Intro

Topological Strings and Mirror Symmetry

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November 7th, 2018

LisMath Seminar, 2017/18
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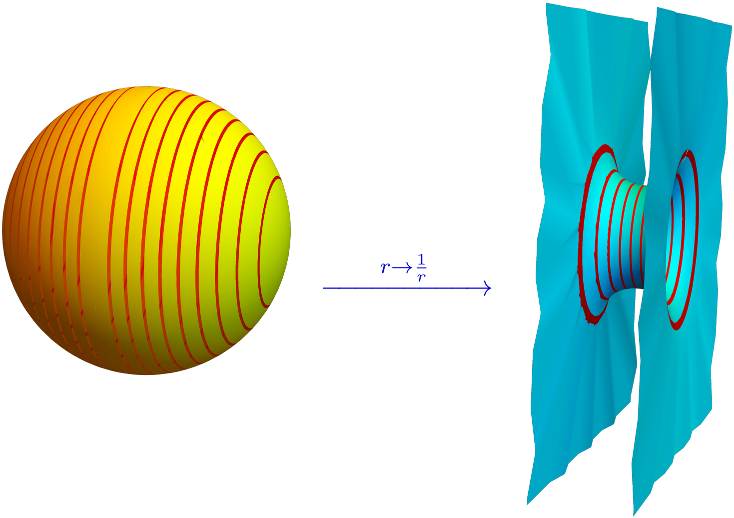
Mirror Symmetry

Mirror Symmetry is the conjecture that the Complex Structure and the Symplectic Structure of Mirror manifolds are equivalent.

Introduction: two easy pictures



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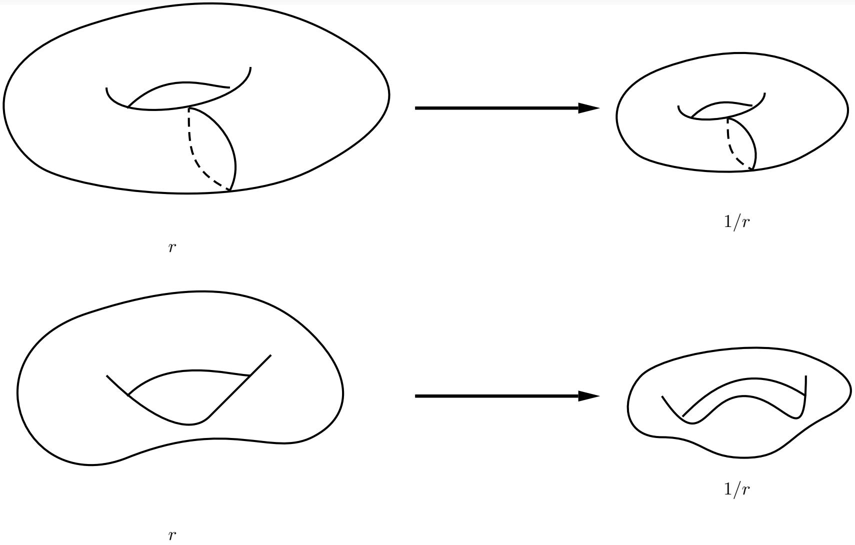
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Introduction: some easy examples

We will study the first cases of Mirror Symmetry through the Strominger-Yau-Zaslow (SYZ) conjecture.

It states that the key to understanding mirror symmetry resides within the submanifolds of a Calabi-Yau and the way in which they are organized.

The focus of this conjecture are special Lagrangian submanifolds, which have special features such as

- having half the dimension of the space
- minimizing length, area or volume
- other properties

The simplest possible case is a two-dimensional torus.

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Calabi-Yau manifolds

For us a Calabi-Yau space is a manifold X with a Riemannian metric g, satisfying three conditions:

1. X is a complex manifold and the metric g should be Hermitian with respect to the complex structure

$$g_{ij} = g_{\bar{i}\bar{j}} = 0.$$

2. X is Kähler. This means that locally on X there is a real function K such that

$$g_{i\bar{j}} = \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}K.$$

We also have a symplectic Kähler form

$$k = g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

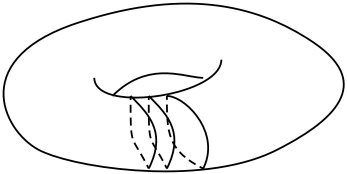
3. X admits a global nonvanishing holomorphic n-form Ω.

The two-dimensional torus

The special Lagrangian submanifold in this case will be a one-dimensional space or object consisting of a loop through the hole of the torus.

The minimizing property tells us that it must be the smallest possible circle going through that hole.

The entire Calabi-Yau, in this case, is just a union of circles.



The moduli space of the torus

The moduli space $\mathcal{M}$ of the torus parameterizes the set of circles. Every point on $\mathcal{M}$ corresponds to a different circle.

Furthermore, it also shows how all these subspaces are arranged.

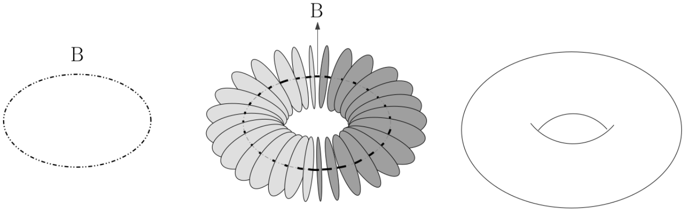


Figure 1: S. Yau, S. Nadis.

T-duality in the two-dimensional torus

The general idea is that one has a manifold X made up of all kinds of submanifolds that are catalogued by the moduli space $\mathcal{X}$.

Next you take all those submanifolds, of radius r, and make them of radius $1/r$. This is known as T-duality and it extends beyond circles to products of circles or tori.

T-duality and Mirror Symmetry go hand in hand. Suppose M is a torus of radii r. Its mirror, M', is also a torus, but of radii $1/r$.

In this case, when we invert the radii we are left with a torus of different radius, but still a torus. In this sense, this example is trivial since the manifold and its mirror are topologically identical.

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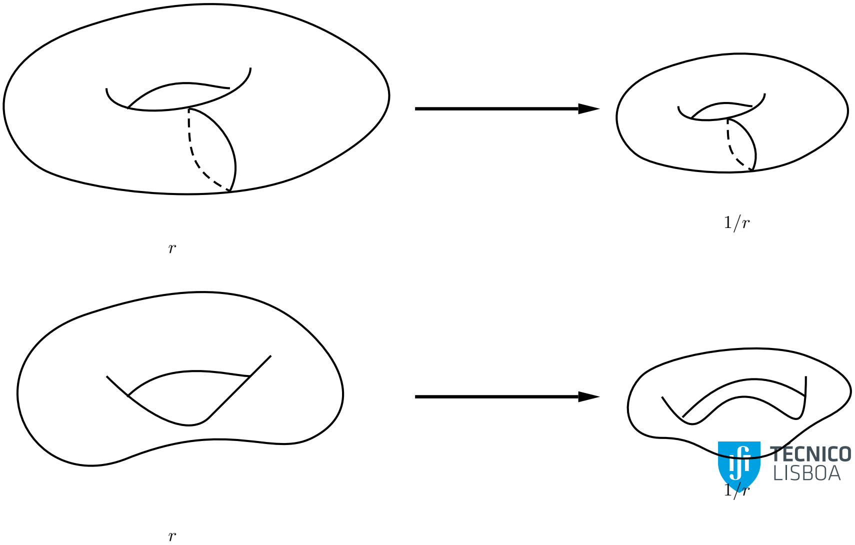
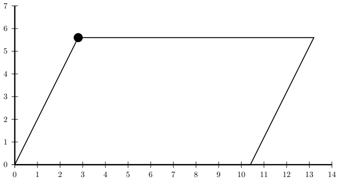
Recall



T-duality in the two-dimensional torus (II)

The flat metric depends on R_1, R_2, so we say we have a 2-dimensional "moduli space" of Calabi-Yau metrics on T^2, parameterized by (R_1, R_2).



It is convenient to repackage the moduli of T^2 into

$$A = iR_1R_2,$$
$$\tau = iR_2/R_1$$

A describes the overall area of the torus, or its "size", while τ describes its complex structure, or its "shape."

T-duality in the two-dimensional torus (III)

String Theory is invariant under the exchange of size and shape, $A \leftrightarrow \tau$, so this is the simplest example of Mirror Symmetry.

τ is called complex modulus of T^2. A is changed by means of the Kähler metric without changing the complex structure, so we call A a Kähler modulus (symplectic flavour).

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Four-dimensional manifold: K3 surface

If we go up by one complex dimension, the Calabi-Yau becomes a K3 surface.

The easiest way to obtain a K3 surface is to quotient $T^4/\mathbb{Z}_2$, using the $\mathbb{Z}_2$ identification

$$(x_1, x_2, x_3, x_4) \sim (-x_1, -x_2, -x_3, -x_4),$$

This quotient gives a singular K3 surface, with 16 singular points which are the fixed points of the $\mathbb{Z}_2$ action. The singular points can be "blown up" to obtain a smooth K3 surface.

Instead of being circles, the submanifolds in are two-dimensional tori.

In this case, $\mathcal{X}$ is just a two-dimensional sphere. Every point on this sphere $\mathcal{X}$ corresponds to a different torus, except for twenty-four points corresponding to pinched tori that have singularities.

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We can construct a K3 surface by taking a two-dimensional sphere and attaching a two-dimensional torus to every point on that sphere.

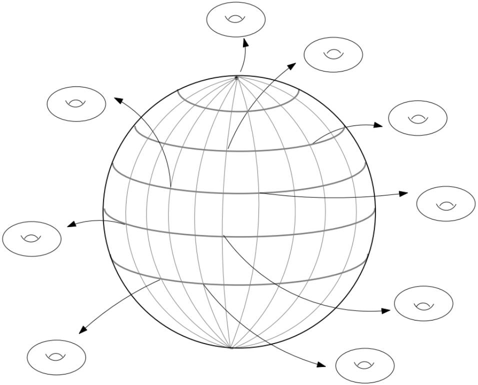

Figure 2: S. Yau, S. Nadis.

T-duality in the K3 surface

This example is also trivial in the same respect because all K3 surfaces are topologically equivalent.

Six-dimensional manifold: The Calabi-Yau threefold

Going up one more complex dimension, the manifold becomes a Calabi-Yau threefold.

$\mathcal{X}$ now becomes a 3-sphere, and the subspaces become three-dimensional tori.

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T-duality in the Calabi-Yau threefold

This example is more interesting. Applying T-duality will invert the radii of the tori. For a nonsingular torus, this radius change will not change the topology.

However, even if all the original submanifolds were nonsingular, changing the radius can still change the topology of the manifolds because the pieces can be put together in a non-trivial way.

Singular points

T-duality interchanges the Euler characteristic of singular submanifolds from $+1$ to -1, or vice versa.

In the mirror manifold, everything is reversed and the Euler characteristic is reversed.

In general, everything interesting in mirror symmetry, all the topological changes, happen at the singular points. This fact puts the moduli space $\mathcal{X}$ at the center of mirror symmetry.

Important remarks

- No trace of String Theory or Physics, even when it was originally discovered in a Physics content.
- Mirror Symmetry is still a conjecture.
- SYZ has only been proven in a few select cases, but not in a general way.
- Even if it is not technically correct, it is likely that some modification would hold.
- It is a nice example of how Physics gives rise to Mathematics, and then mathematics repays its debt.
- If SYZ is correct, it would offer a deeper insight into the geometry of Calabi-Yau spaces, while validating the existence of a Calabi-Yau substructure.

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Toric Geometry

Beyond easy examples

We have seen very particular examples of realizations of Mirror Symmetry through T-duality. We have also seen that Mirror Symmetry in its mathematical core conjectures a duality between Complex Structure and Symplectic Structure of mirror pairs.

In general, one could argue that not every manifold can be given in terms of submanifolds that have intrinsic radii. However, there is an interesting set of manifolds in which this process can be naturally implemented.

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Toric Geometry ($\mathbb{C}^n$)

Consider $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$ and the standard flat metric. Parameterizing

$$z_i = |z_i|e^{i\theta_i},$$

choose the coordinates $((|z_1|^2, \theta_1), \ldots, (|z_n|^2, \theta_n))$. Roughly, with these the coordinates one obtains a "sloppy" factorization

$$\mathbb{C}^n \approx \mathcal{O}^{n+} \times T^n,$$

where $\mathcal{O}^{n+}$ denotes the positive orthant $\{|z_i|^2 \geq 0\}$. At each point of $\mathcal{O}^{n+}$ we have the product of n circles obtained by varying θ_i.

However, when $|z_i|^2 = 0$ the circle $|z_i|e^{i\theta_i}$ degenerates to a single point (that is why we said sloppy).

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Toric Geometry ($\mathbb{CP}^n$)

The toric representation for $\mathbb{CP}^n$ consists on the quotient of the $2n+1$-sphere

$$|z_1|^2 + \cdots + |z_{n+1}|^2 = r$$

by the identification

$$(z_1, \ldots, z_{n+1}) \sim (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1}), \quad \theta \in \mathbb{R}$$

The toric base lies in the space coordinatized by the $|z_i|^2$. In the present case, the base turns out to be an n-dimensional simplex.

In the case of $\mathbb{CP}^2$ it is just a triangle. Over each point then we have a T^2 fiber generated by shifts of θ_i[1]. A cycle of T^2 collapses over each boundary of the triangle.

[1] naively this would give a T^3 for $\theta_1, \theta_2, \theta_3$, but the identification reduces this to T^2.

Toric Geometry (local $\mathbb{CP}^2$, the first Calabi-Yau)

To get a toric presentation of a Calabi-Yau manifold we have to choose a non-compact example. The construction is analogous to that of $\mathbb{CP}^n$. For $r > 0$, we have

$$-3|z_0|^2+|z_1|^2+|z_2|^2+|z_3|^2 = r, \quad (z_0, z_1, z_2, z_3) \sim (e^{-3i\theta} z_0, e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3), \quad \theta \in \mathbb{R}.$$

We can also draw the toric diagram for this case. The condition that all $|z_i|^2 > 0$ becomes

$$\begin{aligned} &|z_1|^2 + |z_2|^2 + |z_3|^2 > r, && |z_2|^2 > 0, \\ &|z_1|^2 > 0, && |z_3|^2 > 0. \end{aligned}$$

So the toric base is the positive octant in $\mathbb{R}^3$ with a corner chopped off.

Toric Geometry

This kind of construction allow us to define these manifolds in terms of the tori defined by the θ_i's, so for the manifolds described by Toric Geometry we have a natural way to apply T-duality (provided they are Calabi-Yau manifolds).

Other examples are local $\mathbb{CP}^1$ or local $\mathbb{CP}^1 \times \mathbb{CP}^1$. All these toric Calabi-Yaus are non-compact, but it is also possible to construct compact Calabi-Yaus using the techniques of toric geometry. For example, starting with $\mathbb{CP}^3$ and $\mathbb{CP}^4$ one can impose some extra algebraic relations on the coordinates to obtain a Calabi-Yau.

Topological Sigma models

Topological Sigma Models

They enable us to interpret the Yukawa couplings used in mirror symmetry as correlation functions of topological field theory.

Two types of topological sigma models obtained by a topological twist of the $\mathcal{N} = 2$ SuperSymmetric Sigma Model:

- A-model: Yukawa couplings associated with Kähler deformations of a a Calabi-Yau 3-fold M can be regarded as 3-point correlation function.
- B-model: Yukawa couplings associated with complex structure deformations of a a Calabi-Yau 3-fold M can be regarded as 3-point correlation function.

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Homological Mirror Symmetry and connection with Topological Strings

Up to now, we have given a geometric interpretation of Mirror Symmetry.

Homological Mirror Symmetry corresponds to the algebraic approach. One of the main ideas is that there are two kinds of submanifolds involved in this phenomenon, and there are branes wrapping this different submanifolds.

We use the nomenclacure A-branes and B-branes. If you have a mirror pair of Calabi-Yau manifolds, X and X', A-branes on X are the same as B-branes on X'.

A-branes are objects defined by symplectic geometry, whereas B-branes are objects of algebraic geometry.

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A- and B-branes

The category of A-branes is an invariant of the symplectic structure on X. It is closely related to the Fukaya category of X.

On the other hand, the category of B-branes is an invariant of the complex structure on X. It has been argued to be equivalent to the derived category of coherent sheaves on the complex manifold X.

Notice that the derived category of coherent sheaves is defined in an essentially algebraic way, while the Fukaya category has a more geometric flavor.

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Topological Sigma Models

These twisted models are particularly interesting because all physical observables can be reduced to classical questions in Geometry.

Mathematically, another interesting point is that the correlation functions of these models can be computed by:

- A-model: counting rational curves, an interesting result for algebraic geometry in the context of enumerative geometry.
- B-model: calculating periods of differential forms.

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The only contribution in these models is structure of zero modes, which corresponds to the solution space of the instanton equation, the moduli space of instantons.

Then, the correlation function of the twisted model is identified with the topological intersection number of the moduli space:

- A-model: the worldsheet instanton is given by a holomorphic map from Σ to M.
- B-model: the instanton is a constant map from Σ to M.

A-Model

A-model

- Φ are map $\Phi : \Sigma \to X$ from a Riemannian 2-dimensional surface Σ to Riemannian manifold X.
- K, $\overline{K}$ are the canonical and anti-canonical line bundle.
- TX is the complexified tangent bundle of X.
- ψ_+^i and $\psi_-^{\bar{i}}$ are regarded as sections of $\Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$ respectively.
- We combine them into a section χ of $\Phi^*(TX)$ as follows:

$$\chi^i = \psi_+^i \qquad \chi^{\bar{i}} = \psi_-^{\bar{i}}.$$

- $\psi_+^{\bar{i}}$ is a (1,0) form on Σ with values in $\Phi^*(T^{0,1}X)$ and is denoted as $\psi_z^{\bar{i}}$.
- ψ_-^i is a (0,1) form on Σ with values in $\Phi^*(T^{1,0}X)$ and is denoted as $\psi_{\bar{z}}^i$.

The key of this model is that we can write the Lagrangian as

$$\mathscr{L} = it \int_\Sigma d^2 z \{Q, V\} + t \int_\Sigma \Phi^*(K)$$

where Q is the BRST operator, with $Q^2 = 0$ and

$$V = g_{i\bar{j}} \left(\psi_z^{\bar{i}} \partial_{\bar{z}} \phi^j + \partial_z \phi^{\bar{i}} \psi_{\bar{z}}^j \right)$$

$$\int_\Sigma \Phi^*(K) = \int_\Sigma d^2 z \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} g_{i\bar{j}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} g_{i\bar{j}} \right),$$

where $K = -i g_{i\bar{j}} dz^i dz^{\bar{j}}$ is the Kähler form. Thus, the integral depends on the cohomology class of both K and the map Φ.

Topological nature

Since K is a representative of $H^2(X)$, $\int_\Sigma \Phi^*(K)$ becomes a topological invariant. In particular, if $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ and the metric g is normalized so that the periods of K are integer multiples of 2π, then

$$\int_\Sigma \Phi^*(K) = 2\pi n,$$

where n is an integer called instanton number or degree.

Correlation function

With insertions of BRST operators we have

$$\langle\prod_a \mathcal{O}_a\rangle_n = e^{-2\pi n t}\int_{B_n} D\phi D\chi D\psi\, e^{-it\{Q,\int V\}}\prod_a \mathcal{O}_a$$

where B_n is the field space of maps of degree n.

For a given n, the bosonic part of the Lagrangian is minimized for holomorphic maps ϕ^i, that is, those fulfilling

$$\partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = 0$$

The weak coupling limit $\mathbb{R}\mathrm{e}\ t \to \infty$ therefore involves a reduction to $\mathcal{M}_n$, the moduli space of holomorphic maps of degree n.

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Final remarks and more on the topological nature

By saying that this is a topological field theory we mean that the correlation functions $\langle\prod_a \mathcal{O}_a\rangle$ are independent of the complex structure of Σ and X and depend only on the cohomology class of the Kähler form.

This can be seen from the fact that all the dependence of the Lagrangian on the complex structure of Σ and X is encoded in V and it only appears in the form $\{Q, V\}$. If we vary the integral with respect to the complex structure we will get factors of the form $\{Q, ...\}$, which are irrelevant.

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In this generic case, our moduli space of instantons $\tilde{\mathcal{M}}_n$ has dimension 0 and consists of a finite number $\#\tilde{\mathcal{M}}_n$ of points and the integration is given by

$$\langle\prod_{a=1}^{s}\mathcal{O}_{H_a}(P_a)\rangle_n = e^{-2\pi nt}\#\tilde{\mathcal{M}}_n \rightarrow \langle\prod_{a=1}^{s}\mathcal{O}_{H_a}(P_a)\rangle = \sum_{n=0}^{\infty}e^{-2\pi nt}\#\tilde{\mathcal{M}}_n$$

In the non generic case, if $\tilde{\mathcal{M}}_n$ has dimension s, the space of fermionic ψ zero modes is also s dimensional and varies as the fibers of a s-vector bundle $\mathcal{V}$ over $\tilde{\mathcal{M}}_n$. The number of points gets substituted by the Euler class $\chi(\mathcal{V})$ of the bundle $\mathcal{V}$

$$\langle\prod_{a=1}^{s}\mathcal{O}_{H_a}(P_a)\rangle = \sum_{n=0}^{\infty}e^{-2\pi nt}\int_{\tilde{\mathcal{M}}_n}\chi(\mathcal{V})$$

B-Model

B-model

- $\psi_{\pm}^{\bar{i}}$ are sections of $\Phi^*(T^{0,1}X)$, ψ_{+}^{i} is a section of $K \otimes \Phi^*(T^{1,0}X)$ and ψ_{-}^{i} is a section of $\bar{K} \otimes \Phi^*(T^{1,0}X)$.
- Define $\eta^{\bar{i}} = \psi_{+}^{\bar{i}} + \psi_{-}^{\bar{i}}$ and $\theta_i = g_{i\bar{i}}\left(\psi_{+}^{\bar{i}} - \psi_{-}^{\bar{i}}\right)$.
- Also, combine $\psi_{\pm}^{i}$ into a one form ρ such that $\rho_z^i = \psi_{+}^{i}$ and $\rho_{\bar{z}}^i = \psi_{-}^{i}$.

The Lagrangian can be written as

$$\mathscr{L} = it\int \{Q, V\} + tW,$$

where

$$V = g_{i\bar{j}}\left(\rho_z^i \partial_{\bar{z}}\phi^{\bar{j}} + \rho_{\bar{z}}^i \partial_z \phi^{\bar{j}}\right)$$

and

$$W = \int_{\Sigma}\left(-\theta_i D\rho^i - \frac{i}{2}R_{i\bar{i}j\bar{j}}\rho^i \wedge \rho^j \eta^{\bar{i}}\theta_k g^{k\bar{j}}\right).$$

The B theory is a topological field theory in the sense that it is independent of the complex structure of Σ and the Kähler metric of X.

The $\{Q, V\}$ will give terms as $\{Q, ...\}$, W is independent of the complex structure of Σ because it is written in terms of differential forms and under the change of Kähler structure W varies by $\{Q, ...\}$.

In the large Re t limit one expands around minima of the bosonic part of the Lagrangian. These are just constant maps $\Phi : \Sigma \to X$. The space of constant maps is a copy of X so the integral reduces to an integral over X.

Recall that in the A-model one had to integrate over the moduli space of holomorphic curves.

Correlation Functions

Pick points $P_a \in \Sigma$ and classes V_a in $H^{p_a}(X, \wedge^{q_a} T^{1,0} X)$. The correlation function

$$\langle \prod_a \mathcal{O}_{V_a}(P_a) \rangle$$

reduces, in the large t limit, to an integral over constant maps $\Phi : \Sigma \to X$.

We can view $\prod_a \mathcal{O}_{V_a}$ as a d form with values in $\wedge^d T^{1,0} X$. The remaining part is to integrate over X the elements of $H^d(X, \wedge^d T^{1,0} X)$ obtained this way.

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Mirror Symmetry. First historical encounter.

If the target space is a complex 3-dimensional Calabi-Yau manifold X and we consider G_X an abelian group determined by X, we have that $X^* = X/G_X$ is a complex 3-dimensional Calabi-Yau manifold with singularities, which is called orbifold.

From orbifold construction one can compute the Hilbert space of a sigma model on X^*. The key part of mirror symmetry is that there is an isomorphism between $\mathcal{H}_X$ and $\mathcal{H}_{X^*}$.

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By defining $h^{p,q}(X) = \dim H^{p,q}(X)$ as de dimension of the vector space of harmonic forms and using the previous result and a bunch of extra tools, one arrives to the Hodge diamond of X

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\ & 0 & & h^{1,1}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{1}$$

Equality $h^{p,q}(X^*) = h^{3-p,q}(X)$ allow us to express the Hodge diamond of X^* as

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,1}(X) & & 0 & \\
1 & & h^{1,1}(X) & & h^{1,1}(X) & & 1 \\
 & 0 & & h^{2,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

This misterious symmetry is what was initially called Mirror Symmetry and the complex 3-dimensional Calabi-Yau manifolds are called Mirror manifolds.

Recall

"Given X and X^* mirror manifolds, the moduli space of the Kähler structure (resp. complex structure) of X coincides with the moduli space of the complex structure (resp. Kähler structure) of X^*"

This comes from the fact that the Mirror Symmetry translates in this case into

$$h^{2,1}(X^*) = h^{1,1}(X) \qquad h^{1,1}(X^*) = h^{2,1}(X)$$

and for complex Calabi-Yau 3-folds:

- $h^{1,1}$ counts the number of parameters that correspond to the size (Kähler structure) and therefore determines the dimension of the moduli space of Kähler structure.
- $h^{2,1}$ accounts for the dimension of the moduli space of the complex structure of the Calabi-Yau, so it counts the number of parameters that correspond to the shape.

Discussion of the Yukawa coupling

Originally, the idea of mirror symmetry was implemented in the context of Heterotic Strings. Compactification by complex 3-dimensional Calabi-Yau manifold produces $h^{2,1}(X)$ massless fermions in $(3, 27)$ representation (with the subscript $\alpha, \beta, \dots$) and $h^{1,1}(X)$ in the $(\bar{3}, \overline{27})$ representation $(a, b, \dots)$.

The Yukawa couplings

$$\lambda_{\alpha\beta\gamma}\phi^{\alpha}\psi^{\beta}\psi^{\gamma} \qquad \lambda_{abc}\phi^{a}\psi^{b}\psi^{c}$$

are related to geometrical quantities of the Calabi-Yau.

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- $\lambda_{\alpha\beta\gamma}$: associate the α generation with a base $u_{(\alpha)}$ of $H^{2,1}(X)$. By identifying with the Dolbeault cohomology $H^1(T^{1,0}X) \simeq H^{2,1}(X)$ we can associate a representative $\tilde{u}^j_{(\alpha),\bar{i}} dx^{\bar{i}}$.
  Recall that a Calabi-Yau has a unique (up to multiplication by a constant) holomorphic 3-form $\Omega = \Omega_{ijk} dx^i \wedge dx^j \wedge dx^k$. Then, up to multiplication

$$\lambda_{\alpha\beta\gamma} = \int_X \Omega_{ijk} \tilde{u}^p_{(\alpha),\bar{l}} \tilde{u}^q_{(\beta),\bar{m}} \tilde{u}^r_{(\gamma),\bar{n}} \Omega_{pqr} dx^i \wedge dx^j \wedge dx^k \wedge dx^{\bar{l}} \wedge dx^{\bar{m}} \wedge dx^{\bar{n}}.$$

- λ_{abc}: the generation a is associated with a base $\nu_{(a)}$ of $H^{1,1}(X)$, which is a closed $(1,1)$-form. λ_{abc} is given by

$$\lambda_{abc} = \int_X \nu_{(a)} \wedge \nu_{(b)} \wedge \nu_{(c)} + \text{instanton corrections.}$$

  The top term can be exactly computed with complex geometry tools.

Example: Quintic Hypersurface

Example: Quintic Hypersurface M_5 in the Complex Projective space $\mathbb{C}P^4$

A quintic hypersurface M_5 in a 4-dimensional complex projective space $\mathbb{C}P^4$ is an example of a complex 3-dimensional Calabi-Yau manifold.

$$\mathbb{C}P^4 = \{(X_1 : X_2 : X_3 : X_4 : X_5) \ : \ X_i \in \mathbb{C}\}$$

The defining equation of the quintic is

$$(X_1)^5 + (X_2)^5 + (X_3)^5 + (X_4)^5 + (X_5)^5 = 0.$$

The abelian group G_{M_5} is given as a finite group isomorphic to $(\mathbb{Z}_5)^3$.

From M_5/G_{M_5} we can define M_5^* as the complex 3-dim Calabi-Yau manifold obtained from resolving the singularities of M_5/G_{M_5}.

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In general, the quintic is represented by a degree 5 polynomial

$$\sum_{\sum_{i=1}^{5} d_i = 5} a_{d_1 d_2 d_3 d_4 d_5} (X_1)^{d_1} (X_2)^{d_2} (X_3)^{d_3} (X_4)^{d_4} (X_5)^{d_5} = 0,$$

where the a coefficients fulfill some conditions so that the quintic does not have singulatities and encode the complex structure.

From Kodaira-Spencer theory the dimension of the moduli space is 101 and it counts the number of substancial parameters of varying the defining equation of M_5.

We can regard M_5^* as a family of spaces obtained from dividing the quintic hypersurface

$$F_\psi := (X_1)^5 + (X_2)^5 + (X_3)^5 + (X_4)^5 + (X_5)^5 + 5\psi X_1 X_2 X_3 X_4 X_5 = 0$$

by the group $G_{M_5} \simeq (\mathbb{Z}_5)^3$ and resolving the singularities.

Here, the parameter ψ represents the degrees of freedom of the moduli space of the complex structure of M_5^*, whose dimension is $h^{2,1}(M_5^*) = h^{1,1}(M_5) = 1$.

The dimension of moduli space of the Kähler structure of M_5 is also 1. We define t as a local coordinate of this space. Then, we have

- t is a subscript of the base of $H^{1,1}(M_5)$
- ψ is a subscript of the base of $H^{2,1}(M_5^*)$

It is feasible to compute all instanton corrections to the Yukawa coupling $\lambda_{ttt}(M_5)$ using the exact result of the Yukawa coupling $\lambda_{\psi\psi\psi}(M_5^*)$:

$$\lambda_{\psi\psi\psi}(\psi) = \int_{M_5^*} \Omega \wedge \frac{\partial^3}{\partial\psi^3}\Omega$$

because this fulfills the equation

$$\frac{dW_3}{d\psi} = -\frac{1}{2}C_3(\psi)W_3(\psi),$$

where $C_3(\psi)$ is the constant appearing in the Picard-Fuchs equation for the periods of a holomorphic 3-form.

From this construction one also obtains the mirror map relating the ψ and t coordinates

$$t = t(\psi).$$

On the other hand, $\lambda_{ttt}(t)$ was conjectured to be given by

$$\lambda_{ttt}(t) = 5 + \sum_{d=1}^{\infty} \alpha_d e^{2\pi i d t} = 5 + 2875\ e^{2\pi i t} + 4876875\ e^{4\pi i t} + \ldots,$$

where α_d is the instanton correction coming from the degree d worldsheet instanton.

These called attention of algebraic geometers interested in enumerative geometry since the number of rational curves of degree d, n_d, were computed to be

$$n_1 = 2875 \qquad n_2 = 609250$$

We see that $\alpha_1 = n_1$, and based on the equality $4876875 = 2^3\ 609250 + 2875$ they proposed the following conjecture

$$\lambda_{ttt}(t) = 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 e^{2\pi i d t}}{1 - e^{2\pi i d t}}.$$

It has been shown that this equality also holds for $d = 3$.

The surprising and interest result is that, if this conjecture is proven right, we would have easily obtained a highly non trivial result of Enumerative Geometry by the means of Mirror Symmetry, which was discovered in the context of String Theory.

Recap

- We have seen that Mirror Symmetry conjectures a relation between the the complex and symplectic nature of two mirror manifolds.
- The easiest realization of Mirror Symmetry is by means of the SYZ conjecture, which intimate relates Mirror Symmetry to T-duality.
- Toric geometry presents itself as a natural candidate for the implementation of SYZ although we still don't know the full reach of this conjecture.
- We have also taken a peek at the A and B models of Topological String Theory, which we have seen are closedly related to Mirror Symmetry.
- Then, we have seen the first encounter of Mirror Symmetry and we have discussed the role of the Yukawa couplings in this duality.
- Finally, we have seen an example that glanced at the possible full power of Mirror Symmetry, allowing us to compute the highly non trivial number of rational curves of degree d by means of an easy mirror computation.

TÉCNICO LISBOA

Conclusions

Conclusions

It is clear that we do not fully understand Mirror Symmetry and its reach. Maybe it will be proved, maybe some modification will hold. However, a possible relation between the complex and symplectic nature of two mirror manifolds is something astonishing.

Complex geometry is in an algebraic way a very rigid world, while symplectic geometry is a more geometric flexible field. The fact that these two worlds may be related is, at least, fascinating.

Furthermore, dualities between different fields are one of the most interesting phenomenons in Mathematics since it incredibly widens the toolkit to tackle problems in the different fields, specially if the fields have very different nature.

TÉCNICO LISBOA

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Proof that $\langle\{Q, U\}\rangle = 0$

Let U be an observable and Q be the BRST charge.

$$\langle\{Q, U\}\rangle := \int D(\text{fields})\, e^{-\mathscr{L}}\, \{Q, U\} = 0$$

Using $\langle U\rangle$, we will rotate the field by using a supersymmetric transformation $\exp(\varepsilon Q)$, with ε a Grassmann number. We will use a the measure is invariant under rotations with Jacobian 1.

$$\begin{aligned}
\langle U\rangle &:= \int D(\text{fields})\, e^{-\mathscr{L}}\, U = \int D(\exp(\varepsilon Q)\text{fields})\, (\exp(\varepsilon Q) e^{-\mathscr{L}}\, U) \\
&= \int D(\text{fields})\, (\exp(\varepsilon Q) e^{-\mathscr{L}}\, U) = \int D(\text{fields})\, e^{-\mathscr{L}}\, (\exp(\varepsilon Q) U) \\
&= \int D(\text{fields})\, e^{-\mathscr{L}}\, (U + \varepsilon\{Q, U\}) \\
&= \langle U\rangle + \langle\{Q, U\}\rangle
\end{aligned}$$

TÉCNICO LISBOA

Proof that the correlation function is invariant under changes in t

We will write $D(\text{fields}) = DX$ for simplicity.

$$\begin{aligned}\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle &= \int DX\ e^{-\mathscr{L}} \mathcal{O}_1 \dots \mathcal{O}_n \\ &= \int DX\ e^{-it\int_\Sigma d^2z\{Q,V\} - t\int_\Sigma \Phi^*(K)} \mathcal{O}_1 \dots \mathcal{O}_n \\ &= \sum_{d=0}^{\infty} e^{-2\pi n d t} \int_{P_d} DX\ e^{-it\int_\Sigma d^2z\{Q,V\} - t\int_\Sigma \Phi^*(K)} \mathcal{O}_1 \dots \mathcal{O}_n,\end{aligned}$$

where P_d are the connected components of the phase space. Now, we implement a variation in the coupling constant $t \to t + \delta t$

$$
\begin{aligned}
&\int_{P_d} DX\, e^{-i(t+\delta t)\int_\Sigma d^2z\{Q,V\}}\mathcal{O}_1\ldots\mathcal{O}_n \\
&\qquad = \int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}e^{-i\delta t\int_\Sigma d^2z\{Q,V\}}\mathcal{O}_1\ldots\mathcal{O}_n \\
&\qquad = \int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}(1-i\delta t\int_\Sigma d^2z\{Q,V\})\mathcal{O}_1\ldots\mathcal{O}_n \\
&\qquad = \int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}\mathcal{O}_1\ldots\mathcal{O}_n \\
&\qquad\qquad - i\delta t\int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}\{Q,\int_\Sigma d^2zV\}\mathcal{O}_1\ldots\mathcal{O}_n \\
&\qquad = \int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}\mathcal{O}_1\ldots\mathcal{O}_n \\
&\qquad\qquad - i\delta t\int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}\{Q,\left(\int_\Sigma d^2zV\right)\mathcal{O}_1\ldots\mathcal{O}_n\} \\
&\qquad = \int_{P_d} DX\, e^{-it\int_\Sigma d^2z\{Q,V\}}\mathcal{O}_1\ldots\mathcal{O}_n
\end{aligned}
$$

Details of the A-model

With insertions of BRST operators we have

$$\langle \prod_a \mathcal{O}_a \rangle_n = e^{-2\pi n t} \int_{B_n} D\phi D\chi D\psi \, e^{-it\{Q, \int V\}} \prod_a \mathcal{O}_a$$

where B_n is the field space of maps of degree n.

We also have that $\langle \{Q, W\} \rangle_n = 0$ for any W.

If $\{Q, \mathcal{O}_a\} = 0$ for all a, then the n-expectation value will be invariant under $\mathcal{O}_a \to \mathcal{O}_a + \{Q, S_a\}$ for any S_a. Thus, $\mathcal{O}_a$ should be considered as representatives of BRST cohomology classes.

TÉCNICO LISBOA

For a given n, the bosonic part of the Lagrangian is minimized for holomorphic maps ϕ^i, that is, those fulfilling

$$\partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = 0$$

The weak coupling limit therefore involves a reduction to $\mathcal{M}_n$, the moduli space of holomorphic maps of degree n.

Then, the entire path integral reduces to an integration over $\mathcal{M}_n$ weighted by one loop determinants of the non-zero modes.

Final remarks and more on the topological nature

By saying that this is a topological field theory we mean that the correlation functions $\langle\prod_a \mathcal{O}_a\rangle$ are independent of the complex structure of Σ and X and depend only on the cohomology class of the Kähler form.

This can be seen from the fact that all the dependence of the Lagrangian on the complex structure of Σ and X is encoded in V and it only appears in the form $\{Q, V\}$. If we vary the integral with respect to the complex structure we will get factors of the form $\{Q, ...\}$, which are irrelevant.

If a_n b_n are, respectively, the number of χ and ψ zero modes, the index theorem tells us that

$$w_n = a_n - b_n$$

is a topological invariant.

The n-correlation function $\langle \prod_a \mathcal{O}_a \rangle_n$ will vanish unless the sum of ghost numbers[2] of the $\mathcal{O}_a$ is equal to w_n.

[2] At the classical level (because at the quantum level we find an anomaly) we have a conservation law for the ghost number coming from assigning the following ghost numbers to our fields: $\chi \to 1$, $\psi \to -1$, $\phi \to 0$ and $Q \to 0$.

Observables of the A-model

The BRST cohomology of the model can be represented by operators that are functions of ϕ and χ only. If we have an n-form $W = W_{I_1,\ldots,I_n} d\phi^{I_1} \ldots d\phi^{I_n}$ on X the corresponding operator is

$$\mathcal{O}_W(P) = W_{I_1,\ldots,I_n} \chi^{I_1} \ldots \chi^{I_n}(P).$$

We also have

$$\{Q, \mathcal{O}_W\} = -\mathcal{O}_{dW}$$

Therefore, taking $W \to \mathcal{O}_W$ gives a natural map from the de Rham cohomology of X to the BRST cohomology.

TÉCNICO LISBOA

Incise

Let H be any cohomology cycle, the Poincaré dual of H is a cohomology class that counts intersections with H. It can be represented by a differential form $W(H)$ that has a delta function support on H.

If we pick some homology cycles H_a of codimensions q_a and points $P_a \in \Sigma$, the quantity

$$\langle \mathcal{O}_{H_1}(P_1) \dots \mathcal{O}_{H_s}(P_s) \rangle_n = e^{-2\pi n t} \int_{B_n} D\phi D\psi D\chi \, e^{-it \int \{Q,V\}} \prod \mathcal{O}_{H_a}(P_a)$$

will vanish unless $\sum_a q_a = w_n$.

In the limit Re $t \to \infty$ it reduces to an integral over the moduli space $\mathcal{M}_n$ of instantons.

Moreover, since we have picked $\mathcal{O}_{H_a}(P_a)$ to have delta function support such that $\Phi(P_a) \in H_a$, the moduli space over which we integrate $\tilde{\mathcal{M}}_n$ turns out to be the moduli space of instantons fulfulling $\Phi(P_a) \in H_a$.

TÉCNICO LISBOA

In a generic situation, the dimension a_n of $\mathcal{M}_n$ coincides with w_n. Moreover, $\Phi(P_a) \in H_a$ imposes q_a conditions. Therefore, the dimension of $\tilde{\mathcal{M}}_n$ should be $w_n - \sum_a q_a = 0$.

In this generic case $\tilde{\mathcal{M}}_n$ consists of a finite number $\#\tilde{\mathcal{M}}_n$ of points and the integration is given by

$$\langle \prod_{a=1}^{s} \mathcal{O}_{H_a}(P_a) \rangle_n = e^{-2\pi n t} \#\tilde{\mathcal{M}}_n \to \langle \prod_{a=1}^{s} \mathcal{O}_{H_a}(P_a) \rangle = \sum_{n=0}^{\infty} e^{-2\pi n t} \#\tilde{\mathcal{M}}_n$$

In the non generic case, if $\tilde{\mathcal{M}}_n$ has dimension s, the space of ψ zero modes is also s dimensional and varies as the fibers of a s-vector bundle $\mathcal{V}$ over $\tilde{\mathcal{M}}_n$. The number of points gets substituted by the Euler class $\chi(\mathcal{V})$ of the bundle $\mathcal{V}$

$$\langle \prod_{a=1}^{s} \mathcal{O}_{H_a}(P_a) \rangle = \sum_{n=0}^{\infty} e^{-2\pi n t} \int_{\tilde{\mathcal{M}}_n} \chi(\mathcal{V})$$

Details on the B-model

The theory is independent of t except for a trivial factor as long as Re $t > 0$ so that the integral converges.

For the $t\{Q, V\}$ term, the variation w. r. t. t will give $\{Q, ...\}$ terms.

For the tW term, it can be removed by $\theta \to \theta/t$.

If $\mathcal{O}_a$ are BRST invariant operators that are homogeneus in t of degree k_a, then the dependence of $\langle \prod_a \mathcal{O}_a \rangle$ is a factor $t^{-\sum_a k_a}$ coming from the rescaling of θ to remove the t in tW.

TÉCNICO LISBOA

In the large Re t limit one expands around minima of the bosonic part of the Lagrangian. These are just constant maps $\Phi : \Sigma \to X$. The space of constant maps is a copy of X so the integral reduces to an integral over X.

Recall that in the A-model one had to integrate over the moduli space of holomorphic curves. This comes from the fact that the t dependence in the A-model allows for more general setups satisfying that the bosonic part is zero. That is, holomorphic curves rather that only constant maps.

The fermion determinant in the A-model is real and positive so it is a well-defined quantum field theory even before taking BRST cohomology. This justifies that in the A-model the correlation functions don't use the Calabi-Yau condition.

However, for the B-model, the Calabi-Yau condition becomes crucial. The fermion determinant is complex. To make sense of it as a quantum field theory we need an anomaly cancellation condition that makes it possible to define fermion determinants as a functions. This condition is $c_1(X) = 0$, that is, X has to be a Calabi-Yau manifold.

As the A-model, the B-model has an $\mathbb{Z}$ grading by a quantum ghost number[3]. If X is a Calabi-Yau of complex dimension d, and $\mathcal{O}_a$ are BRST invariant operators of ghost number w_a, then $\langle \mathcal{O}_a \rangle$ vanishes in genus g unless

$$\sum_a w_a = 2d(1-g)$$

[3] $\eta \to 1$, $\theta \to 1$, $\rho \to -1$, $\phi \to 0$ and $Q \to 1$.

Consider $(0,p)$ forms X with values in $\wedge^q T^{0,1}X$

$$V = d\bar{z}^{i_1}\dots d\bar{z}^{i_p} V_{\bar{i}_1\dots\bar{i}_p}{}^{j_1\dots j_q}\frac{\partial}{\partial z_{j_1}}\dots\frac{\partial}{\partial z_{j_q}}$$

For each V and every $P\in\Sigma$ we can form a quantum field theory operator

$$\mathcal{O}_V = \eta^{\bar{i}_1}\dots\eta^{\bar{i}_p} V_{\bar{i}_1\dots\bar{i}_p}{}^{j_1\dots j_q}\psi_{j_1}\dots\psi_{j_q}$$

that fulfills

$$\{Q,\mathcal{O}_V\} = -\mathcal{O}_{\bar{\partial}V},$$

where $\bar{\partial}$ is the exterior operator of the sheaf cohomology group $H^p(X,\wedge^q T^{1,0}X)$. Consequently we have a natural map $V\to\mathcal{O}_V$ from $\oplus_{p,q}H^p(X,\wedge^q T^{1,0}X)$ to the BRST cohomology.

Correlation Functions

Pick points $P_a \in \Sigma$ and classes V_a in $H^{p_a}(X, \wedge^{q_a} T^{1,0} X)$. The correlation function

$$\langle \prod_a \mathcal{O}_{V_a}(P_a) \rangle$$

vanishes unless $\sum_a p_a = \sum_a q_a = d$. In the large t limit, the calculation reduces to an integral over constant maps $\Phi : \Sigma \to X$.

In addition to the bose zero modes, displacements of the constant map Φ, there are fermi zero modes, constant modes of η and θ.

The non-zero modes just go into the definition of the string coupling constant t.

We can view $\prod_a \mathcal{O}_{V_a}$ as a d form with values in $\wedge^d T^{1,0} X$. The remaining part is to integrate over X the elements of $H^d(X, \wedge^d T^{1,0} X)$ obtained this way.

The Calabi-Yau condition then becomes essential to ensure that $H^d(X, \wedge^d T^{1,0} X)$ is nonzero and one-dimensional.

TÉCNICO LISBOA