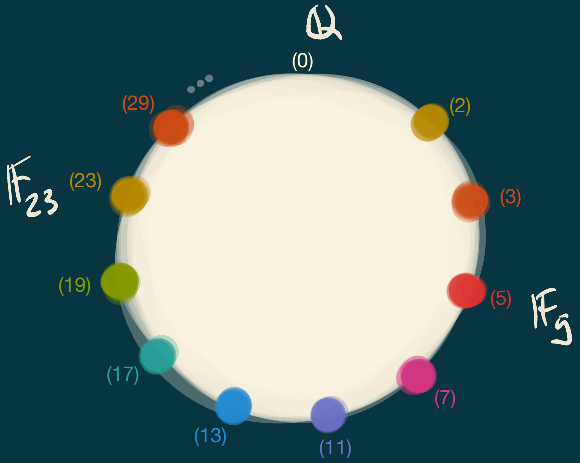
Factorization algebras in quite a lot of generality

Today’s goal:

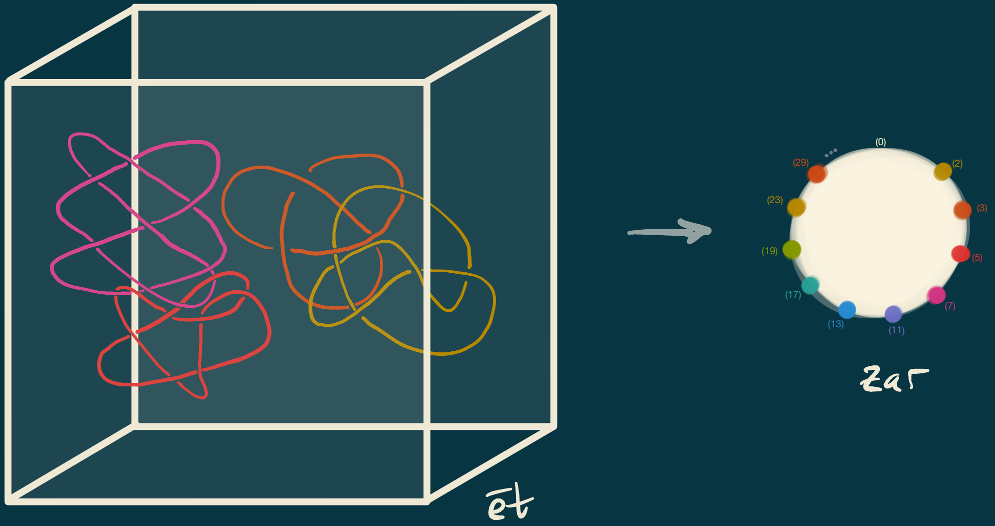
to motivate and describe
a formalism for factorization algebras
in very general geometric contexts.

Spec $\mathbb{Z}$, as seen through the lens of the Zariski topology:

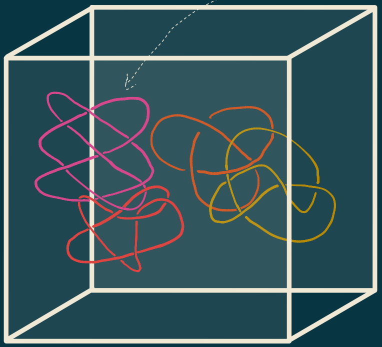


(profinite poset)

Spec $\mathbf{Z}$, as seen through the lens of the étale topology:



Why are primes knots?



Étale homotopy groups:

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

$$\|$$

$$\pi_n^{\acute{e}t}(\mathrm{Spec}\,\mathbb{F}_p) = \begin{cases} \hat{\mathbb{Z}} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

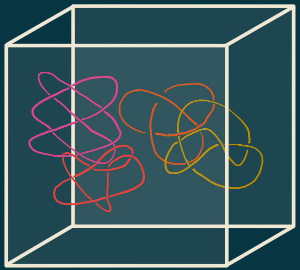
Étale homotopy type:

$$(\mathrm{Spec}\,\mathbb{F}_p)_{\acute{e}t} = \widehat{S^1}$$

Why is the ambient space 3-dimensional?

$\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t^{\pm 1}]$
$=$ "$U(1)$"

Global duality: a perfect pairing

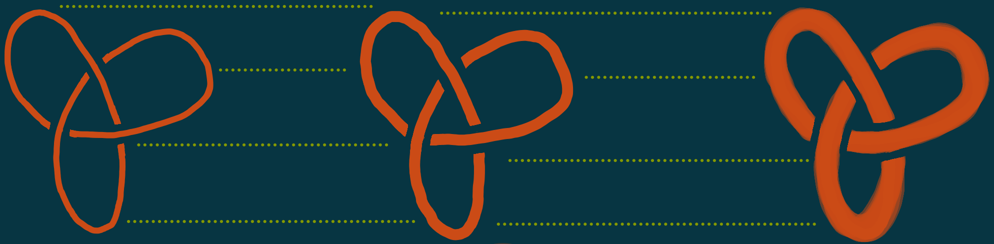


$$H^{*}_{et}(\overline{\operatorname{Spec}\mathbb{Z}}, \mathcal{F}) \otimes H^{-*}_{et}(\overline{\operatorname{Spec}\mathbb{Z}}, \mathcal{F}^{\vee}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

- $\overline{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$
- $\mathcal{F}$ = constructible sheaf
- $\mathcal{F}^{\vee} = \mathrm{Hom}(\mathcal{F}, \mathbb{G}_m[3])$

… a consequence of global class field theory!

What happens if we thicken one of our knots …



Spec $\mathbb{F}_p$ Spec $\mathbb{Z}/p^k$ Spec $\mathbb{Z}_p$

$\mathbb{Z}/p$

p-adic integers.

… and then delete the interior?

$\mathrm{Spec}\, \mathbb{Q}_p$

p-adic rational numbers

Local duality: a perfect pairing

$$H^*(G_{\mathbb{Q}_p}, A) \otimes H^{-*}(G_{\mathbb{Q}_p}, A^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

- $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$
- $A =$ finite Galois-module
- $A^\vee = \mathrm{Hom}(A, \mu[2])$

… a consequence of local class field theory!

Arithmetic topology

Mumford Mazur '60s
Morishita '90s

Number ring R	$\mathring{M}$	Open 3-manifold
Prime $\mathfrak{p} \triangleleft R$	$K \subset \mathring{M}$	Embedded knot
Archimedean places v	$x \in M \setminus \mathring{M}$	Points at infinity
Global duality class field thy		3-dimensional Poincaré duality
Completion of number ring $R_{\mathfrak{p}}$	NK	Tubular neighborhood
Completion of number field $\mathrm{Frac}(R)_{\mathfrak{p}}$	∂NK	Boundary of tubular neighborhood
Local duality		2-dimensional Poincaré duality

Schemes, rigid analytic spaces, adic spaces, diamonds

How do we 'do geometry' with arithmetic objects?

The geometry of arithmetic objects is controlled by categories of sheaves on them.

$$\mathrm{Sh} : \{\text{arithmetic objects}\}^{op} \longrightarrow \{\text{symmetric monoidal categories}\}$$

Arithmetic topology reflects the cohomological (or homotopical) properties of these sheaves.

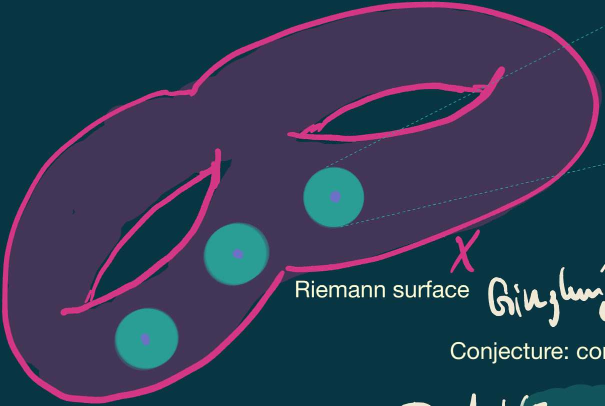
$$\mathrm{Sh}(\overline{\mathrm{Spec}\,\mathbb{Z}}) \underset{p^!}{\overset{p_*}{\rightleftarrows}} \mathrm{Sh}(*)$$

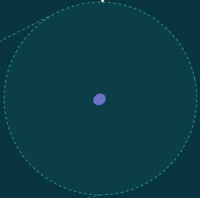
$$\text{“} p^!(\mathbb{Q}/\mathbb{Z}) \simeq \mathbb{G}_m[3] \text{”}$$

What about more serious number theory, like Langlands?

Geometric Langlands

$\mathrm{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$





Theorem: geometric Satake

$P(\mathrm{Gr}_G) \simeq \mathrm{Rep}(\check{G})$

Ginzburg, Mirković - Vilonen

Conjecture: compatible global equivalence

$\text{D-Mod}(\mathrm{Bun}_G(X)) \simeq \mathrm{Coh}(\mathrm{LocSys}_{\check{G}}(X))$

(not quite)

Derived geometric Satake and geometric Langlands

Campbell & Raskin:

equivalence of factorization monoidal categories on X

$$\text{D-Mod}^{sph}(\mathrm{Gr}_{G,X}) \simeq \mathrm{Sph}^{spec}(\mathrm{Gr}_{\check{G}})$$

'spherical' D-modules on the Beilinson-Drinfeld Grassmannian

'renormalized' factorization modules

Campbell, Chen, Gaitsgory, Raskin:

factorization homology of this equivalence gives geometric Langlands

Fargues & Scholze:

local Langlands = geometric Langlands on the Fargues-Fontaine curve

$X_{FF} = \mathrm{Spd}\, \mathbb{Q}_p / \varphi^{\mathbb{Z}}$

points = char 0 untilts

$\mathbb{Q}_p$ diamond = quotient of a perfectoid space

$$D(\mathrm{Bun}_G(X_{FF}), \mathbb{Z}_\ell) \simeq \mathrm{Coh}(Z^1(W_{\mathbb{Q}_p}, \check{G})/\check{G})$$

Factorization algebras in new geometric contexts

category of 'geometric objects'

$\mathcal{X}$ finite product, finite limits?

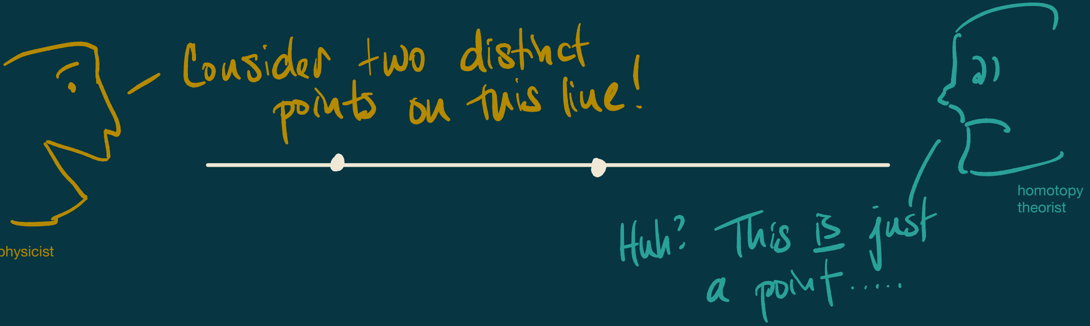
symmetric monoidal categories of sheaves

$\mathcal{X}^{op} \xrightarrow{Sh} SMCat_{(\infty,1)}$

... What else do we need?

Description.

Answer: a notion of distant or distinct points



Locality, cluster decomposition (Haag-Kästler, Weinberg, …)

Description.

$\mathbb{D}$ = the category of finite sets with apartness relations

inequivalence

irreflexive

symmetric

cotransitive: $x \neq y,\ z \Rightarrow (z \neq x \text{ or } z \neq y)$

$\langle 3 \rangle = (\{1,2,3\}, \emptyset)$

$\text{disj} \downarrow$

$\langle 2 \oplus 1 \rangle = (\{1,2,3\},$

$1 \neq 3$

$2 \neq 3$

$(1\ 2) \quad (3)$

Isolability objects: $X^{\bullet} : \mathbb{D}^{op} \longrightarrow \mathcal{X}$

X^1

$X^{1 \oplus 1} \subset X^2$

$X^{3 \oplus 4}$

Examples:

M - manifold

$M^{\bullet} : \mathbb{D}^{op} \longrightarrow \mathrm{Mfld}$

$\langle 1 \oplus 1 \rangle \rightsquigarrow M \times M \setminus \Delta$

$f : M \longrightarrow N$ smooth

$x \neq y \Leftrightarrow f(x) \neq f(y)$

Description.

Factorization algebras in quite a lot of generality:

Description.

Examples:

$\mathcal{X}$	$\mathrm{Sh}(X)$	$\mathrm{FA}(X^\bullet)$
$\mathrm{Var}_{\mathbb{C}}$	$D\text{-}\mathrm{Mod}(X)$	
Mfld	$\mathrm{coSh}_{\mathrm{Weiss}}(X)$	
Animae	$\mathrm{LocSys}(X, \mathrm{Vect}_k)$	
$\mathrm{Diamonds}$	$\mathrm{Sh}^{\square}(X, \mathbb{Z}_\ell)$	

Description.

Thank you.