Geometric quantization, fusion categories, and Rozansky–Witten theory

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Projective/anomalous TQFTs

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Summary:

- Naturally occurring classical symmetry groups typically act *projectively* on the associated quantized theories.

Alternative title

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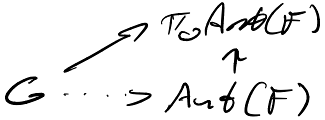
- In many examples this projectivity can be understood as appearing via the homotopy theory of the higher automorphism group of the quantum theory itself.



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  - Also see [Teleman] and [Braverman-Dhillon-Finkelberg-Raskin-Travkin-Johnson-Freyd].

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    - Also see [Teleman] and [Braverman-Dhillon-Finkelberg-Raskin-Travkin-Johnson-Freyd].
    - Relatedly, **RW** is the B-side of 3d mirror symmetry. See e.g. [Raskin-Hilburn, Gammage-Hilburn-Mazel-Gee].

Table of contents

We are entering:

Deformation and geometric quantization

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  - The group $\mathrm{Sp}_{2n}(\mathbb{R})$ only acts *projectively* on $\mathcal{H}$. Equivalently, a central extension, classified by $w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$, acts linearly on $\mathcal{H}$:

$g, h \mapsto T_g, T_h$

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Mp} \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{R})$$

$T_h \circ T_g = c\, T_{gh}$

Table of analogies

$d = 1$ (QM)	(V, ω)	$\mathcal{O}(V), *_\omega$	$L^2(\ell)$	$w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$
$d = 3$ (TV)				
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Two equivalent ways of encoding this notion:

$F\colon \mathbf{Bord}_d^X \to \mathcal{T}$	$F\colon 1 \to \sigma_X^{d+1}$

- Notation: The $(d+1)$-dimensional gauge theory associated to X as in [Freed-Moore-Teleman] is $\sigma_X^{d+1}\colon \mathbf{Bord}_{d+1} \to \mathbf{Alg}(\mathcal{T})$.

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- The RHS is an (op)lax natural transformation as in [Johnson-Freyd-Scheimbauer].

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- On the other hand, the theory σ_{BG}^2 is a functor valued in the Morita category of algebras:

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- On the other hand, the theory σ^2_{BG} is a functor valued in the Morita category of algebras:

$$\begin{aligned} \mathbf{Bord}_2 &\xrightarrow{\sigma^2_{BG}} \mathbf{Alg} \\ * &\longmapsto (\mathbb{C}[G], *) \end{aligned}$$

$1 \longrightarrow \sigma^2_{BG}$

$*: \mathbb{C} \xrightarrow[M]{} \mathbb{C}[G]$

- A theory defined relative to the theory σ^2_{BG} is a morphism in the Morita category from $\mathbb{C}$ to the group algebra, i.e. a module over the group algebra.

Anomalies

Let X be a (π-finite) groupoid (space), and now consider *projectivity data on* X:

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Projective/anomalous d-dimensional TQFT with background X-fields:

Theorem (VD23)

TFAE:

$F\colon 1_X \to \alpha$	$F\colon 1 \to \sigma_{X,c}$
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E.g. $X = BG$, $\widetilde{X} = B\widetilde{G}$ for $\widetilde{G}$ a central extension of G classified by α.

projective rep	mod over the twisted gp alg
rep of the ext'n	mod over gp alg of the ext'n

Anomalies

- We will focus on the analogue of a module over the twisted group algebra:

$$\sigma^{d+1}_{X,c} \colon \mathbf{Bord}_{d+1} \to \mathbf{Alg}(\mathcal{T}) \qquad\qquad F \colon 1 \to \sigma_{X,c}$$

- Caveat: For some of the examples we will consider, the theory $\sigma^{d+1}_{X,c}$ has not been formally constructed.
- In these cases, one can consider the analogue of a projective representation instead: $1 \to \alpha$
  where everything is rigorous. See [VD23, Hypothesis Q] for more details.

Recasting quantization

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$$
\begin{array}{ccccc}
\mathrm{U}(1) & \longrightarrow & \mathrm{U}(\mathcal{H}) & \longrightarrow & \mathrm{U}(\mathcal{H})\,/\,\mathrm{U}(1) \\
\uparrow & & \uparrow & \nwarrow & \uparrow \\
\mathbb{Z}/2 & \longrightarrow & \mathrm{Mp} & \longrightarrow & \mathrm{Sp}
\end{array}
$$

$$
\begin{array}{ccc}
B\,\mathrm{Mp} & & \\
\downarrow & & \\
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\end{array}
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$c \in U(1)$

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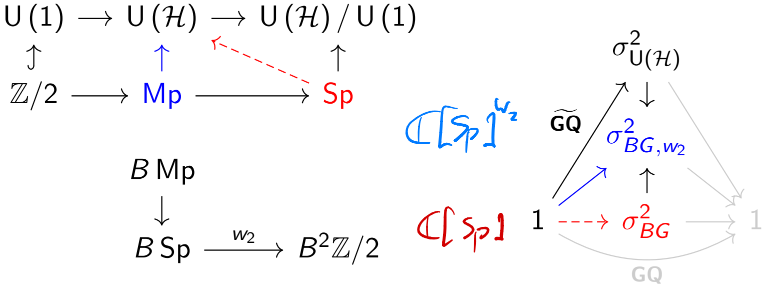

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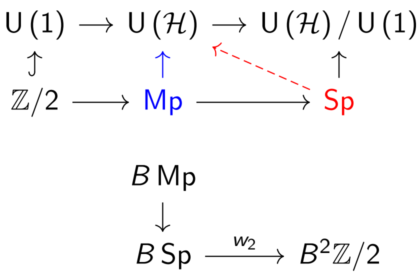

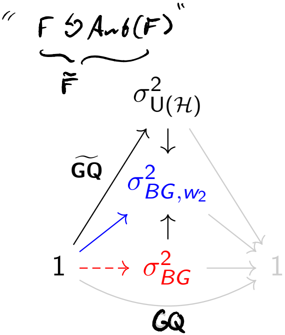

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Fusion categorical analogue of deformation quantization

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$$(\mathbf{Vect}[\Lambda], *, \beta_q) \qquad \beta_{\mathbb{C}_\ell, \mathbb{C}_k} : \mathbb{C}_\ell * \mathbb{C}_k \xrightarrow{\langle \ell, k \rangle_q \,\mathrm{id}} \mathbb{C}_k * \mathbb{C}_\ell \ .$$

$$\langle -, - \rangle_q = \frac{q(- + -)}{q(-)\,q(-)}$$

$$\mathbb{C}_\ell * \mathbb{C}_k = \mathbb{C}_{\ell+k} = \mathbb{C}_k * \mathbb{C}_\ell$$

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$$\mathcal{A} = (\mathbf{Vect}[\Lambda], *, \beta_q) \qquad \beta_{\mathbb{C}_\ell, \mathbb{C}_k} \colon \mathbb{C}_\ell * \mathbb{C}_k \xrightarrow{\langle \ell, k \rangle_q \,\mathrm{id}} \mathbb{C}_k * \mathbb{C}_\ell \,.$$

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- Analogue of geometric quantization: fusion category

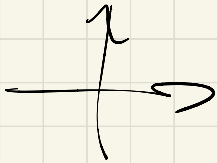
$$\mathcal{Z}(\mathrm{Vect}[L]) \simeq \mathcal{A} \qquad \circlearrowleft \qquad (\mathbf{Vect}[L], *) = \mathcal{C} \qquad \mathrm{TV}: * \mapsto \mathcal{C}, \; S^1 \mapsto \mathcal{A}$$

where we have chosen $\Lambda \simeq L \oplus L^\vee$ such that $q = \mathrm{ev}$.

3-type $B^3\mathbb{C}^\times \hookrightarrow B\mathrm{Aut}_{\mathrm{Fus}}(\mathrm{Vect}[L])$

$\downarrow \simeq$

2-type $B\mathrm{Aut}_{\mathrm{EqBr}}(\mathcal{Z}(\mathcal{C})) \simeq \pi_{\leq 2}$ " "



Br Eq cl of $\mathcal{Z}(\mathcal{C})$

deformations

Morita class of $\mathcal{C} = \mathrm{Vect}[L]$

has only up to scalar

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$$\mathbf{Vect}[L]$$

  where we have chosen $\Lambda \simeq L \oplus L^\vee$ such that $q = \mathrm{ev}$.
  - The group $\mathrm{O}(\Lambda)$ only acts projectively on $\mathbf{Vect}[L]$.

The fusion categorical anomaly

Recasting the previous slide in terms of TQFTs:

- There is a (framed, fully-extended) Turaev-Viro theory, which sends the point to **Vect** $[L]$, in the Morita 3-category of fusion categories [Douglas-Schommer-Pries-Snyder].

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- Using the obstruction theory of [Etingof-Nikschych-Ostrik]:

Theorem (VD23)

The Turaev-Viro theory for **Vect** $[L]$ *can be upgraded to a theory defined relative to a twisted gauge theory for* $\mathrm{O}(\Lambda)$*:*

$$\mathbf{TV} : 1 \to \sigma_{B^4 \mathrm{O}(\Lambda), c} = \mho_4$$

Table of analogies (reprise)

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$d = 3$ (TV)	(Λ, q)	$\mathbf{Vect}[\Lambda], \beta_q$	$\mathbf{Vect}[L]$	$O_4 \in H^4(B\,\mathrm{O}(\Lambda))$
$d = 3$ (RW)				

3-gp of inv. bimod/Vect[L]s

$$\mathrm{Aut}_{\mathrm{Fus}}(\mathrm{Vect}[L]) = \mathrm{BrPic}(\mathrm{Vect}[L]) \cong \mathrm{Pic}(Z(\mathcal{C}))$$

$$\downarrow$$

$$G = \mathcal{O}(L \oplus L^\vee, \omega)$$

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$U(n)/U(1)$

$\uparrow$

Sp

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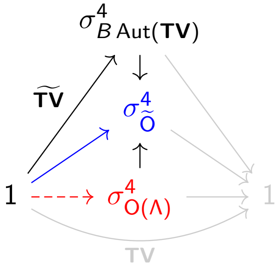

We are entering:

1. Quantum mechanics
2. Interlude: Families, symmetries, and anomalies
3. Fusion categories
4. Rozansky-Witten theory

$\Lambda \rightsquigarrow M$

Shifted deformation quantization

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- This will be the assignment of the Rozansky-Witten theory to the circle. [Roberts-Willerton]
  - Compare: The braided fusion category $\mathbf{Vect}[\Lambda]$ was assigned to the circle by Tuaev-Viro.

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- The deformation quantization can be thought of as a braided deformation of the derived category $\mathbf{QC}(M)$ of quasi-coherent sheaves.
  - Think: the deformation quantization of k-shifted is an $\mathbb{E}_{k+1}$-algebra, modules over this form an $\mathbb{E}_k$-category.
- This will be the assignment of the Rozansky-Witten theory to the circle. [Roberts-Willerton]
  - Compare: The braided fusion category $\mathbf{Vect}[\Lambda]$ was assigned to the circle by Tuaev-Viro.

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- The theory should fit into the framework of the AKSZ construction. [Alexandrov-Kontsevich-Schwarz-Zaboronsky, Qiu-Zabzine, Scheimbauer-Calaque-Haugseng, Stefanich, Riva]

Table of analogies (reprise^2)

$d = 1$ (QM)	(V, ω)	$\mathcal{O}(V), *_\omega$	$L^2(\ell)$	$w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$
$d = 3$ (TV)	(Λ, q)	$\mathbf{Vect}[\Lambda], \beta_q$	$\mathbf{Vect}[L]$	$O_4 \in H^4(B\,\mathrm{O}(\Lambda))$
$d = 3$ (RW)	(M, ω)	$\mathbf{QC}(M), \beta_\omega$	$\mathbf{RW}(*)$	$? \in H^4(B\,\mathrm{Sp}(M))$

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$$
\begin{array}{ccccc}
K & \longrightarrow & \mathrm{Aut}(\mathbf{RW}) & \longrightarrow & \pi_{\leq 1}\,\mathrm{Aut}(\mathbf{RW}) \\
\| & & \uparrow & \nwarrow & \uparrow \\
K & \longrightarrow & \widetilde{\mathrm{Sp}} & \longrightarrow & \mathrm{Sp}(V)
\end{array}
$$

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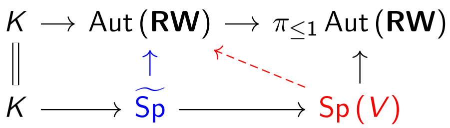

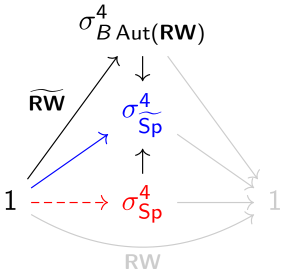

(V, q) $\quad Cl(V, q)$ $\quad M \simeq \Lambda\ell$ $\quad H^2(BSO)$

(Λ, q) $\quad \mathrm{Vect}[\Lambda]$ $\quad \mathrm{Vect}[\ell]$ $\quad H^4(BSO)$

$\Lambda = \text{v.s.}/\mathbb{F}_p$ $\qquad q \in H^4(B^2\Lambda, \mathbb{C}^\times)$

$\simeq H^2(B^2\Lambda, B^2\mathbb{C}^\times)$

$H^3_{cts}(M, \mathbb{C}^\times)$

My current work in this direction

- The analogue of Crane-Yetter in the **RW** context.
  - One strategy for accessing this projective Sp-action.
- General relationship between the prequantum k-gerbe and the anomaly $(k+1)$-gerbe.
  - One strategy for understanding the impact of changing the polarization on the shifted geometric quantization.

Thank you!

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