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TQFT Club
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Thank You for this invitation!

Ask a question any time!

Title: Factorization Homology of Higher Categories

(Full) "beta" **Factorization Homology** is a construction:

Input
- M a (framed) k-manifold ($k \le n$) — Domain of a (T)QFT
- $1 \in \mathcal{R}$ a (pointed) rigid (∞,n)-category (finiteness)
  - (enriched over $\mathcal{V}$) ($\mathcal{V}$ = Vect, Ch, Spaces, …)
  - $\mathcal{R}$ organizes the pt-operators, line-operators, … (topological)

Rmk: For A a rigid monoidal dg-cat, $\ast \in B^n A$ is a pointed rigid $(\infty,2)$-cat enriched over Ch

Output
- $\int_M \mathcal{R}$ an object in $\mathcal{V}$ — $\mathbb{C}$-vector space, Chain complex$/\mathbb{C}$, … — All observables/operators (topological)

Key Novelty: $\int_M \mathcal{R}$ is manifestly well-defined and derived. It does not involve a choice of triangulation, etc.

Typically, take $\mathcal{V} = Ch_{\mathbb{C}}$.

Interested in $H_0\left(\int_M \mathcal{R}\right) \in Vect_{\mathbb{C}}$

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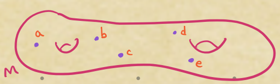
commutative alpha version = ordinary homology

Homology w/ coefs in abel gps defines a pairing:
$$M, A \longmapsto H_*(M; A)$$

Idea: Let A be a com monoid. Let M be a top spc (eg, mfld). Consider the set
$$\{(S \subset M, S \xrightarrow{l} A)\}_{S \text{ finite}} \overset{def}{=:} \int^{(\alpha)}_M A$$



$S \xrightarrow{l} A$, $a,b,c,d,e \in A$

w/ tplgy so that paths witness
- pts collide and labels add in A ($a, b \to a{+}b$)
- pts appear labeled by $0 \in A$ ($\emptyset \to 0$)

Rmk: So, a pt $x \in M$ determines a map $A \longrightarrow \int^{(\alpha)}_M A$, $a \longmapsto (x, a)$,

A physics-interpretation may regard $\int^{(\alpha)}_M A$ as the Observables of a (T)QFT, and thus map as the insertion of a pt-operator.

Rmk: $\int^{(\alpha)}_M A$ is perfectly functorial and continuous in M and A!

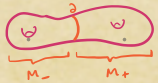
Eg: $\mathrm{Diff}(M) \curvearrowright M \rightsquigarrow \mathrm{Diff}(M) \curvearrowright \int^{(\alpha)}_M A$ (continuous gp)

Thm (Dold–Thom): $\pi_* \int^{(\alpha)}_M A \cong H_*(M; A)$ For A a gp.

Cor: $\int^{(\alpha)}_{S^n} A \simeq K(A,n) = B^n A$ (reduced), $\pi_r B^n A \cong \begin{cases} A, & r=n \\ 0, & r \ne n \end{cases}$

Key Lemma: $\int^{(\alpha)}_M A$ satisfies excision in M:

$M = M_- \cup M_+ \rightsquigarrow \int^{(\alpha)}_{M_-} A \underset{\int^{(\alpha)}_{\partial} A}{\otimes} \int^{(\alpha)}_{M_+} A \xrightarrow[\sim]{\text{"excision"}} \int^{(\alpha)}_M A$



Each pt $x \in M$ (and choice of o.emb $T_x M \xrightarrow{\varphi} M$, $0 \mapsto x$) determines a Collapse-map
$$M \longrightarrow (T_x M)^+$$
$$y \longmapsto \begin{cases} \varphi^{-1}(y), & y \in \mathrm{Im}(\varphi) \\ +, & y \notin \mathrm{Im}(\varphi) \end{cases}$$

$\rightsquigarrow \int^{(\alpha)}_M A \longrightarrow \int^{(\alpha)}_{(T_x M)^+} A \simeq B^n A$ (A framing of M $\to T_x M \cong \mathbb{R}^n$)

vary $x \in M$: $\rightsquigarrow M \times \int^{(\alpha)}_M A \longrightarrow B^n A \rightsquigarrow \int^{(\alpha)}_M A \longrightarrow \mathrm{Map}_c(M, B^n A)$ — LHS: "state-sum TQFT", RHS: "sigma-model", observe

Thm (Non Abelian Poincaré duality) This map is an equiv. (A gp)

Apply π_* $\rightsquigarrow$ Cor: $H_*(M; A) \underset{\text{Dold-Thom}}{\cong} \pi_* \int^{(\alpha)}_M A \xrightarrow{\simeq} \pi_* \mathrm{Map}_c(M, B^n A) \cong H^{n-*}_c(M; A)$ — Poincaré duality

"Poincaré/Koszul duality" Linear version of w/ RHS = Obbls of a quantized sigma model

Eg: $n=1$, full beta version, the coefs $A = \mathrm{End}(\mathcal{V})$: $\int^{(\beta)}_{S^1} A \simeq (\mathcal{V} \otimes \mathcal{V}^\vee)^{\otimes ?}$ excision, state-sum, dualizable

Eg: $n=2$, $A = \mathrm{End}(\mathcal{C}^{\otimes})$ … dualizable

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Generalize locally in M,

if M is 1-dimensional, then A need only be associative to define $\int^{(\alpha)}_M A$ ($\{S^0\text{-family of binary operations}\}$, excision)

if M is 2-dimensional, then A need only have an S^1-family of binary operations

if M is n-dimensional, then A need only have an S^{n-1}-family of binary operations

ie A need only be an $\mathcal{E}_n$-algebra

Def: Consider Mfld^{fr}_n, the sym monoidal topological category ($(\infty,1)$-category) in which
- an object is a framed n-mfld
- the top spc of morphisms is $\mathrm{Emb}^{fr}(M, N)$
- the sym mon strctr is $\sqcup$.

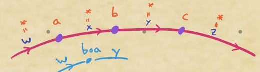
$\mathrm{Disk}^{fr}_n \subseteq \mathrm{Mfld}^{fr}_n$ is the full sym mon subcat consisting of $(\mathbb{R}^n)^{\sqcup I}$ for I a finite set.

Def: An $\mathcal{E}_n$-algebra in $\mathcal{V}$ is a sym mon fctr $\mathrm{Disk}^{fr}_n \xrightarrow{A} \mathcal{V}$.

$S^{n-1} \simeq \mathrm{Emb}^{fr}(\mathbb{R}^n \sqcup \mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathrm{Hom}_{\mathcal{V}}(A \otimes A, A)$

Beta version

Spse $\dim(M) = 1$.



For $\mathcal{R}$ a category, $w, x, y, z \in \mathrm{Obj}(\mathcal{R})$ ($\mathcal{R} = \ast \Rightarrow \mathcal{R} = BA$); $a, b, c \in \mathrm{Mor}(\mathcal{R})$ w/ $w = \mathrm{source}(a)$, $\mathrm{target}(a) = x = \mathrm{source}(b)$, $\mathrm{target}(b) = y = \mathrm{source}(c)$, $\mathrm{target}(c) = z$.

- As pts collide compose morphisms
- Insert pts labeled by identity morphisms.

is a stratified space X equipped w/ an injection $T_x X \xrightarrow{\varphi_x} \mathbb{R}^1$ (Target space of the stratum of $X \ni x$) } a solid 1-framing

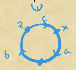
Say a solidly 1-framed strat spc (X, φ) is disk-stratified if each stratum is $\mathbb{R}^0$ or $\mathbb{R}^1$.

Obs: A cmpt disk-strat solidly 1-fr spc is just a finite directed graph.

determines a quiver: a category freely generated by such.

Fact: An $(\infty,1)$-category $\mathcal{R}$ is characterized in terms of its (linear) quiver representations:

the functor $\mathrm{Cat}_{(\infty,1)} \longrightarrow \mathrm{Fun}(\mathrm{Quiv}^{op}, \mathrm{Spcs})$, $\mathcal{R} \longmapsto (\Gamma \mapsto \mathrm{Rep}_\Gamma(\mathcal{R}))$ is fully faithful ($\mathrm{Hom}_{\mathrm{Cat}}(\Gamma, \mathcal{R})$)



$x, y, z \in \mathrm{Obj}(\mathcal{R})$, $a, b, c \in \mathrm{Mor}(\mathcal{R})$

Def: (full) (beta) factorization homology of a $(\infty,1)$-cat $\mathcal{R}$ is

$$\begin{array}{ccc} \mathrm{Quiv}^{op} \cong \mathrm{Disk}^{sfr}_{(1,1)} & \xrightarrow{\Gamma \mapsto \mathrm{Rep}_\Gamma(\mathcal{R})} & \mathrm{Spaces} \\ \downarrow & \nearrow_{\int^{(\beta)} \mathcal{R} := \mathrm{LKE}} & \\ \mathrm{Mfd}^{sfr}_{(1,1)} & & \end{array}$$

the left Kan extension

A $(\infty,1)$-category of solidly 1-fr strat spcs

Eg: $\int^{(\beta)}_M B A \underset{\textbf{Thm}}{\simeq} \int^{(\alpha)}_{M \setminus Sk_0(M)} A$ for A a monoid "Symmetries of the cyclic nerve" (ie, $\mathcal{E}_1$-alg in spaces)

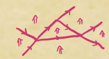
$\int^{(\beta)}_{S^1} \mathcal{R} \underset{\text{Thm}}{\simeq} HH(\mathcal{R})$

Key Features
- Continuously functorial in M and $\mathcal{R}$
- Excisive in M
- Values on $\mathcal{R} = \mathrm{Free}(\Gamma)$ are familiar
- Preserves sifted colims in $\mathcal{R}$

Higher Dimensions

Rmk: The full beta version is complicated and technical.

The idea is: For $\mathcal{R}$ an (∞,n)-category (n-category), a disk-stratn of an n-mfld w/ framing data



can be labeled by objects, 1-morphisms, 2-morphisms, etc in $\mathcal{R}$.

Better, a suitably framed disk-stratified n-mfld determines an (∞,n)-category via "free" constrctn (like above).
$$D \longmapsto \mathrm{Free}(D)$$

Then, $\mathcal{R}$ determines
$$\mathrm{Disk}^{sfr}_{(n,n)} \xrightarrow{\mathrm{Rep}_-(\mathcal{R})} \mathrm{Spaces}$$
$$D \longmapsto \mathrm{Hom}_{\mathrm{Cat}_{(\infty,n)}}(\mathrm{Free}(D), \mathcal{R}) =: \mathrm{Rep}_D(\mathcal{R})$$ (The space of $\mathcal{R}$-labelings of D)

And (full) beta fact hmlgy of $\mathcal{R}$ is

$$\begin{array}{ccc} \mathrm{Disk}^{sfr}_{(n,n)} & \xrightarrow{\mathrm{Rep}_-(\mathcal{R})} & \mathrm{Spaces} \\ \downarrow & \nearrow_{\int^{(\beta)} \mathcal{R} := \mathrm{LKE}} & \\ \mathrm{Mfd}^{sfr}_{(n,n)} & & \end{array}$$

the left Kan extension

So
$$\int^{(\beta)}_M \mathcal{R} = \mathrm{Colim}\left(\mathrm{Disk}^{sfr}_{(n,n)/M} \xrightarrow{\text{fgt}} \mathrm{Disk}^{sfr}_{(n,n)} \xrightarrow{\mathrm{Rep}_-(\mathcal{R})} \mathrm{Spaces}\right)$$

$\int^{(\beta)}_M B^n A \simeq \int^{(\alpha)}_M A$ ($\mathcal{E}_n$-alg)

Rmk: Much simpler for $\mathcal{R}$ the $(n-1)$-fold deloop $B^{n-1}\mathcal{A}$ of an $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-category.

In that full beta case, only strata of $\dim = n$ or $n-1$ matter.

Cat	$[0,n]$	$[1,n]$	…	$[n-1,n]$	$\cap$ alpha
Strat	n-cat	Mon $(n-1)$-cat		$\mathcal{E}_{n-1}$-mon 1-cat; Strat spcs w/ strata $\dim = n-1$ or n	$\mathcal{E}_n$-alg $\equiv$ n-mflds; n-cat concent in deg n

Def (sketch): $\mathrm{Mfd}^{sfr}_{[n-1,n]}$ is an $(\infty,1)$-cat in which an objct is a strat spc X equipped w/ a solid n-framing such that each stratum is $(n-1)$- or n-dimensional. ($\int^{(\beta)}_M B^n A \overset{?}{\simeq} \int^{(\alpha)}_M A$ Yes; Eg a framed n-mfld)

$\mathrm{Disk}^{sfr}_{[n-1,n]} \subseteq \mathrm{Mfd}^{sfr}_{[n-1,n]}$ consists of those each of whose strata is Euclidean.

Eg:

$\in \mathrm{Disk}^{sfr}_{[n-1,n]}$ ($\circ\{$ an n-dimensionally thickened graph$\}$)

(Technical)

Lem: An $\mathcal{E}_{n-1}$-monoidal $(\infty,1)$-category $\mathcal{A}$ in which each object has a monoidal dual determines a functor

$$\mathrm{Disk}^{sfr}_{[n-1,n]} \xrightarrow{\mathrm{Rep}(\mathcal{A})} \mathrm{Spaces}.$$

"Definition" An (∞,n)-category $\mathcal{R}$ consists of

0) a moduli space $Obj(\mathcal{R})$ of "objects"

Eg) $\overset{n=1}{\mathcal{R}} = Vect^{fin.dim}$ $\rightsquigarrow$ $Obj(\mathcal{R}) = \coprod_{n\geq 0} BGL_n(\mathbb{C})$

moduli space of f.dim $\mathbb{C}$-v.sps

1) for each $u, v \in Obj(\mathcal{R})$, a moduli space

$1Hom_{\mathcal{R}}(u, v)$ of 1-morphisms.

2) For each $f, g \in 1Hom_{\mathcal{R}}(u, v)$, a moduli space

$2Hom_{\mathcal{R}}(f, g)$ of 2-morphisms.

⋮

n) For each $\alpha, \beta \in (n-1)Hom_{\mathcal{R}}(x, y)$, a Chain cplx (enrichment)

$nHom_{\mathcal{R}}(\alpha, \beta)$ of n-morphisms.

Composition Rules! $kHom(x,y) \times kHom(y,z) \xrightarrow{\circ} kHom(x,z)$

Def (alpha) factorization homology of A ($\mathcal{E}_n$-alg) is the left Kan ext

$$\begin{array}{ccc} Disk_n^{fr} & \xrightarrow{A} & \mathcal{V} \\ \downarrow & \nearrow_{\int^{(\alpha)} A := LKE} & \\ Mfld_n^{fr} & & \end{array}.$$

So, $$\int_M^{(\alpha)} A = Colim\left(Disk_{n/M}^{fr} \xrightarrow{fgt} Disk_n^{fr} \xrightarrow{A} \mathcal{V} \right).$$

And (for $\mathcal{V} = Ch_{\Bbbk}$),

$$H_0 \int_M^{(\alpha)} A \cong \frac{\Bbbk\langle (M \supset \overset{fin}{S} \xrightarrow{\ell} A)\rangle}{\left\langle \begin{matrix} \text{collision \& } \mathcal{E}_n\text{-multiply} & \text{and} & \text{insertion of pt \& label by unit} \end{matrix} \right\rangle}$$ } local relations

Key obs

By construction $\int_M^{(\alpha)} A$ is continuously functorial in M and A.

- **In M** $Diff^{fr}(M) \curvearrowright \int_M^{(\alpha)} A \rightsquigarrow H_* Diff^{fr}(M) \curvearrowright H_* \int_M^{(\alpha)} A$

  $\rightsquigarrow H_r Diff^{fr}(M) \otimes H_{-r} \int_M^{(\alpha)} A \longrightarrow H_0 \int_M^{(\alpha)} A$

  ↑ a choice of $* \in M$

  $H_r Diff^{fr}(M) \otimes H_{-r} A$ $\quad A \in Alg_{\mathcal{E}_n}(Ch_{\Bbbk})$

  (Being a space, has positive hmlgy) $\quad$ ($End^R(V)$ has negative homology)

  Also, $H_* Emb^{fr}(S^1 \times \mathbb{R}^{n-1}, M) \otimes H_* \int_{S^1 \times \mathbb{R}^{n-1}}^{(\alpha)} A \longrightarrow H_* \int_M^{(\alpha)} A$

  (space of framed knots in M) $\quad \int_{S^1\times\mathbb{R}^{n-1}}^{(\alpha)} A \cong HH_* A$

- **In A** $O(n) \curvearrowright S^{n-1}$ (orthog gp) $\rightsquigarrow O(n) \curvearrowright Alg_{\mathcal{E}_n}(\mathcal{V})$

  $\rightsquigarrow \Omega O(n) \curvearrowright A \rightsquigarrow \Omega O(n) \curvearrowright \int_M^{(\alpha)} A$

  $\rightsquigarrow H_r \Omega O(n) \otimes H_{-r} \int_M^{(\alpha)} A \longrightarrow H_0 \int_M^{(\alpha)} A$

Key Technical results

- $\int_M^{(\alpha)} A$ satisfies excision in M:

  $M_- \cup_\partial M_+ = M \rightsquigarrow \int_{M_-}^{(\alpha)} A \underset{\int_{\partial\times\mathbb{R}}^{(\alpha)} A}{\otimes} \int_{M_+}^{(\alpha)} A \xrightarrow{\simeq} \int_M^{(\alpha)} A.$

- $\int_M^{(\alpha)} \Omega^n X \xrightarrow{\simeq} Map_c(M, X)$ provided $* \longrightarrow X$ is $(n-1)$-connected

  ie $\pi_r X = 0$ $\forall\, r \leq n-1$.

- $\int_M^{(\alpha)} Free_{\mathcal{E}_n}(V) \simeq \coprod_{r\geq 0} Conf_r(M) \underset{\Sigma_r}{\otimes} V^{\otimes r}.$

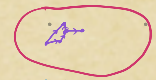
- $\int_M^{(\alpha)} A$ preserves sifted colimits in A.

  Eg) Free resolutions

Def (n-1)-beta factorization hmlgy of $\mathcal{A}$ is

$$\begin{array}{ccc} Disk_{[n-1,n]}^{sfr} & \xrightarrow{Rep(\mathcal{A})} & Spaces \\ \downarrow & \nearrow_{\int^{(\beta)}\mathcal{A} := LKE} & \\ Mfd_{[n-1,n]}^{sfr} & & \end{array}$$ the left Kan extension.

So $$\int_M^{(\beta)} \mathcal{A} = Colim\left(Disk_{[n-1,n]/M}^{sfr} \xrightarrow{fgt} Disk_{[n-1,n]}^{sfr} \xrightarrow{Rep(\mathcal{A})} Spaces \right)$$



an embedded graph in M

So $$\pi_0 \int_M^{(\beta)} \mathcal{A} = \frac{\{\mathcal{A}\text{-labeled embedded graphs in } M \text{ (w/ framing data)}\}}{\begin{matrix} \text{edges contract and morphisms compose} & \text{edges snap together and } \mathcal{E}_{n-1}\text{-monoidal mult} \\ \text{and identity relations} & \end{matrix}}$$

String Nets

An enriched version achieves,

for $\mathcal{A}$ a braided-monoidal ($\mathcal{E}_2$) $\mathbb{C}$-linear category

Eg) $\mathcal{A} = Rep_q^{f.d}(sl_2(\mathbb{C}))$

$\int_M^{(\beta)} \mathcal{B}^2\mathcal{A} \in Ch_{\mathbb{C}}$ (M fr 3-mfld) such that $H_0 \int_M^{(\beta)} \mathcal{B}^2\mathcal{A} = Skein(M, \mathcal{A})$.

Key Features

- $\int_M^{(\beta)} \mathcal{A}$ is continuously functorial in M and $\mathcal{A}$

- $\int_M^{(\beta)} \mathcal{A}$ is excisive in M (Technical!)

- $\int_M^{(\beta)} \mathcal{A}$ is explicit for $\mathcal{A}$ "free"

- $\int_M^{(\beta)} \mathcal{A}$ preserves sifted colims in $\mathcal{A}$.

Take $\mathcal{A} = Rep_q^{f.d}(sl_2(\mathbb{C}))$

$\int_{\mathbb{R}^2}^{(\beta)} \mathcal{B}^2\mathcal{A} \overset{!}{\simeq} Obj(\mathcal{A})$

$\int_{\mathbb{R}^3}^{(\beta)} \mathcal{B}^2\mathcal{A} \overset{!}{\simeq} End_{\mathcal{A}}(\mathbb{C}) \simeq \mathbb{C}(q)$

in $Mfd_{[2,3]}^{sfr}$, a framed knot $S^1 \xhookrightarrow{K} \mathbb{R}^3$

determines $\mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}^2 \xrightarrow{K} \mathbb{R}^3$

$V \longmapsto J^V(K)$

$\rightsquigarrow Obj(\mathcal{A}) \simeq \int_{\mathbb{R}^2}^{(\beta)} \mathcal{B}^2\mathcal{A} \longrightarrow \int_{\mathbb{R}^3}^{(\beta)} \mathcal{B}^2\mathcal{A} \simeq End_{\mathcal{A}}(\mathbb{C}) = \mathbb{C}(q)$

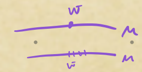
Forthcoming papers!

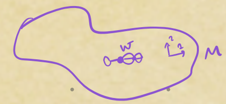
Q) Is TV fact homlgy?

Guess) Yes! · of End($\mathcal{C}$)

· using excision

$$\Theta_k^{op} \xrightarrow{\langle - \rangle \times W^{n-k}} \mathrm{Mfd}^{sfr}_{[0,n]}$$





1-cat

$$\left.\begin{array}{l} \mathcal{C} \xrightarrow{L} \mathcal{V} \\ \mathcal{C}^{op} \xrightarrow{R} \mathcal{V} \end{array}\right\} \rightsquigarrow (\Delta_{/[1]})^{op} \longrightarrow \mathcal{V}$$

$R \otimes \mathcal{C} \otimes L$

$$\rightsquigarrow \quad R \otimes_{\mathcal{C}} L \simeq \mathrm{Colim}\left((\Delta_{/[1]})^{op} \longrightarrow \mathcal{V} \right)$$

$$(\Theta_2/_{C_2})^{op} \longrightarrow \mathcal{V}$$