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Assembly of constructible factorization algebras

(based on joint work with E. Karlsson and C. I. Scheimbauer)

Factorization algebras

Conventions: $(\mathcal{C}, \otimes)$ symm monoidal ∞-cat

$\otimes$ pres. colims in each variable

X, Y, Z Hausdorff spaces

Def [Costello - Gwilliam]:
A fact algebra on X (in $\mathcal{C}$) $\mathcal{A}$ is a map of operads $\mathcal{A} : \mathrm{Open}(X)^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$
(= $\mathrm{Open}(X)$-algebra in $\mathcal{C}$)

where $\mathrm{Open}(X)$ is operad

with colors = open subsets of X

$$\mathrm{Mul}(U_1, \dots, U_n; V) = \begin{cases} \emptyset & \text{otherwise} \\ * & \text{if } U_i \text{ are pw. disjoint and } U_i \subseteq V \end{cases}$$

(i.e. assignment $U_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} U_n \subseteq V$

$\rightsquigarrow \mathcal{A}(U_1) \otimes \dots \otimes \mathcal{A}(U_n) \to \mathcal{A}(V)$)

s.th.: • $\mathcal{A}(U_1) \otimes \mathcal{A}(U_2) \xrightarrow{\simeq} \mathcal{A}(U_1 \mathbin{\dot\cup} U_2)$

(multiplicative)

• $\operatorname{colim}_{U \in \mathcal{W}_\downarrow} \mathcal{A}(U) \xrightarrow{\simeq} \mathcal{A}(V)$

where $\mathcal{W}$ is a Weiss cover of V

($\mathcal{W} \subseteq \mathrm{Open}(X)$ sth

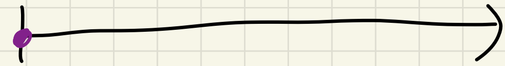
$\forall S \subseteq V$ fin $\exists W \in \mathcal{W}: S \subset W$)

$$\mathcal{W}_\downarrow := \{ U \in \mathrm{Open} \mid \exists W \in \mathcal{W} : U \subseteq W \}$$

Example: $\mathcal{C} = \mathrm{Vect}$, A ass. alg, M right module

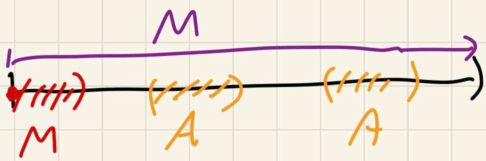
$\rightsquigarrow$ fact on $[0,\infty) = X$

with



$$\mathcal{A}(I) = \begin{cases} M & \text{if } 0 \in I \\ A & \text{otherwise} \end{cases}$$

$I \subseteq X$ interval



$$M \otimes A \otimes A \longrightarrow M$$

$$\mathrm{RMod}(\mathcal{C}) \longrightarrow \mathrm{Fact}_{[0,\infty)}(\mathcal{C})$$

From now on X, Y, Z "stratified manifolds"

<u>Stratified manifolds</u> [Ayala - Francis - Tanaka]

C^0-stratified spaces / conically smooth strat. spaces

A strat. manifold is a stratified space which locally looks like

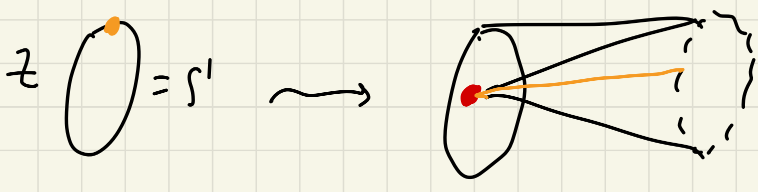
$$D(n, Z) := \mathbb{R}^n \times CZ$$

"a basic open disk"

where Z is a strat. manifold (compact)

and

$$CZ := Z \times [0,\infty) \big/ Z \times \{0\}$$



<u>Smooth</u> stratified manifold.

(conical smoothness)

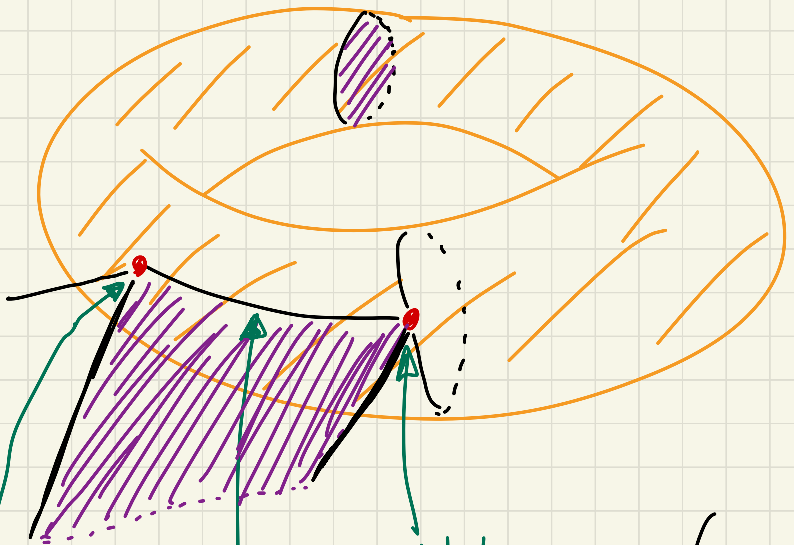
Example: 2-dim'l strat space

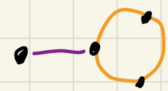
Strata:

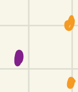
2-dim :

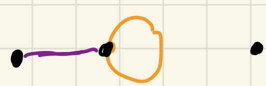
1-dim :

0-dim :



nbhd here : $C\Big($  $\Big)$

nbhd here : $\mathbb{R} \times C\Big($  $\Big)$

nbhd here : $C\Big($  $\Big)$

Note: if all basic disks are $D(n, \emptyset) = \mathbb{R}^n$

$\rightsquigarrow$ usual notion of top manifold

Constructible factorization algebras

"the value only changes when changing strata"

Two approaches:

[CG, Ginot, AFT]

(A) A factorization algebra $\mathcal{A}$ (as before) on X is called constructible if

$$\mathcal{A}(U) \xrightarrow{\simeq} \mathcal{A}(V)$$

whenever $U \subseteq V$ are both basic disks of the same type

i.e., $U \cong \mathbb{R}^n \times CZ \cong V$.

(B) [Lurie, AFT]

There is an ∞-operad $\mathrm{Disk}(X)$ with colors $= (U, \varphi)$ where $U := D(n, Z)$

$$\varphi : U \hookrightarrow X$$

spaces of operations

$$\mathrm{Mul}((U_1, \varphi_1), (U_2, \varphi_2); (V, \psi))$$

(unstratified : constructible = locally constant)

$$\begin{tikzcd}
& \parallel & & \\
& U_1 \sqcup U_2 \arrow[rr, hook] \arrow[dr] \arrow[rr, phantom, bend right=40, "\simeq"] & & V \arrow[dl, "\psi"] \\
U_i \arrow[ur] \arrow[rr, hook, "\varphi"] \arrow[r, phantom, shift left=4, "\simeq"] & & X &
\end{tikzcd}$$

where "$\simeq$" means isotopy

consider $\quad \mathrm{Alg}_{\mathrm{Disk}(X)}(\mathcal{C})$

<u>Convention</u> $\quad X, Y, Z \quad$ smooth strat manifolds

+ assume that there are "enough good disks"

i.e. $\exists$ basis of X by basic disk

which is closed under intersections.

(known X unstratified)

<u>Thm</u>

$$\underset{\substack{\parallel \\ \infty\text{-cat of cstr. Fact} \\ \text{algebras}}}{\mathrm{Fact}_X^{\mathrm{cstr}}(\mathcal{C})} \simeq \mathrm{Alg}_{\mathrm{Disk}(X)}(\mathcal{C})$$

[AFT (incomplete), Mazel-Gee
Arakawa (unstratified), KSW]

Assembly of (cstr) fact alg

$$
\begin{tikzcd}
\text{fact. alg.}\ X \arrow[d, rightsquigarrow, "\text{locally looks like}"] & \mathrm{Fact}_X \\
\mathbb{R}^n \times CZ \arrow[d, rightsquigarrow, "\text{split off } \mathbb{R}^n"] & \mathbb{E}_n\text{-alg}(\mathrm{Fact}_{CZ}) \arrow[u, rightsquigarrow, "\text{gluing}"'] \\
CZ \arrow[d, rightsquigarrow] & \mathrm{Fact}_{CZ} \arrow[u, rightsquigarrow, "\text{additivity}"'] \\
Z & \mathrm{Fact}_Z \arrow[u, rightsquigarrow, "??"']
\end{tikzcd}
$$

(1) Local-to-global

Thm [Ginot (unstratified), KSW]
A fact. algebra $\mathcal{A}$ is constructible
$\iff$ $\mathcal{A}$ is locally constructible
($i.e.\ \forall x \in X\ \exists\, x \subseteq U : \mathcal{A}|_U$ cstr)

Question: Is the assignment

$$\mathrm{Open}(X)^{op} \longrightarrow \mathrm{Cat} \qquad \text{a sheaf}$$
$$U \longmapsto \mathrm{Fact}_U$$

i.e. $$\mathrm{Fact}_V \xrightarrow{\sim} \lim_{\{U_1, \dots, U_n\} \subseteq \mathcal{U}} \mathrm{Fact}_{U_1 \cap \dots \cap U_n}$$

$\mathcal{U}$ cover V

Answer: do not know!

Thm [Matsuoka (unstratified), KSW]

Fact^{cstr} is a sheaf i.e.

$$\mathrm{Fact}^{cstr}_V \xrightarrow{\sim} \lim_{\{U_1,\dots,U_n\}\subseteq \mathcal{U}} \mathrm{Fact}^{cstr}_{U_1\cap\dots\cap U_n}$$

whenever $\mathcal{U}$ is cover of V.

Additivity: open

unstratified : Lurie, Berry (PhD-thesis)

general case : work in progress (Anja Svraka)

Conjecture:

$$\mathrm{Fact}^{cstr}_{X\times Y}(\mathcal{C}) \simeq \mathrm{Fact}^{cstr}_X\left(\mathrm{Fact}^{cstr}_Y(\mathcal{C})\right)$$

Base case [Lurie]

$$\mathrm{Fact}^{\mathrm{cstr}}_{\mathbb{R}^n}(\mathcal{C}) \simeq \mathbb{E}_n\text{-algebras in } \mathcal{C}$$

$$\mathrm{Fact}^{\mathrm{cstr}}_{[0,\infty)}(\mathcal{C}) \simeq \mathrm{RMod}(\mathcal{C})$$

Cones?

Can we describe $\mathrm{Fact}^{\mathrm{cstr}}_{CZ}$ in terms of $\mathrm{Fact}^{\mathrm{cstr}}_{Z}$?

Theorem [Brav-Rozenblyum, KSW]

The square

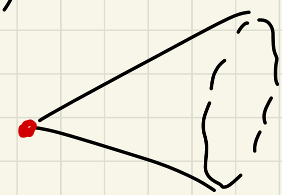
$$CZ = Z \times [0,\infty) / Z \times \{0\}$$

$$\begin{tikzcd}
 & CZ \arrow[dr] & \\
Z \times (0,\infty) \arrow[ur] \arrow[dr, "q"'] & & {[0,\infty)} \\
 & (0,\infty) \arrow[ur] &
\end{tikzcd}$$

induces a pullback of categories

additivity

$$
\begin{tikzcd}
& \mathrm{Fact}^{cstr}_{CZ}(\mathcal{C}) \arrow[dl, "rest"'] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & \\
\mathrm{AssFact}^{cstr}_{Z} \simeq \mathrm{Fact}^{cstr}_{Z\times(0,\infty)} \arrow[dr, "q_*"] & & \mathrm{Fact}^{cstr}_{[0,\infty)} \simeq \mathrm{RMod}(\mathcal{C}) \arrow[dl, "restr"'] \\
& \mathrm{Ass}(\mathcal{C}) \simeq \mathrm{Fact}^{cstr}_{(0,\infty)} &
\end{tikzcd}
$$

Giving cstr fact on CZ 

= giving an associative algebra A in Fact^{cstr}_Z

+ a right module (in $\mathcal{C}$) for $\int_Z A \in \mathrm{Ass}(\mathcal{C})$

Special case of Additivity:

Dunn's Additivity:

$$\mathbb{E}_n\text{-alg}(\mathbb{E}_m\text{-alg}) \overset{?}{=} \mathbb{E}_{n+m}\text{-alg}$$

$$\text{Fact}^{\text{cstr}}_{\mathbb{R}^n}(\text{Fact}^{\text{cstr}}_{\mathbb{R}^m}) \overset{?}{=} \text{Fact}^{\text{cstr}}_{\mathbb{R}^{n+m}}$$

We do not know:

$$X = X_1 \amalg X_2 \qquad \text{(strat. manifold)}$$

$\mathcal{A}$ $\underbrace{\text{multiplicative prefact}}_{\text{fact. alg} - \text{Weiss}}$ alg on X

<u>Q</u>: If $\mathcal{A}|_{X_1}$ is Weiss and $\mathcal{A}|_{X_2}$ Weiss

$\overset{?}{\Longrightarrow}$ $\mathcal{A}$ is Weiss