Localization in Supersymmetric Quantum Field Theories

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References

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Outline

1. Introduction

2. Equivariant cohomology and localization

3. Supersymmetry and QFT

4. Supersymmetric localization

1. Introduction

Introduction

Main object of study in physics

$$\int [\mathcal{D}\phi] e^{-\lambda \int \mathcal{L}[\phi]}$$

Infinite dimensional

Usually computed *perturbatively*

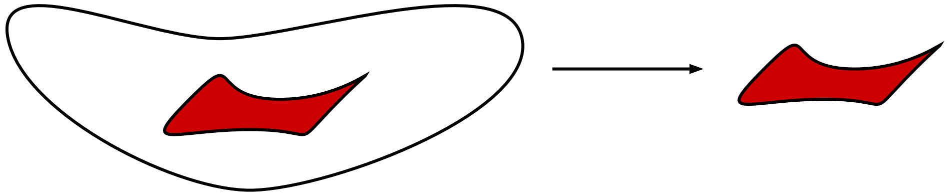
How to obtain exact results?

Introduction

The integration over manifolds with a group action can display

Localization

I.e., the value of an integral is given by a modified integral on a subset.



We'll see that this is a phenomenon which appears when studying the equivariant cohomology of a manifold with a group action

For (some) supersymmetric quantum field theories

Supersymmetric Localization

Historical introduction

First localization result: Duistermaat-Heckman (1982)

$$\int_M e^{-itf} \frac{\omega^n}{n!} = \sum_p \frac{e^{-itf(P)}}{(it)^n e(P)}$$

Stationary-phase approximation is exact

(Conditions: global Hamiltonian torus action over symplectic manifold)

Historical introduction

Atiyah-Bott (1982) showed that it was a particular case of more general localization property of equivariant cohomology.

Berline-Vergne (1982) used it to derive an integration formula for Killing vectors in compact Riemannian manifolds.

Several generalizations to infinite dimensions for particular cases (Atiyah, Witten, Bismut , Picken…).

2. Equivariant cohomology and localization

Cohomology

(de Rham)

Idea: On a smooth manifold, closed forms which are not exact

$\mathcal{M}$ n-dim Manifold, $\Omega^k(M)$ Space of k-forms $\qquad \Omega^\bullet(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$

Three operations:

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$$

Three sets:

$$Z^k(M) = \{\omega \in \Omega^k(M) : \ d\omega = 0\}$$

$$B^k(M) = \{\omega \in \Omega^k(M) : \ \omega \in d(\Omega^{k-1}(M))\}$$

$$H^k(M) = Z^k(M)/B^k(M)$$

$d^2 = 0$

Cohomology ring

$$H^\bullet(M) = \bigoplus_{k=0}^{n} H^k(M)$$

Integration

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega$$

(For the case of no boundary, we only care about the cohomology class)

Topological definition

G compact Lie group acting on a smooth manifold M by

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ (g,p) & \longmapsto & g \cdot p \end{array}$$

The action is called free if $\forall p \in M,\ g \cdot p = p \Rightarrow g = e$

That is, if no element of G different from the identity leaves some point fixed.

If G acts freely, M/G is also a smooth manifold. Then, one can define its equivariant cohomology as the usual cohomology:

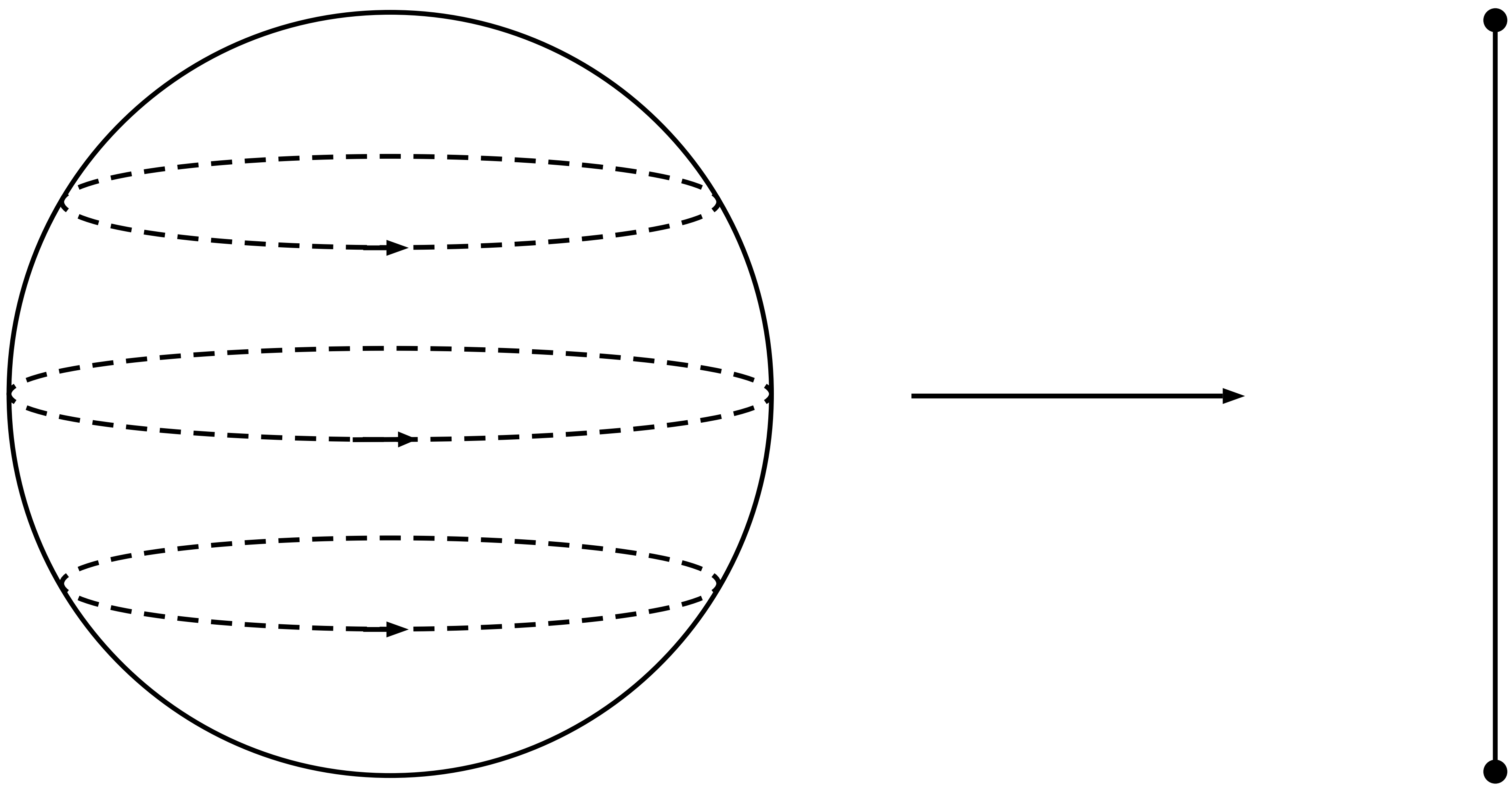
$$H^\bullet_G(M) = H^\bullet(M/G)$$

Ex: with left multiplication

$$H^\bullet_G(G) = H^\bullet(pt.)$$

When not free?

Example



$S^1 \times S^2 \to S^2$

S^2/S^1

Topological definition

When the action is not free, M/G can be *pathological* (not a manifold)

$H^\bullet_G(M)$ is the *right* substitute for $H^\bullet(M/G)$

Equivariant cohomology is the generalization of the usual cohomology

Topological definition

Recall, two homotopical manifolds have the same cohomology. Hence, we want to find a homotopy equivalent space on which the group acts freely: Take EG such that

1. The space EG is contractible
2. The group G acts freely on EG

and define

$$H^\bullet_G(M) = H^\bullet(M \times_G EG) = H^\bullet((M \times EG)/G)$$

$$\uparrow$$

$$(g \cdot p, q) \sim (p, g \cdot q)$$

Topological definition

Equivariant cohomology:

$$H_G^\bullet(M) = H^\bullet(M \times_G EG) = H^\bullet((M \times EG)/G)$$

- EG exists and is called the universal bundle associated to G

- The definition of the equivariant cohomology does not depend on the choice of EG

- The quotient $EG/G = BG$ is called the classifying space

Example: $H_G^\bullet(pt.) = H^\bullet(EG/G) = H^\bullet(BG)$

Topological definition

Equivariant cohomology:

$$H^\bullet_G(M) = H^\bullet(M \times_G EG) = H^\bullet((M \times EG)/G)$$

Example: $G = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ acts freely on $S^{2n+1} \subset \mathbb{C}^{n+1}$

by $z \cdot (w_0, \ldots, w_n) = (zw_0, \ldots, zw_n)$

Then, $S^{2n+1}/S^1 = \mathbb{C}P^n$ would be the classifying space if S^{2n+1} was contractible. However, we can take the limit $n \longrightarrow \infty$ and get

$$ES^1 = S^\infty \qquad \text{where} \qquad S^\infty = \lim_{n\longrightarrow\infty} S^{2n+1}$$

$$BS^1 = \mathbb{C}P^\infty \qquad \qquad \mathbb{C}P^\infty = \lim_{n\longrightarrow\infty} S^{2n+1}/S^1$$

Cartan model

One can define

$$\Omega^\bullet_G(M) = (\Omega^\bullet(M) \otimes S\mathfrak{g}^*)^G$$

and the exterior equivariant derivative on it

$$d_G = d \otimes 1 + \iota_\alpha \otimes \phi^\alpha$$

There is an isomorphism

$$H^\bullet_G(M) = (\Omega^\bullet_G(M), d_G)$$

Cartan model

Idea

$$M \times_G EG$$

is a *twisted* product so its cohomology has to be a *twisted* cohomology

$$\Omega^\bullet_G(M) = (\Omega^\bullet(M) \otimes S\mathfrak{g}^*)^G$$

Localization via an example: S^1

Consider:

Symplectic manifold (M, ω)

Hamiltonian map $H : M \longrightarrow \mathbb{R}$ which generates a

Circle action $S^1 \times M \to M$

I.e., $S^1 \to \mathrm{Ham}(M) : t \to \psi_t$ $\partial_t \psi_t = X_{\psi_t}$ $\psi_0 = \mathrm{id} = \psi_1$

$$\iota_X \omega = dH$$

Localization via an example: S^1

Also,

Invariant k-forms $\alpha \in \Omega^k_{S^1}(M)$ $\quad \mathcal{L}_X \alpha = d\iota_X \alpha + \iota_X d\alpha = 0$

Equivariant k-forms $\alpha \in \Omega^k_{S^1}(M)[\hbar]$ $\quad \alpha = \alpha_k + \hbar \alpha_{k-2} + \hbar^2 \alpha_{k-4} + \ldots$

$$\deg(\hbar) = 2$$

Equivariant exterior differential $d_\hbar = d + \iota_X \hbar$ $\quad d_\hbar^2 = 0$

$d^2 = 0, \iota_X^2 = 0$

In particular, $\tau \in \Omega^{2n}_{S^1}(M)[\hbar]$ $\quad d_\hbar \tau = 0 \iff d\tau_{2k-2} + \iota_X \tau_{2k} = 0, \ \forall k$

Localization via an example: S^1

Localization Lemma *Assume that the action has isolated fixed points and*

$$\tau = \tau_{2n} + \hbar\tau_{2n-2} + \cdots + \hbar^n \tau_0$$

is a $d_\hbar$-closed $2n$-form such that τ_0 vanishes on the fixed points of the action. Then τ is $d_\hbar$-exact. In particular,

$$\int_M \tau_{2n} = 0.$$

Idea: integral only cares about the fixed points!
(on the bottom component of the form τ_0)
Also,

$$\tau \; d_\hbar\text{-exact} \Rightarrow \tau \; d\text{-exact}$$

We have Stokes theorem for the equivariant case

$$d_\hbar \tau = 0 \iff d\tau_{2k-2} + \iota_X \tau_{2k} = 0, \; \forall k$$

Localization via an example: S^1

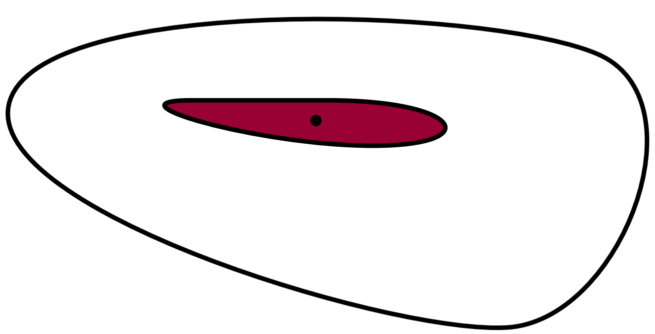
Lemma *Assume that the circle action is Hamiltonian and the critical points of H are all nondegenerate. Then for every fixed point p there exists an equivariant differential form*

$$\tau_p = \tau_{p,2n} + \hbar\tau_{p,2n-2} + \cdots + \hbar^n \tau_{p,0} \in \Omega^{2n}_{S^1}(M)[\hbar]$$

which is supported in an arbitrarily small neighbourhood of p and satisfies

$$\int_M \tau_{p,2n} = 1, \qquad \tau_{p,0}(p) = e(p), \qquad d_\hbar \tau_p = 0.$$

Euler class = product of weights of the action



Idea: For each fixed point there is a volume form such that the integral over the whole manifold is *localized* around the fixed point, at the point has bottom value the Euler class and is closed

Localization via an example: S^1

Lemma *Assume that the circle action is Hamiltonian and the critical points of H are all nondegenerate. Then for every fixed point p there exists an equivariant differential form*

$$\tau_p = \tau_{p,2n} + \hbar\tau_{p,2n-2} + \cdots + \hbar^n \tau_{p,0} \in \Omega^{2n}_{S^1}(M)[\hbar]$$

which is supported in an arbitrarily small neighbourhood of p and satisfies

$$\int_M \tau_{p,2n} = 1, \qquad \tau_{p,0}(p) = e(p), \qquad d_\hbar \tau_p = 0.$$

Actually, this says that the form is the *pushforward* of $1 \in H^0(N_p)$ at each fixed point:

$$\begin{array}{c} M \times_{S^1} ES^1 \\ \downarrow \uparrow f_p \\ \mathbb{C}P^\infty \end{array}$$

$$N_p = f_p(\mathbb{C}P^\infty)$$

and the pullback is

$$f_p^* f_{p_*} 1 = e(p)\hbar^n \in H^{2n}(\mathbb{C}P^\infty)$$

Localization via an example: S^1

Theorem (Duistermaat-Heckman) *Consider a circle action on a closed manifold* (M,ω) *that is generated by a Morse function* $H : M \to \mathbb{R}$. *Then,*

$$\int_M e^{-\hbar H}\frac{\omega^n}{n!} = \sum_p \frac{e^{-\hbar H(p)}}{\hbar^n e(p)}$$

for every $\hbar \in \mathbb{C}$, *for* p *critical points of* H *and* $e(p) \in \mathbb{Z}$ *is the product of weights at* p.

Localization via an example: S^1

Idea of proof:

Consider the closed form $\omega - \hbar H \in \ker d_\hbar$

Define

$$\sigma = \hbar^{n-k}(\omega - \hbar H)^k - \sum_p \frac{(-H(p))^k}{e(p)}\tau_p \qquad k \geq n$$

Is equivariantly closed and the degree 0 term vanishes on fixed points. Then, by first **Lemma**, the integral of its degree *2n* term is zero.

Then,

$$\binom{k}{n}\int_M (-H)^{k-n}\omega^n = \sum_p \frac{(-H(p))^k}{e(p)}$$

so, since for $k < n$ one can show that the integral will vanish, one has

$$\int_M (\omega - \hbar H)^k = \sum_p \frac{-\hbar H(p))^k}{\hbar^n e(p)} \qquad \blacksquare$$

Only integrating the degree *2n* term

Localization via an example: S^1

We have actually seen

$$(\Omega^\bullet_{S^1}(M)[\hbar], d_\hbar) \quad \text{is} \quad H^\bullet_{S^1}(M)$$

$$\Omega^\bullet_G(M) = (\Omega^\bullet(M) \otimes S\mathfrak{g}^*)^G$$

We were working with the bundle

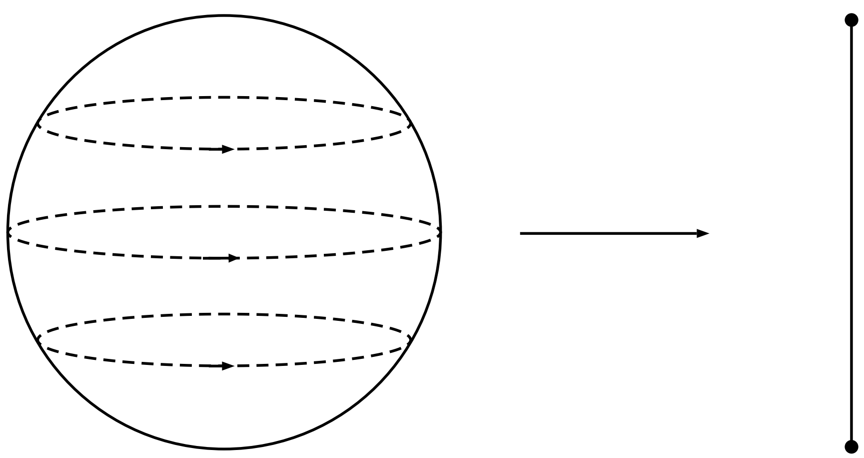
$$\begin{array}{c} M \times_{S^1} ES^1 \\ \downarrow \\ \mathbb{C}P^\infty \end{array}$$

And we can understand the *Planck constant* as

$$\hbar \in H^2(\mathbb{C}P^\infty)$$

Example

$$X = \frac{\partial}{\partial \phi} \qquad \omega = d(\cos\theta)d\phi$$



$S^1 \times S^2 \to S^2$ $\qquad$ S^2/S^1

$$\iota_X \omega = dH \Rightarrow H = -\cos\theta = -h$$

$$\text{period 1: } H = -2\pi h$$

$$\int_{S^2} d\cos\theta \, d\phi \, e^{-\hbar H} = \frac{e^{2\pi\hbar} - e^{-2\pi\hbar}}{\hbar} = 4\pi \frac{\sin(t)}{t}$$

$2\pi\hbar = it$

Localization

Theorem (**Berline-Vergne, Atiyah-Bott**) Let T be a torus acting on a manifold M, and let $\mathcal{F}$ index the components of F of the fixed point set M^T of the action of T on M. Let $\phi \in H_T^\bullet(M)$. Then,

$$\pi_*^M \phi = \sum_{F \in \mathcal{F}} \pi_*^F \left(\frac{\iota_F^* \phi}{e(\nu_F)} \right)$$

Which for the de Rham version gives

$$\int_M \phi = \sum_{F \in \mathcal{F}} \int_F \frac{\iota_F^* \phi}{e(\nu_F)}$$

3. Supersymmetry and QFT

QFT

Main ingredients:

(spacetime) (target space)

Fields $\phi : \mathcal{M} \to \mathcal{F}$

Lagrangian $\mathcal{L}(\phi, \partial\phi, \partial\partial\phi, \ldots; x)$

Action $S[\phi] = \int_{\mathcal{M}} \mathrm{dx}\, \mathcal{L}(\phi, \partial\phi, \partial\partial\phi, \ldots; x)$

Partition function $Z = \int_{\mathcal{F}} \mathcal{D}\phi\, e^{\lambda S[\phi]}$

QFT

Partition function

$$Z = \int_{\mathcal{F}} \mathcal{D}\phi \, e^{\lambda S[\phi]}$$

Expectation values of operators computed by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{F}} \mathcal{D}\phi \, \mathcal{O} \, e^{\lambda S[\phi]}$$

Supersymmetry

$$\delta(\text{fermions}) = \text{bosons}$$

$$\delta(\text{bosons}) = \text{fermions}$$

Bosonic

Supersymmetry operator Q $\quad Q^2 = B$

$$QS[\phi] = 0$$ Supersymmetric QFT

4. Supersymmetric localization

Supersymmetric localization

Writing $Q^2 = \mathcal{L}_\phi$, *

Consider an action invariant under Q , $QS = 0$, and a functional $V(\phi)$ invariant under $\mathcal{L}_\phi$, $Q^2 V = 0$.

The deformation of the action by a *Q*-exact term does not change the integral

$$\frac{d}{dt}\int e^{S+tQV} = \int \{Q, V\} e^{S+tQV} = \int \{Q, V e^{S+tQV}\} = 0$$

up to b.c.

For $t \to \infty$, the integral localizes to the critical set of QV , and for sufficiently *nice* V , it is given by a 1-loop *super*determinant.

* This notation is related to $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu$ $\qquad P_\mu = -i\partial_\mu$

Supersymmetric localization

Of course, even in this cases, the integral

$$\int_{\mathcal{M}} dx\,(S + tQV_{\rm loc})$$

can diverge. Then, a way to solve this problem is to consider $\mathcal{M}$ to be a compact manifold.

PROBLEM: we had $$QS[\phi] = \int_{\mathcal{M}} dx\,\partial_\mu(\dots)^\mu = 0$$

now we need $$QS[\phi] = \int_{\mathcal{M}} dx\,\nabla_\mu(\dots)^\mu = 0$$

Need to work with Supersymmetry in curved manifolds

Supersymmetric localization

We have the following analogy

$\mathrm{d}_\hbar$	Q
$\mathrm{d}_\hbar^2 = \mathcal{L}_X$	$\mathrm{Q}^2 = \mathcal{L}_\phi$
$\mathrm{e}(\nu_F)$	1-loop S-Det

Toy model

Poincaré-Hopf theorem's field theoretical version

Riemannian manifold X of even dimension n
Metric $g_{\mu\nu}$, a vector field V_μ on X .

Supercoordinates on the tangent bundle TX

$$(x^\mu, \psi^\mu) \qquad (\overline{\psi}_\mu, B_\mu)$$

Grassmannian

Related by *supersymmetry*:

$$\delta x^\mu = \psi^\mu \quad \delta \overline{\psi}_\mu = B_\mu$$
$$\delta \psi^\mu = 0 \quad \delta B_\mu = 0$$

$$\delta^2 = 0$$

Toy model

Poincaré-Hopf theorem's field theoretical version

Construct '*action*':

$$S(t) = \delta\Psi \qquad \Psi = \frac{1}{2}\overline{\psi}_\mu(B^\mu + 2itV^\mu + \Gamma^\sigma_{\tau\nu}\overline{\psi}_\sigma\psi^\nu g^{\mu\tau})$$

So we have the following partition function:

$$Z_X(t) = \frac{1}{(2\pi)^n}\int_X dx\, d\psi\, d\overline{\psi}\, dB\, e^{-S(t)}$$

Toy model

Poincaré-Hopf theorem's field theoretical version

Use Riemannian geometry *technology* to get

$$Z_X(t) = \frac{1}{(2\pi)^n/2} \int_X dx\, d\psi\, d\chi\, e^{-S'(t)}$$

$$S'(t) = -\frac{t^2}{2} g_{\mu\nu} V^\mu V^\nu + \frac{1}{4} R^{ab}_{\mu\nu} \chi_a \chi_b \psi^\mu \psi^\nu + it \nabla_\mu V^\nu e^a_\nu \chi_a \psi^\mu$$

It is independent of t because

$$S(t) = S(0) + t\delta V \qquad V = i\overline{\psi}_\mu V^\mu$$

We can evaluate at $t = 0$

$$Z_X(0) = \int_X dx \frac{\mathrm{Pf}(R)}{(2\pi)^{n/2}} = \chi(X)$$

Toy model

Poincaré-Hopf theorem's field theoretical version

We can also evaluate $\lim_{t\to\infty} Z_X(t)$

Assuming isolated zeroes, expanding around each saddle point and rescaling the variables, we get

$$\lim_{t\to\infty} Z_X(t) = \sum_{p_k} \frac{1}{(2\pi)^{n/2}} \int_X d\xi\, d\psi\, d\chi\, e^{-\frac{1}{2} g_{\mu\nu} H_\alpha^{(k)\mu} H_\beta^{(k)\nu} \xi^\alpha \xi^\beta + i H_\mu^{(k)\nu} e_\nu^a \chi_a \psi^\mu}$$

where $H_\sigma^{(k)\mu} = \partial_\sigma V^\mu|_{p_k}$

So, $\lim_{t\to\infty} Z_X(t) = \sum_{p_k} \frac{\det H^{(k)}}{|\det H^{(k)}|}$

$$Z_X(0) = \lim_{t\to\infty} Z_X(t)$$

$$\chi(X) = \sum_{p_k} \frac{\det H^{(k)}}{|\det H^{(k)}|}$$

Poincaré-Hopf theorem

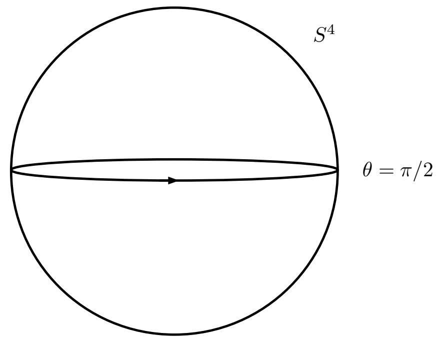
Pestun's localization

Action of N=4 SuperYangMills on a 4-sphere

$$S_{\mathcal{N}=4} = \frac{1}{g_{YM}^2} \int_{S^4} \sqrt{g} d^4 x \left(\frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi^A \Phi_A \right)$$

Want to compute the expectation value of a Wilson loop

$$W_R(C) = \mathrm{tr}_R P \exp \oint_C (A_\mu dx^\mu + i \Phi_0^E ds)$$



(supersymmetric and circular)

Pestun's localization

Get the localization locus

$$S_{bos}^{Q} = 0 \Rightarrow \begin{cases} A_\mu = 0 \quad \mu = 1, \ldots, 4 \\ \Phi_i = 0 \quad i = 5, \ldots, 9 \\ \Phi_0^E = a_E \quad \text{constant over } S^4 \\ K_i^E = -\omega_i a_E \\ K_I = 0 \end{cases}$$

Perform computations
(involve gauge fixing, index theorems, instanton corrections, etc.)

Pestun's localization

Result

$$\langle W_R(C)\rangle = \frac{1}{Z_{S^4}}\frac{1}{\mathrm{vol}(G)}\int_{\mathfrak{g}}[da]e^{-\frac{8\pi^2 r^2}{g_{YM}^2}(a,a)} Z_{1-\mathrm{loop}}(ia)|Z_{inst}(ia, r^{-1}, r^{-1}, q)|^2 \mathrm{tr}_R e^{2\pi ria}$$

Proves the conjecture that the expectation value of the Wilson loop operator in N=4 SU(N) is given by a Gaussian matrix model

Exact result highly non-trivial
Allows for checks in the AdS/CFT correspondence

SuperObrigado