_{f,k}$$

Example: G-graded complex vector spaces

Label set $L = \{g \mid g \in G\}$

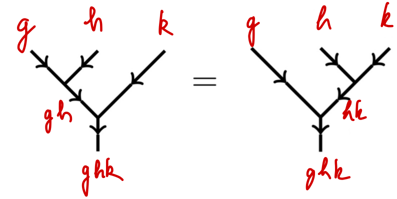
Fusion rules $g \otimes h = gh \quad \text{for } g,h \in G$

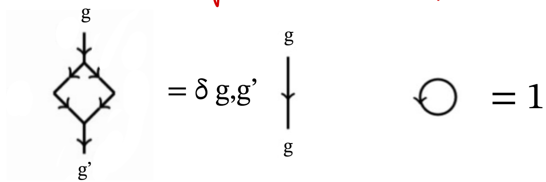
F-symbols $[F^{ghk}_{l}]_{e,f} = \delta_{l,\,ghk}\,\delta_{e,\,gh}\,\delta_{f,hk}$

Pivotal coefficents $t_g = 1 \qquad \text{for } g \in G$

Quantum dimensions $d_g = 1 \qquad \text{for } g \in G$

$D = \sqrt{|G|}$

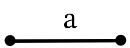




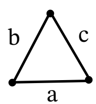
State-sum invariants of triangulated 3-manifolds

Start with a *state* θ i.e. a labeling of the edges of the triangulation with simple objects in $\mathcal{C}$ (elements of L)

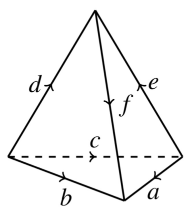
$\mapsto$ $$D = \sum_{a\in L} d_a^2$$



$\mapsto$ d_a

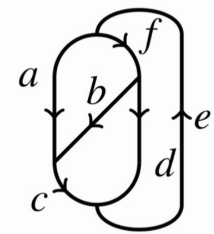


$\mapsto$ $\sqrt{d_a d_b d_c}$

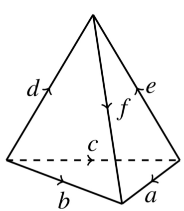


$(\Delta, +)$

$\mapsto$ $$|\Delta|^+_\theta = \begin{vmatrix} \theta_a & \theta_b & \theta_c \\ \theta_d & \theta_e & \theta_f \end{vmatrix}^+ :=$$

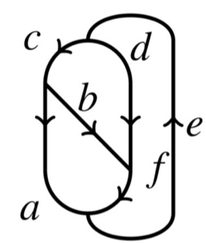


$$= [F^{abd}_e]_{c,f}\sqrt{d_a d_b d_d d_e}$$



$(\Delta, -)$

$\mapsto$ $$|\Delta|^-_\theta = \begin{vmatrix} \theta_a & \theta_b & \theta_c \\ \theta_d & \theta_e & \theta_f \end{vmatrix}^- :=$$



$$= [F^{abd}_e]^{-1}_{f,c}\sqrt{d_a d_b d_d d_e}$$

State-sum invariants of 3-manifolds

State sum formula:

$$|M|_{\mathscr{C},\mathfrak{T},\{I_t\}_{t\in T}} = \sum_{\theta\in \mathrm{Adm}_L(\mathfrak{T})} \frac{\prod_{t\in T} |t|_{\theta}^{O(t)} \prod_{e\in E} d_{\theta e}}{\prod_{f\in F} \sqrt{d_{\theta e_1} d_{\theta e_2} d_{\theta e_3}} \prod_{v\in V} D}$$

UFC

TRIANGULATION

Theorem: [Barrett-Westbury, Turaev-Viro]
The state sum is independent of the choice of triangulation, defines an invariant of closed, oriented 3-manifolds, and defines a fully-extended (2+1)D TQFT.

Computing the state-sum invariant of a 3-manifold

$$|M|_{\mathscr{C}} = \sum_{\theta \in \mathrm{Adm}_L(\mathfrak{T})} \frac{\prod_{t \in T} |t|_\theta^{O(t)} \prod_{e \in E} d_{\theta e}}{\prod_{f \in F} \sqrt{d_{\theta e_1} d_{\theta e_2} d_{\theta e_3}} \prod_{v \in V} D}$$

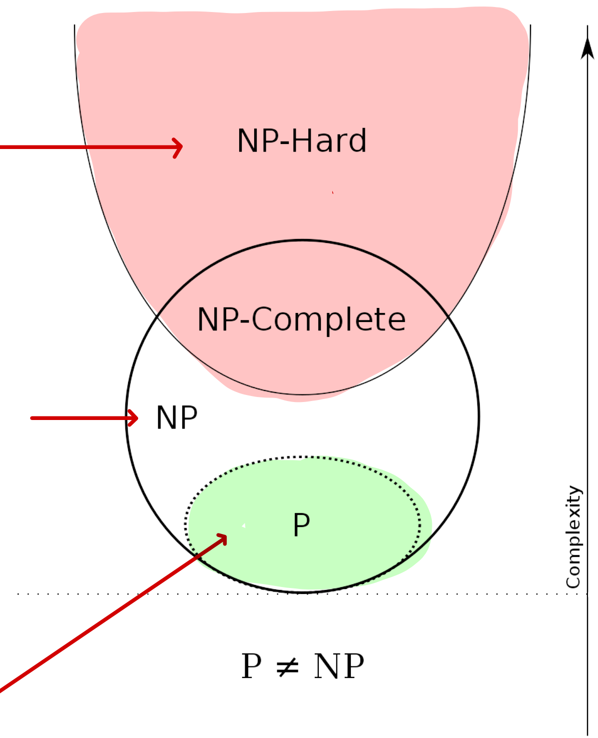
Note: The naive algorithm to compute the state sum invariant of a 3-manifold using this formula is exponential in the number of tetrahedra.

Complexity classes appearing in this talk

at least as hard as problems in NP

given a solution, can verify in polynomial time

can find a solution in polynomial time



Computing state-sum invariants for the simplest nontrivial fusion categories

$\mathcal{C} = \mathrm{Vec}_A$

$$|M|_{\mathscr{C}} = \sum_{\theta \in \mathrm{Adm}_L(\mathfrak{T})} \frac{\prod_{t \in T} |t|_\theta^{O(t)} \prod_{e \in E} d_{\theta e}}{\prod_{f \in F} \sqrt{d_{\theta e_1} d_{\theta e_2} d_{\theta e_3}} \prod_{v \in V} D}.$$

$$\sim |H^1(M,A)|$$

which can be computed in polynomial time

How hard is it to compute the state-sum for other fusion categories?

Vec_G

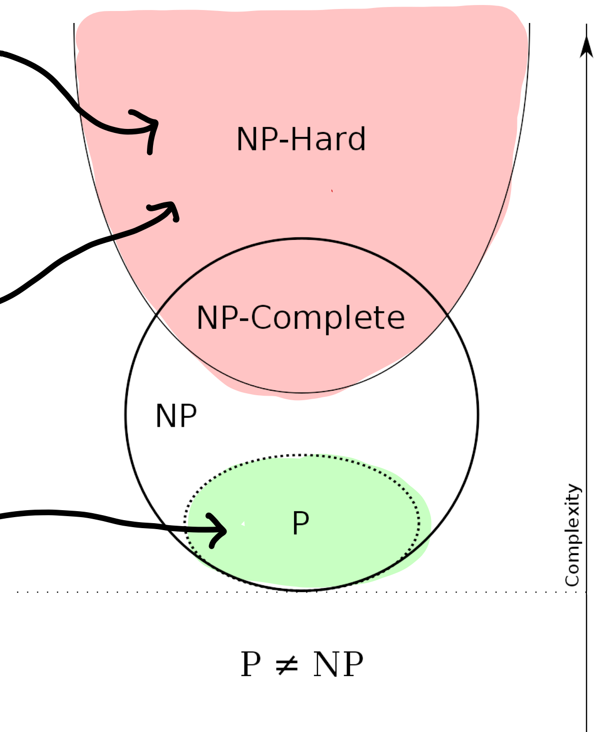
G nonabelian, simple

[Kuperberg-Samperton '17]

$TY(\mathbb{Z}_2, \chi, \nu) \simeq$ $SU(2)_2$

[Kirby-Melvin '04]

Vec_A



Tambara-Yamagami fusion categories

[Tambara-Yamagami]

$$\mathrm{TY}(A, \chi, \nu)$$

finite abelian group A

bicharacter $\chi: A \times A \to U(1)$

a square root $|A|^{-1/2}$

Example: when $A = \mathbb{Z}/2\mathbb{Z}$, $\chi(a,b) = \exp(\pi i a b)$, $\nu = \pm 1/\sqrt{2}$

$L = \{1, \sigma, \psi\}$

$$\begin{cases} \sigma \otimes \sigma = \mathbf{1} \oplus \psi \\ \sigma \otimes \psi = \psi \otimes \sigma = \sigma \\ \psi \otimes \psi = \mathbf{1} \end{cases}$$

Explicit description of Tambara-Yamagami fusion categories TY(A, χ, ν)

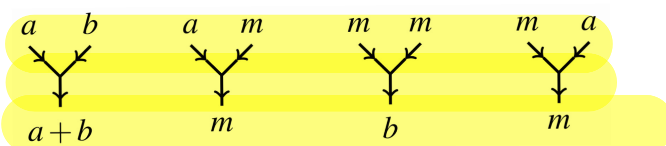
$$\{N_c^{ab}, [F_d^{abc}]_{e,f}, t_a : a,b,c,d,e,f \in L\}$$

Fusion rules

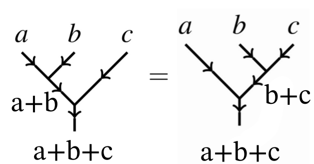
$a \otimes b = b \otimes a = a + b \quad a, b \in A$

$a \otimes m = m \otimes a = m \qquad a \in A$

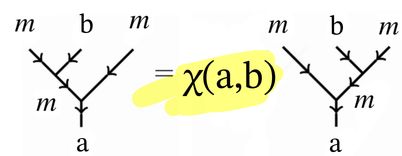
$m \otimes m = \bigoplus_{a \in A} a$



F-symbols



e.g.



Label set

$L = \{ a : a \in A \} \sqcup \{m\}$

Pivotal coeffcients

$t_a = 1$ for all $a \in A$, and

$t_m = \nu/|\nu| = \mathrm{sign}(\nu)$.

$a \circlearrowleft \ = 1 \qquad m \circlearrowleft \ = \sqrt{|A|}$

NP hardness of Tambara-Yamagami state-sum invariants

Vec_G

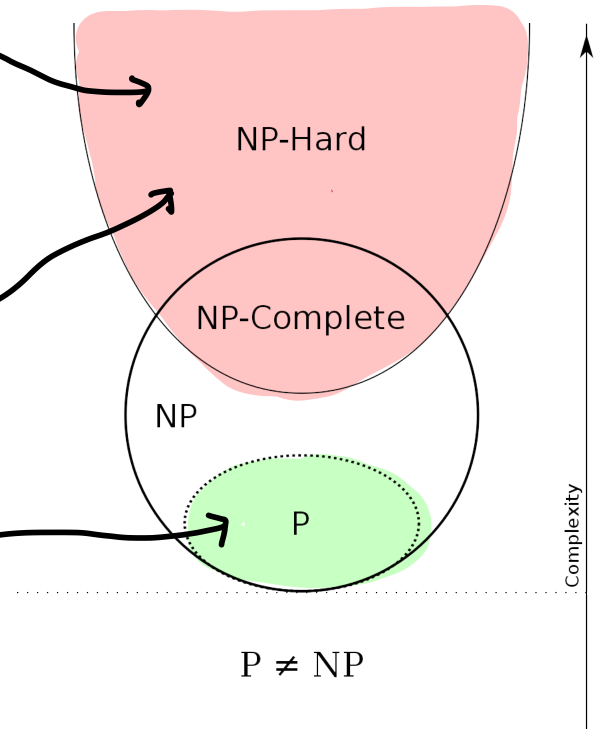
G nonabelian,
simple

[Kuperberg-Samperton ‘17]

$\mathrm{TY}(A,\chi,\nu)$

[work in progress with
Maria, Samperton]

Vec_A



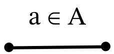
A fixed parameter tractable algorithm for Tambara-Yamagami quantum invariants of 3-manifolds

State sum data for Tambara-Yamagami categories

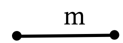
Vertex weights

• $\mapsto$ D=2|A|

Edge weights



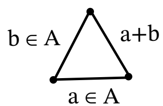
$\mapsto$ 1



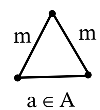
$\mapsto$ $\sqrt{|A|}$

$$\sum_{\theta\in \mathrm{Adm}_L(\mathfrak{T})} \frac{\prod_{t\in T}|t|_\theta^{O(t)}\prod_{e\in E} d_{\theta e}}{\prod_{f\in F}\sqrt{d_{\theta e_1}d_{\theta e_2}d_{\theta e_3}}\prod_{v\in V} D}.$$

Face weights

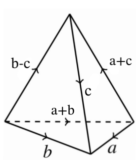


$\mapsto$ 1



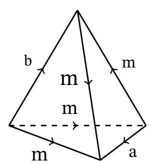
$\mapsto$ $\sqrt{|A|}$

Tetrahedral weights



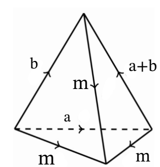
$(\Delta,+)$

$\mapsto$ 1



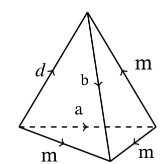
$(\Delta,+)$

$\mapsto$ $\sqrt{|A|}\,\chi(a,b)$



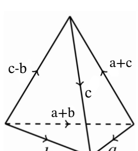
$(\Delta,+)$

$\mapsto$ $\sqrt{|A|}$



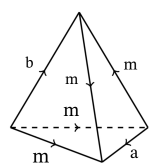
$(\Delta,+)$

$\mapsto$ $\nu|A|\,\overline{\chi(a,b)}$



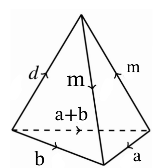
$(\Delta,-)$

$\mapsto$ 1



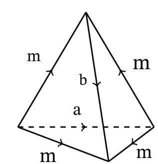
$(\Delta,-)$

$\mapsto$ $\sqrt{|A|}\,\chi(a,b)$



$(\Delta,-)$

$\mapsto$ $\sqrt{|A|}$



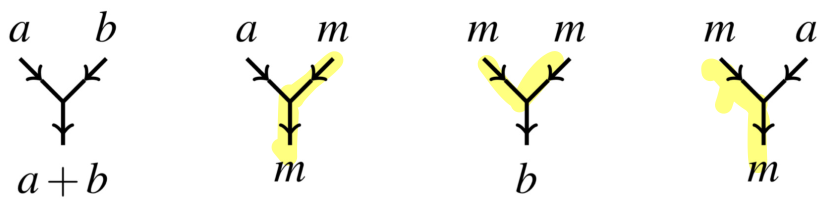
$(\Delta,-)$

$\mapsto$ $(1/\nu)|A|\,\chi(a,b)$

Key idea of FPT algorithm for Tambara-Yamagami state-sum invariants

Generalizes FPT algorithm known in the A= $\mathbb{Z}/2\mathbb{Z}$ case **[Maria-Spreer]**

↳ Recall the admissible labelings for general A:



↳ Observe that every admissible labeling determines a 1-cocycle on the triangulation via

$$\phi: \ \mathrm{Adm}_A(\mathfrak{T}) \to C^1(\mathfrak{T}, \mathbb{Z}/2\mathbb{Z})$$

$$\phi(\theta)(e) = \begin{cases} 1 & \text{if } \theta(e) = m, \\ 0 & \text{otherwise.} \end{cases}$$

FPT algorithm for Tambara-Yamagami state-sum invariants

Step 1: partition the state sum into terms by cocycle

$$|M|_{\mathrm{TY}(A,\chi,\nu)} = \sum_{\alpha \in Z^1(\mathfrak{T}_*, \mathbb{Z}/2\mathbb{Z})} |\mathfrak{T}, \alpha|_{\mathrm{TY}(A,\chi,\nu)}$$

$$|\mathfrak{T}, \alpha|_{\mathrm{TY}(A,\chi,\nu)} := \sum_{\theta \in \phi^{-1}(\alpha)} \frac{\prod_{t \in T} |t|_\theta^{O(t)} \prod_{e \in E} d_{\theta e}}{\prod_{f \in F} \sqrt{d_{\theta e_1} d_{\theta e_2} d_{\theta e_3}} \prod_{v \in V} D}$$

Step 2: collect terms representing same cohomology class

Step 3: identify the partial state sum at a cohomology class with a Gauss sum

$$\Theta(A', q) = \frac{1}{\sqrt{|A'|}} \sum_{a \in A'} \exp(2\pi i q(a))$$

(for some other abelian group A′, q a quadratic form on this A′)

which can be computed in polynomial time

FPT algorithm for Tambara-Yamagami state-sum invariants

Let β_1 be the first Betti number of M with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Theorem: Given a fixed Tambara-Yamagami fusion category and a triangulation of a 3-manifold M with n tetrahedra, there is an algorithm to compute $|M|_{TY(A,\chi,\nu)}$ in $O(2^{\beta_1} n^3)$ operations

↳ These quantum invariants are efficient to compute classically for 3-manifolds with bounded first Betti number

FPT algorithm for Tambara-Yamagami state-sum invariants

Interpretation of main result:

↳ The failure of these quantum invariants to be computed in polynomial time is for reasons intrinsic to 3-manifold topology, rather than for "quantum" reasons.

Practical considerations:

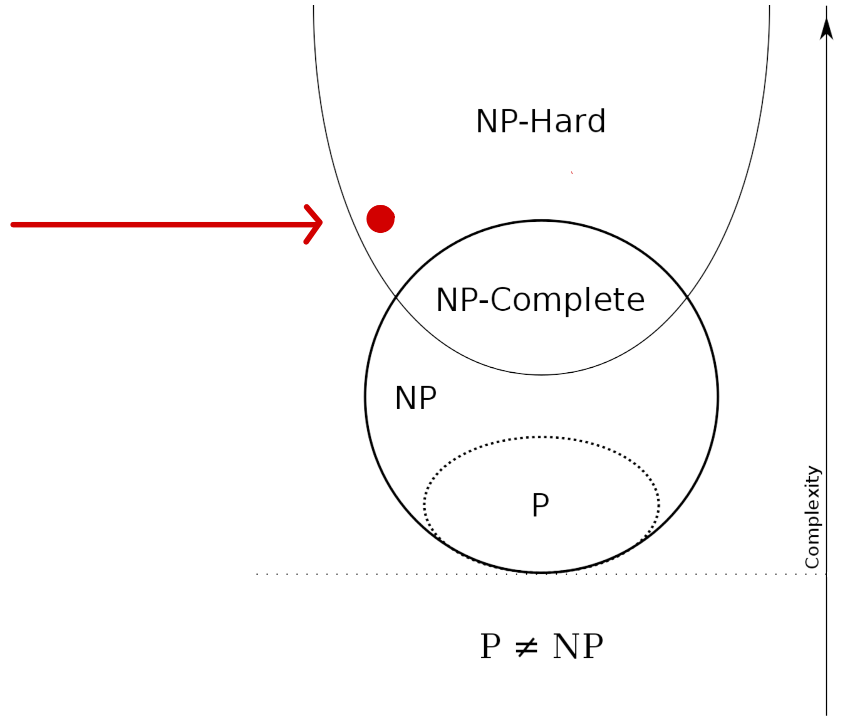
↳ β_1 can be computed in polynomial time in the size of the triangulation, so for a fixed 3-manifold you "know" the algorithm runtime.

↳ Good enough for Regina implementation on a 3-manifold database.

NP hardness of Tambara-Yamagami state-sum invariants

Recall that we expect to be able to prove these invariants are NP-hard, generalizing work of Kirby and Melvin in the A=$\mathbb{Z}/2\mathbb{Z}$ case

If you can put the problem here, it strongly suggests that our FPT algorithm is the nicest thing you can hope to say



Natural questions

Can you extend these results to near-group fusion categories?

↳ **No, at least not using our methods.**

What can you say about Turaev-Viro with defects?

↳ **Not much: the problem can only get harder, but maybe there's still a nice algorithm.**

Algorithms/complexity for state-sum invariants of 4-manifolds for some easy fusion 2-categories?

Thanks!