The categorical spin-statistics theorem

Based on arXiv:2403.02282

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March 6, 2024

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Outline

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   2. anti-involutions & hermitian pairings
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   3. $B\mathbb{Z}/2$-actions
3. The spin-statistics theorem
   1. fermionically dagger compact categories
   2. the proof

Unitary TQFT

- State spaces have a Hilbert space structure
- Hilbert spaces form a †-category

dagger category

Definition

A †-category is ... a category $\mathcal{D}$ together with a contravariant $(\cdot)^\dagger : \mathcal{C} \to \mathcal{C}^{op}$ s.t. $((\cdot)^\dagger)^2 = id_{\mathcal{C}}$ & $x^\dagger = x$ if $x \in ob\,\mathcal{D}$

A †-functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ between †-categories $\mathcal{D}_1, \mathcal{D}_2$ is a functor such that ... $F(f^\dagger) = F(f)^\dagger$

Idea: a d-dim unitary TQFT is ... a (sym mon) † - functor

$Bord_d \to Hilb$

Hermitian pairings on vector spaces

joint with Jan Steinebrunner

$\overline{V} = \{\bar{v} : v \in V\}$

$\lambda \cdot \bar{v} := \overline{\bar{\lambda} v}$

Consider the functor

$$\mathrm{Vect}^{\mathrm{fd.}}_{\mathbb{C}} \to \mathrm{Vect}^{\mathrm{op}} \qquad V \mapsto \overline{V}^*$$

Observation

A nondegenerate Hermitian form on V is equivalent to an isomorphism $h : V \to \overline{V}^*$ such that

$\langle v, w \rangle := h(v)(\bar{w})$

$\langle v, w \rangle = \overline{\langle w, v \rangle} \iff$

$$\begin{tikzcd}
V \arrow[r, "h"] \arrow[d] & \overline{V}^* \\
\overline{\overline{V}^*}^* \arrow[ur, "\overline{h}^*"'] &
\end{tikzcd}$$

commutes.

Hermitian pairings on categories

Definition

An *anti-involution* on a category $\mathcal{C}$ is a fixed point for the $\mathbb{Z}/2$-action $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ on Cat.
A *Hermitian pairing* is a fixed point for the $\mathbb{Z}/2$-action on $\mathcal{C}^{\cong}$ given by composing the anti-involution with the inverse.

Explicitly, ...

$\mathcal{C}^{op} \xrightarrow[d]{\sim} \mathcal{C}$ $\quad \eta : id_{\mathcal{C}} \Rightarrow d^2$ s.t. ... $\quad$ "$d = \overline{(\cdot)}^{*}$"

$x \xrightarrow[\sim]{h} dx$ s.t.

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[dr, "\eta_x"'] & dx \\ & d^2x \arrow[u, "dh"'] \end{tikzcd}$$

The Hermitian completion

If (d, η) is an anti-involution on $\mathcal{C}$, $\mathrm{Herm}\,\mathcal{C}$ is the category with

- objects: Hermitian pairings
- morphisms: morphisms in $\mathcal{C}$

$\mathcal{C}^{op} \underset{\sim}{\xrightarrow{d}} \mathcal{C}$ $\quad d = (\cdot)^*$ $\quad \mathrm{V} \overset{*}{\mapsto} \mathrm{V}$ f.d.

$x \in \mathcal{C}$

$(x, h : x \to dx)$

$\mathrm{Hom}_{\mathrm{Herm}\,\mathcal{C}}((x_1, h_1), (x_2, h_2)) = \mathrm{Hom}_{\mathcal{C}}(x_1, x_2)$

This is a †-category:

$$\left(f : (x_1, h_1) \to (x_2, h_2)\right)^{\dagger} = x_2 \xrightarrow{h_2} dx_2 \xrightarrow{df} dx_1 \xrightarrow{h_1^{-1}} x_1$$

Positivity structures

If $P \subseteq \mathrm{ob}(\mathrm{Herm}\,\mathcal{C})$ is a collection of Hermitian pairings, let $\mathcal{C}_P \subseteq \mathrm{Herm}\,\mathcal{C}$ denote the full subcategory on P.

Definition

hermitian

The *transfer* of the pairing $h : x \to dx$ under the isomorphism $f : y \to x$ is

$$y \xrightarrow{f} x \xrightarrow{h} dx \xrightarrow{df} dy$$

We say P, P' are *equivalent* if they have the same closure under transfer.

Definition

A *positivity structure* on $\mathcal{C}$ is an equivalence class of Hermitian pairings $P \subseteq \mathrm{ob}(\mathrm{Herm}\,\mathcal{C})$ which surjects onto $\mathrm{ob}(\mathcal{C})$.

Non-evil †-categories

$\mathcal{C}_{all} = \mathrm{Herm}\,\mathcal{C}$

Example

Let $\mathcal{D}$ be a †-category. Define

$f = g^\dagger g$ for $g : x \to y$ iso

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$$\mathrm{Pos}(\mathcal{D}) := \{f : x \to x \mid \text{iso-positive}\}$$

ob $\mathcal{D} = (\mathbb{C}^n, \langle \cdot, \cdot \rangle_{st})$ $\mathrm{Herm}(\mathcal{D}) \ni (x, h)$ $h : x \to dx = x$ self-adjoint $h^\dagger = h$

$\simeq \mathcal{D}_{\mathrm{Pos}(\mathcal{D})}$ $\subset (\mathbb{C}^n, \langle \cdot, A \cdot \rangle_{st})$ " A

Theorem ([2]) A is a pos operator

There is an equivalence of 2-categories

$$\begin{tikzcd} \mathrm{alCat}^{pos} \arrow[rr, bend left, "(\mathcal{C},P)\mapsto\mathcal{C}_P"] & & \dagger\,\mathrm{Cat}\,. \arrow[ll, bend left, "(\mathcal{D},\mathrm{Pos}(\mathcal{D}))\leftarrow\!\shortmid\mathcal{D}"] \end{tikzcd}$$

Duality

From now on $\mathcal{C}$ symmetric monoidal. Recall duals: ...

$x^* \otimes x \to 1$

$\Rightarrow (\cdot)^* : \mathcal{C} \to \mathcal{C}^{op}$ sym mon

Dave Penneys

Duality

From now on $\mathcal{C}$ symmetric monoidal. Recall duals: ...

Definition

A dual functor is called *unitary* if it is a symmetric monoidal dagger functor. $(.)^*: \mathcal{C} \to \mathcal{C}^{op}$

$$Fd \simeq dF$$

Lemma

If $d : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is an anti-involution, $(.)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ has canonical commutation data with d. d sym mon $\Rightarrow$ preserves dual

Dagger compactness

Definition

The *standard unitary dual functor* on $\mathcal{C}$ ($\mathrm{Herm}\,\mathcal{C}$) is the symmetric monoidal dagger functor

$$\mathcal{C}_P \dashrightarrow \mathcal{C}_P^{op}$$

$$(.)^* : \mathrm{Herm}\,\mathcal{C} \to \mathrm{Herm}\,\mathcal{C}^{\mathrm{op}}$$

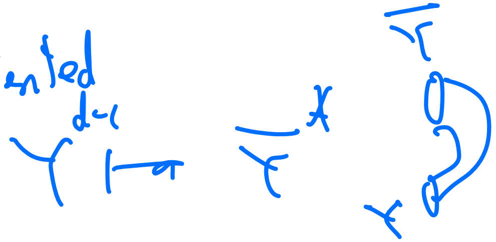
induced by the commutation data in the previous lemma.
A dagger category $\mathcal{C}_P$ is $\dagger$-*compact* if $(.)^*$ restricts to $\mathcal{C}_P$.

$\mathrm{Herm}\,\mathcal{C} \ni$

Explicitly, $(x, h) \mapsto \left(x^*,\ x^* \xrightarrow{h^*} (dx)^* \simeq dx^*\right) \overset{?}{\in} P$

P — d preserves duals

Ex Hilb Bord_d^{SO} ← oriented $d-1$



Fermions, spin and TQFTs

Idea

$$\begin{tikzcd} \mathrm{Bord}_d^{SO} \arrow[r] \arrow[d, squiggly] & \mathrm{Vect} \arrow[d, squiggly] \\ \mathrm{Bord}_d^{Spin} \arrow[r] & \mathrm{sVect} \end{tikzcd}$$

Statistics: super vector spaces

$V = V_0 \oplus V_1$ $\circlearrowleft$ $v_0 \mapsto v_0$ $\quad v_1 \mapsto -v_1$

Physically:

$|v| = 0, 1$

sVect ←
- State spaces are graded by $(-1)^F$ fermion parity

degree

- Bose statistics $\phi_1\phi_2 = \phi_2\phi_1$

$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$

- Fermi statistics $\psi_1\psi_2 = -\psi_2\psi_1$

Mathematically:

- symmetric monoidal category sVect of super vector spaces

Spin and the spin group

$d=3$

- for $d = 3$, irreps of $\mathrm{Spin}(d)$ are given by spins $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$
- integer spin $\iff$ lift to $\mathrm{SO}(d)$
- for arbitrary d, the kernel $\mathrm{Spin}(d) \to \mathrm{SO}(d)$ is generated by an element $(-1)^{2s} \in \mathrm{Spin}(d)$
- if (V, R) is an irrep of $\mathrm{Spin}(d)$, we say $v \in V$ has *half-integer spin* if... $R((-1)^{2s}) \cdot v = -v$
  and *integer spin* if ... $R((-1)^{2s})v = v$

Spin and statistics for TQFTs

Spin structures on spacetime allow for definitions of spinors in QFT.

Definition

A *fermionic TQFT* is a symmetric monoidal functor

$$P \xrightarrow{\cdot(-1)^{2s}} P$$

Spin principal

$$Z : \mathrm{Bord}_d^{\mathrm{Spin}} \to \mathrm{sVect}\,.$$

- $(-1)^{2s} \in \mathrm{Spin}(d) \to SO(d)$ induces an involution $(-1)^{2s}_Y$ of every time slice $Y^{d-1} \in \mathrm{Bord}_d^{\mathrm{Spin}}$
- The involution $Z((-1)^{2s})$ on the state space $Z(Y)$ determines... which states have half-integer spin
- The involution $(-1)^F_{Z(Y)}$ on the state space $Z(Y)$ determines... which states are fermionic

Idea: $Z((-1)^{2s}) = (-1)^F_{Z(Y)} \iff$ connection between spin & statistics

Super Hilbert spaces

- There is an anti-involution

$$d : \mathrm{sVect} \to \mathrm{sVect}^{\mathrm{op}} \qquad dV = \overline{V}^*$$

- Herm(sVect) is the dagger category of super Hermitian vector spaces

$\overline{\langle v, w\rangle} = (-1)^{|v||w|}\langle w, v\rangle \Rightarrow \overline{\langle v, v\rangle} = (-1)^{|v|}\langle v, v\rangle$

$\langle v, v\rangle \in i\mathbb{R}$

- sHilb $\subseteq$ Herm(sVect) given by the positivity structure...

$\langle v, v\rangle \in i\mathbb{R}_{\geq 0}$ if v odd

$B\mathbb{Z}/2$-actions

Definition

A $B\mathbb{Z}/2$-action on a category $\mathcal{C}$ is a nat iso $\mathrm{id}_{\mathcal{C}} \overset{(-1)^F}{\Longrightarrow} \mathrm{id}_{\mathcal{C}}$
s.t. $\left((-1)^F_x\right)^2 = \mathrm{id}_x \quad \forall x$

$B\mathbb{Z}/2$-equivariance $\Longleftrightarrow$ spin-statistics connection

Spin $B\mathbb{Z}/2$ Statistics $B\mathbb{Z}/2$

$$\begin{tikzcd}
\mathrm{Bord}_d^{\mathrm{Spin}} \arrow[loop above, "Y \mapsto (-1)^{2s}_Y"] \arrow[r, "Z"] & \mathrm{sHilb}\,. \arrow[loop above, "V \mapsto (-1)^F_V"]
\end{tikzcd}$$

Formulating the spin-statistics theorem

Theorem

[1, Theorem 3.23.] Every symmetric monoidal dagger functor

$$Z\colon \mathrm{Bord}_d^{\mathrm{Spin}} \to \mathrm{sHilb}$$

is $B\mathbb{Z}/2$-equivariant.

Spin str

Fermionic †-compact categories

Lemma

The standard unitary dual functor maps $V \in \mathrm{sHilb}$ to a super Hermitian vector space that is negative definite on V_{odd}.

Definition

Let $\mathcal{C}_P$ be a †-category with unitary $B\mathbb{Z}/2$-action $(-1)^F$.
Then $\mathcal{C}_P$ is called fermionically †-compact if the standard unitary dual functor restricts to

$$(.)^* : \mathcal{C}_P \to \mathcal{C}^{\mathrm{op}}_{P_{(-1)^F}}.$$

Fermionic †-compactness geometrically

Lemma

A †-category $\mathcal{D}$ is fermionically †-compact if and only if every object $x \in \mathcal{D}$ admits a dual $\mathrm{ev} : x^* \otimes x \to 1, \mathrm{coev} : 1 \to x \otimes x^*$ *such that the diagram*

$$\begin{tikzcd} 1 \arrow[rr, "\mathrm{ev}^\dagger"] \arrow[d, "\mathrm{coev}"] & & x^* \otimes x \arrow[d, "\sigma_{x^*,x}"] \\ x \otimes x^* \arrow[rr, "(-1)^F_x \otimes \mathrm{id}_{x^*}"] & & x \otimes x^* \end{tikzcd} \tag{1}$$

commutes.

Proof of the spin-statistics theorem

Theorem

[1, Theorem 3.19.] Let

$$F : \mathcal{D}_1 \to \mathcal{D}_2$$

$x \xrightarrow{h} dx$

$\mathrm{TopVect} \xrightarrow{(\,)^*} \mathrm{TopVect}$

$V \simeq \overline{V}^*$

be a symmetric monoidal †-functor between fermionically †-compact categories. Suppose the only iso-positive involution in $\mathcal{D}_2$ is the identity. Then F is $B\mathbb{Z}/2$-equivariant.

Idea

1. prove $\mathrm{Bord}_d^{\mathrm{Spin}}$ is fermionically †-compact
2. the only iso-positive involution in sHilb is the identity
3. prove the above

Proof of the spin-statistics theorem

If $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor between categories with anti-involutions, uniqueness of duals gives a 2-cell in aICat

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[d, "(.)^*"] & \mathcal{C}_2 \arrow[d, "(.)^*"] \arrow[dl, Rightarrow] \\ (\mathcal{C}_1)^{\mathrm{op}} \arrow[r, "F"] & (\mathcal{C}_2)^{\mathrm{op}} \end{tikzcd} .$$

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If P, Q are positivity structures on $\mathcal{C}_1, \mathcal{C}_2$ respectively and $F : (\mathcal{C}_1)_P \to (\mathcal{C}_2)_Q$ is a $\dagger$-functor, we get $F(P_{(-1)^F}) \subseteq Q_{(-1)^F}$.

Proof of the spin-statistics theorem

If $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor between categories with anti-involutions, uniqueness of duals gives a 2-cell in aICat

$$\begin{tikzcd}
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\end{tikzcd} .$$

If P, Q are positivity structures on $\mathcal{C}_1, \mathcal{C}_2$ respectively and $F : (\mathcal{C}_1)_P \to (\mathcal{C}_2)_Q$ is a $\dagger$-functor, we get $F(P_{(-1)^F}) \subseteq Q_{(-1)^F}$. Concretely, if $(h : x \to dx) \in P$, then

$$\underbrace{(F(x) \xrightarrow{F((-1)^F_x)} F(x) \xrightarrow{(-1)^F_{F(x)}} F(x)}_{\text{iso-positive involution}} \xrightarrow{F(h)} F(dx) \cong dF(x)) \in Q$$

iso-positive involution $\Rightarrow$ $F((-1)^F_x) = (-1)^F_{F(x)}$

1. Freed-Hopkins: $\mathbb{Z}/2$-equivariant $\Leftrightarrow$ reflection ($\Leftrightarrow$ anti-involution pres)

$Z(\overline{Y}) \simeq \overline{Z(Y)}$ preserve pos pairings $\Leftrightarrow$ reflection pos ($\Leftrightarrow$ $*$-functor)

$\Updownarrow$

$Z(Y^*) \simeq \overline{Z(Y)}^*$

ITQFT proof due to

2. Higher dagger n-cats

2-cat → 2-op dagger

→ 1-op dagger

→ "$O(2)$-dagger"

→ $O(\infty)$-dagger

spin statistics?

$B\mathbb{Z}_r$

$\mathrm{Bord}_2^{\mathrm{Spin}^r}$

$\mathrm{Bord}_1^{\mathbb{Z}/r}$

Pin^r ← orientation reversal

$\mathrm{Bord}_2^{\mathrm{Spin}}$ v.s. $\mathrm{Bord}_2^{SO \times \mathbb{Z}/2}$

$\mathrm{Bord}_1^{\mathrm{Spin}_1} = \mathrm{Bord}_1^{\mathbb{Z}/2}$

G Liegroup & $\theta: G \to \mathbb{Z}_2$ → $H_n(G) =$ "$G \times \mathrm{Spin}$"

$(-1)^F$ central symm $\qquad$ s.t. $\theta((-1)^F) = 0$

D super div alg $/\mathbb{R}$ $\Rightarrow$ $\dfrac{D^\times}{\mathbb{R}_{>0}}$ compact fermionic group $(-1)^F = [-1]$, θ is the $\mathbb{Z}/2$-grading

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