Bounds on the strong spherical maximal functions

Sanghyuk Lee

Seoul National University

joint work with Juyoung Lee and Sewook Oh

Harmonic Analysis Summer Workshop
Instituto Superior Técnico, 2024, July 23

Strong maximal function

- The Hardy–Littlewood maximal function

$$M_{HL}f(x) = \sup_{r>0} \fint_{B(0,r)} |f(x-y)|dy = \sup_{r>0} \fint_{B(0,1)} |f(x-ry)|dy.$$

- (Hardy–Littlewood) $\|M_{HL}f\|_p \le C\|f\|_p$ for $1 < p \le \infty$.
- **Strong maximal function:**

$$M_{str}f(x) = \sup_{R} \fint_{R} |f(x-y)|dy,$$

where R are rectangles centered at the origin with sides parallel to the coordinates axis. Equivalently,

$$M_{str}f(x) = \sup_{t_1,\dots,t_d>0} \fint_{[-1,1]^d} |f(x-(t_1y_1, t_2y_2, \cdots, t_dy_d))dy$$

- $M_{str}f(x) \le M_1M_2\dots M_df(x)$.
- (Jessen–Marcinkiewicz–Zygmund, '35)

$$\|M_{str}f\|_p \le C\|f\|_p, \; 1 < p \le \infty.$$

- No L^1 bound is possible but weak estimate holds for $f \in L(\log^+ L)^{d-1}$.
- For a arbitrary collection of rectangles no nontrivial L^p, $p \neq \infty$, maximal estimate holds as can be shown using Besicovitch's construction.

Multi-parametric maximal functions

- For $\mathfrak{t} = (t_1, t_2, \cdots, t_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, let

$$y_{\mathfrak{t}} = (t_1 y_1, t_2 y_2, \cdots, t_d y_d).$$

For a given finite positive measure μ,

$$\mu_{\mathfrak{t}}(f) = \fint f(y_{\mathfrak{t}}) d\mu(y).$$

- Let $\delta = (\delta_1, \ldots, \delta_k) \in \mathbb{R}^k_+$ and anisotropic dilation parameter $t_j(\delta) = \prod_{l=1}^k \delta_l^{a_{j,l}}$, $a_{j,l} \geq 0$. Consider

$$Mf(x) = \sup_{\delta \in \mathbb{R}^k_+} |f * \mu_{\mathfrak{t}(\delta)}|(x).$$

Theorem (Ricci–Stein, '92)

Let μ be a positive measure. If there is a finite positive (dominating) measure ν such that

$$\sup_{1 \leq \delta_j \leq 2} \mu_{\mathfrak{t}(\delta)} \leq \nu,$$

i.e.,

$$\sup_{1 \leq \delta_j \leq 2} \int f \, d\mu_{\mathfrak{t}(\delta)} \leq \int f \, d\nu$$

for all positive continuous function with compact support, then the maximal function M is bounded on L^p for $1 < p \leq \infty$.

• The theorem also implies L^p bound on the strong maximal function. It also contains some of earlier results regrading maximal average over surfaces

• (Carlsson–Sjogren–Stromberg, '85) Let p be a homogenous function and

$$\sup_{h\in\mathbb{R}^n_+} \frac{1}{h_1\dots h_n}\int_{|y_j|\le h_j} f(x-(y,p(y))dy.$$

In this case, we may take the measure ν as follows:

$$\int f\, d\nu = \int_{|y_j|\le 2} f(y,p(y))dy.$$

• The proof of Ricci–Stein's result basically relies on L^p bounds on the maximal function along well curved homogenous curve.

Spherical and circular maximal functions (1-parameter)

• There are maximal functions for which the previous Ricci–Stein's theorem does not work since there is no dominating measure.

• **Spherical maximal function:** For $d \geq 2$, let

$$M_{sphere}f(x) = \sup_{t>0} \left| \int_{\mathbb{S}^{d-1}} f(x-ty)d\sigma(y) \right|,$$

where σ is the normalized surface measure on $\mathbb{S}^{d-1}$.

• There is no finite measure ν such that

$$\sup_{1 \leq t \leq 2} \int f(y)d\sigma_t(y) \leq \int f\, d\nu.$$

Theorem (Stein '76, $d \geq 3$; Bourgain '86, $d = 2$)

M_{sphere} is bounded on L^p if and only if $p > d/(d-1)$.

• Both results rely on smoothing property of the spherical averages (decay of Fourier transform of σ).

Maximal averages over hypersurfaces (1-parameter)

• The same results hold also for smooth compact surfaces with nonvanishing Gaussian curvature.

• There are **various results concerning degenerate surfaces** whose curvature vanishes (Sogge-Stein, '85; ... Muller–Ikromov–Kempe, '10; etc). The problem is better understood in $\mathbb{R}^3$ since degeneracy Δ of 2-dimensional surfaces is easier to characterize. However, in higher dimensions these problems are largely open except for special cases, for example, finite type convex surfaces (Nagel-Seeger-Wainger, '93)

• There are a lot of generalizations such as $L^p - L^q$ bounds on the local maximal function

$$M_{sphere}^{local} f(x) = \sup_{1 \le t \le 2} \Big| \int_{\mathbb{S}^{d-1}} f(x - ty) d\sigma(y) \Big|,$$

maximal bound depending on the dimension of dilation set, estimates relative to fractal measure, and problems in different settings such as certain groups and discrete settings.

Multi-parameter maximal averages over hypersurfaces

• Unlike the Hardy–Littlewood maximal function, obtaining L^p bound for the maximal functions given by hypersurfaces is far less trivial. However, there have been attempts to obtain multiparametric extensions of those maximal functions.

Theorem (Marletta–Ricci,'98)

Let $d \geq 2$. Let $\Gamma : \mathbb{R}^{d-1} \setminus \{0\} \to \mathbb{R}$ be a smooth function homogeneous degree $a > 0$. Suppose the hypersurface $x_d = \Gamma(\bar{x})$ has non-vanishing Gaussian curvature, the maximal operator

$$\sup_{a,b>0} \left| \int_{y \in \mathbb{R}^{d-1}} f(x - (ay, b\Gamma(y)))dy \right|$$

is bounded in L^p if and only if $p > d/(d-1)$.

• This result heavily relies on homongeneity of the surface, which make it possible to deduce the parameter maximal bounds from that of one parameter.

Results based on L^2 argument

- (Yongkum Cho '98, Yaryong Heo '16) If

$$|\widehat{d\mu}(\xi)| \lesssim \prod_{i=1}^{d}(1+|\xi_i|)^{-a_i}, \quad a_i > 1/2,$$

then the maximal operator

$$f \mapsto \sup_{t\in\mathbb{R}^d_+} |f * d\mu_{\mathrm{t}}|$$

is bounded in L^p for a "suitable" range of p depending $a_1, \ldots, a_d$.

- The results rely on the L^2 argument and the assumption is somewhat too strong to give L^p bound on the maximal function on hypersufaces.

Main topic: Strong spherical maximal function

- The strong (d-parametric) spherical maximal function

$$\mathcal{M}f(x) = \sup_{\mathfrak{t}\in\mathbb{R}_+^d} |f * \sigma_{\mathfrak{t}}(x)| = \sup_{\mathfrak{t}\in\mathbb{R}_+^d} \Big|\int_{\mathbb{S}^{d-1}} f(x-(t_1y_1,\dots,t_dy_d))d\sigma(y)\Big|$$

Theorem (Lee– L.–Oh, '23)

Let $d \geq 3$. Then, the strong spherical maximal function $\mathcal{M}$ is bounded on L^p if $p > 2(d+1)/(d-1)$.

- The range of p is far from being optimal. A Knapp type example shows $\mathcal{M}_d$ fails to be bounded on L^p if $p < (d+1)/(d-1)$.

- When $d = 2$, $\mathcal{M}$ is bounded on L^p if and only if $p > 3$.

• Consequences of the strong maximal estimate that strengthen the earlier results:

- Let $p > 2(d+1)/(d-1)$. For all $f \in L^p$,

$$f * \sigma_{\mathfrak{t}} \to f \quad \text{a.e. as} \quad \mathfrak{t} \to 0.$$

- Let E be a set of measure zero and $\tilde{\mathfrak{t}} : \mathbb{R}^d \to \mathbb{R}^d_+$ and $\mathbb{S}_{\tilde{\mathfrak{t}}(x)}$ denote the ellipsoid with axis length $\tilde{\mathfrak{t}}_1(x), \dots, \tilde{\mathfrak{t}}_d(x)$. Then,

$$|E \cap (\mathbb{S}_{\tilde{\mathfrak{t}}(x)} + x)|_{d-1} = 0 \;\text{ a.e. } x,$$

where $|\cdot|_{d-1}$ denotes $(d-1)$ dimensional Hausdorff measure. Indeed,

$$|E \cap (\mathbb{S}_{\tilde{\mathfrak{t}}(x)} + x)|_{d-1} \le \int \chi_E(x - \tilde{\mathfrak{t}}(x)y) d\sigma(y) \le M\chi_E(x).$$

By the maximal bounds $\|M\chi_E\|_p \lesssim |E|^{\frac{1}{p}}$. Since E is of measure zero, $M\chi_E = 0$ a.e. x.

t–localized maximal function

- Multi-parameter Littlewood–Paley decomposition:

$$\mathcal{M}f(x) = \sup_{k\in\mathbb{Z}^d} \sup_{t_j\in[2^{-k_j},2^{-k_j+1}]} \Big| \sum_{l\in\mathbb{Z}^d} P_l f * \sigma_t(x) \Big|,$$

where

$$P_l f(\xi) = \Big(\prod_{j=1}^{d} \beta(2^{-l_j}\xi_j)\widehat{f}(\xi) \Big)^{\vee}$$

- For fixing k and $t_j \in [2^{-k_j}, 2^{-k_j+1}]$,

$$\Big| \sum_{l\in\mathbb{Z}^d} P_l f * \sigma_t(x) \Big| \le \Big| \sum_{l_j\le k_j} P_l f * \sigma_t(x) \Big| + \Big| \sum_{l_j> k_j} P_l f * \sigma_t(x) \Big|$$

The lower frequency part is bounded by the strong maximal function. For the high frequency part we use the high frequency decay in maximal bounds after scaling.

High frequency decay estimates

- Via scaling

$$\sum_{l_j > k_j} P_{\mathfrak{l}} f * \sigma_{\mathfrak{t}}(x) = \sum_{l_j > k_j} P_{\mathfrak{l}-\mathfrak{k}} f(2^{-\mathfrak{k}} \cdot) * \sigma_{2^{-\mathfrak{k}}\mathfrak{t}}(2^{\mathfrak{k}} x),$$

where $2^{-\mathfrak{k}}\mathfrak{t} = (2^{-k_1} t_1, \dots, 2^{-k_d} t_d)$. The proof of theorem essentially reduces to showing bound on a local maximal operator

$$\mathcal{M}_{loc} f(x) = \sup_{\mathfrak{t} \in [1,2]^d} |f * \sigma_{\mathfrak{t}}(x)|.$$

Proposition

Let $d \geq 3$ and $p > 2(d+1)/(d-1)$. Then, for some $\delta_0 > 0$,

$$\|\mathcal{M}_{loc} f\|_{L^p} \lesssim 2^{-\delta_0 j} \|f\|_{L^p}, \quad j \geq 0$$

whenever $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j := \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$.

- Asymptotic expansion:

$$\widehat{\sigma_{\mathfrak{t}}}(\xi) = e^{i|\xi_{\mathfrak{t}}|} a_+(\xi_{\mathfrak{t}}) + e^{-i|\xi_{\mathfrak{t}}|} a_-(\xi_{\mathfrak{t}}), \quad |\xi| \geq 1$$

and $a_\pm$ satisfies

$$|a_\pm(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}.$$

- **Multi-parameter wave operators:**

$$\mathcal{U}_\pm f(x, \mathfrak{t}) = a(x, \mathfrak{t}) \int e^{i\Phi_\pm(x,\mathfrak{t},\xi)} \widehat{f}(\xi) d\xi,$$

where $a \in C_c^\infty(B(0,2) \times [2^{-1}, 2^2]^d)$ and

$$\Phi_\pm(x, \mathfrak{t}, \xi) = x \cdot \xi \pm |\xi_{\mathfrak{t}}| = x \cdot \xi \pm \sqrt{(t_1\xi_1)^2 + \cdots + (t_d\xi_d)^2}.$$

• **(1-parameter) sharp local smoothing estimate for the wave operator** (Bourgain–Demeter, '15)

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^d\times[1,2])} \lesssim 2^{(\frac{d-1}{2}-\frac{d}{p}+\epsilon)j}\|f\|_{L^p}$$

for $p \ge 2(d+1)/(d-1)$ whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_j$. Sogge's local smoothing conjecture tells that the estimate holds for $p \ge 2d/(d-1)$.

• Changing variables $\mathfrak{t} = (t, tt_2, \dots, tt_d)$ and scaling give

$$\|\mathcal{U}_{\pm}f\|_{L^p_{x,\mathfrak{t}}(\mathbb{R}^d\times[1,2]^d)} \lesssim 2^{(\frac{d-1}{2}-\frac{d}{p}+\epsilon)j}\|f\|_{L^p}, \quad \epsilon > 0$$

for $p \ge 2(d+1)/(d-1)$ whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_j$. Thus,

$$\|f * \sigma_{\mathfrak{t}}\|_{L^p_{x,\mathfrak{t}}(\mathbb{R}^d\times[1,2]^d)} \lesssim 2^{(-\frac{d}{p}+\epsilon)j}\|f\|_{L^p}, \quad \epsilon > 0.$$

• This and Sobolev imbedding give

$$\|\mathcal{M}_{loc}f\|_{L^p} \lesssim \|\prod_{j=1}^{d}(1+|\partial_{t_j}|)^{\frac{1}{p}+}f * \sigma_{\mathfrak{t}}\|_{L^p_{x,\mathfrak{t}}} \lesssim 2^{\epsilon j}\|f\|_{L^p}$$

whenever $\operatorname{supp}\widehat{f} \subset \mathbb{A}_j$.

Multiparameter local smoothing: extra smoothing

• **Do more averaging parameter gives additional smoothing?** To get the desired maximal estimate, we only need to get additional smoothing from more than 1 parameters.

Proposition

Let $p > 2(d+1)/(d-1)$ and $j \geq 0$. Then, for some $\delta_0 = \delta_0(p) > 0$,

$$\|\mathcal{U}_{\pm} f\|_{L^p_{x,t}} \lesssim 2^{(\frac{d-1}{2} - \frac{d}{p} - \delta_0)j} \|f\|_{L^p}, \quad \operatorname{supp} \widehat{f} \subset \mathbb{A}_j.$$

• For a small constant $c > 0$, set

$$\mathbb{A}_j^{near}(c) = \bigcup_{k=1}^{d} \{\xi \in \mathbb{A}_j : ||\xi|^{-1}\xi - e_k| < c\},$$

where e_k denotes the k-th standard unit vector. We also set

$$\mathbb{A}_j^{away}(c) = \mathbb{A}_j \setminus \mathbb{A}_j^{near}(c).$$

Decoupling inequalities

- Let $\mathfrak{S}(2^{-j/2}) = \{\nu\}$ denote a partition of unity on $\mathbb{S}^{d-1}$ subordinated to boundedly overlapping caps of diameter $\sim 2^{-j/2}$ with $\partial^\alpha \nu = O(2^{j|\alpha|/2})$. For $\nu \in \mathfrak{S}(2^{-j/2})$,

$$\widehat{f_\nu}(\xi) = \widehat{f}(\xi)\nu(\xi/|\xi|).$$

Proposition

Let $p \ge 6$ and $j \ge 0$. Suppose $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j^{near}(c)$. Then, if $c > 0$ is small enough, for any $\epsilon > 0$ and $M > 0$ we have

$$\|\mathcal{U}_\pm f\|_{L^p_{x,t}} \lesssim 2^{(\frac{d-1}{2} - \frac{2d-2}{p} + \epsilon)j} \Big(\sum_{\nu \in \mathfrak{S}(2^{-j/2})} \|\mathcal{U}_\pm f_\nu\|^p_{L^p_{x,t}} \Big)^{1/p} + 2^{-Mj} \|f\|_{L^p}.$$

- The smoothing estimate follows since

$$\sum_\nu \|\mathcal{U}_\pm f_\nu\|^p_{L^p_{x,t}} \lesssim \|f\|_p^p, \quad 2 \le p \le \infty.$$

Proposition

Let $p \ge 6$ and $j \ge 0$. Suppose $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j^{away}(c)$ for a constant $c > 0$. Then, for any $\epsilon > 0$ and $M > 0$ we have

$$\|\mathcal{U}_\pm f\|_{L^p_{x,t}} \lesssim 2^{(\frac{d-1}{2} - \frac{d+1}{p} + \epsilon)j} \Big(\sum_{\nu \in \mathfrak{S}(2^{-j/2})} \|\mathcal{U}_\pm f_\nu\|^p_{L^p_{x,t}} \Big)^{1/p} + 2^{-Mj} \|f\|_{L^p}.$$

- **Linearization.** Observe that

$$\nabla_{x,t}(x\cdot\xi+|\xi_t|)=\Big(\xi,\frac{t_1\xi_1^2}{|\xi_t|},\frac{t_2\xi_2^2}{|\xi_t|},\cdots,\frac{t_d\xi_d^2}{|\xi_t|}\Big).$$

If ξ is contained in a narrow conic neighborhood of e_d, setting

$$u_j=\xi_j/\xi_d,\quad j=1,\dots,d-1,$$

the left hand side can be regarded as (essentially) a conic (homogeneous) extension of the surface

$$(u_1,u_2,\cdots,u_{d-1},1,u_1^2,u_2^2,\cdots,u_{d-1}^2,|u|^4),\quad |u|\ll 1,$$

- Standard strategy to obtain decoupling inequality:
  - Quadratic surfaces (Guo–Oh–Zhang–Zorin-Kranich, '23)
  - Conic extension (Bourgain–Demeter, '15)
  - Variable coefficient generalization (e.g., Beltran–Hickman–Sogge, '20): Suppose decoupling inequality for the surface $\xi\mapsto\nabla_z\Phi(z_0,\xi)$ for each z_0 in a uniform manner (Stability of decoupling bounds). Then, the corresponding decoupling inequality holds for the operator

$$\int e^{i\Phi(z,\xi)}a(z,\xi)f(\xi)d\xi.$$

Linearization and conical extension

- When $\operatorname{supp}\widehat{f} \subset \mathbb{A}_j^{near}(c)$ (near e_d):
  - (fixing s) consider
$$\Phi_s^{near}(\mathfrak{t}', u) = \sqrt{s + \sum_{j=1}^{d-1} t_j u_j^2}, \quad \mathfrak{t}' = (t_1, \dots, t_{d-1}).$$
  - Essentially, $\nabla_{\mathfrak{t}'}\Phi_s^{near}(u) = (2\sqrt{s})^{-1}(u_1^2, \dots, u_{d-1}^2) + O(c|u|^4)$.
  - Decoupling bound is given by
$$C2^{j(d-1)(\frac{1}{2}-\frac{2}{p})}.$$
- When $\operatorname{supp}\widehat{f} \subset \mathbb{A}_j^{away}(c)$:
  - Fixing (d-2 variables) $\mathfrak{t}''$, consider $f \mapsto \mathcal{U}_+ f(\cdot, t(\mathfrak{t}'', 1), ts)$.
  - After suitable change of variables (2-parameter smoothing):
$$\Phi^{away}(t, s, u) = t\sqrt{|u|^2 + s^2},$$
  - $\nabla_{s,t}\Phi^{away}(t, s, u) \sim (u_2^2 + \cdot + u_{d-1}^2, a^2 u_1^2)$ with $a \neq 0$.
  - Decoupling bound is given by
$$C2^{j(\frac{d-2}{2}-\frac{d-1}{p})}2^{j(\frac{1}{2}-\frac{2}{p})} = C2^{(\frac{d-1}{2}-\frac{d+1}{p})j}.$$

Multiparameter maximal circular averages: $d = 2$

- R_θ denotes a rotation in $\mathbb{R}^2$ such that $R_\theta(1,0) = (\cos\theta, \sin\theta)$. Define a measure on the rotated ellipse

$$\mathbb{E}_{\mathfrak{t}}^\theta = \{R_\theta y_{\mathfrak{t}} : y \in \mathbb{S}^1\}$$

by

$$\langle f, \sigma_{\mathfrak{t}}^\theta \rangle = \int_{\mathbb{S}^1} f\big(R_\theta y_{\mathfrak{t}}\big) d\sigma(y).$$

- **Elliptic maximal function:** Let $\mathbb{J} \subset \mathbb{R}_+$ be a compact interval and

$$\mathfrak{M}f(x) = \sup_{(\theta,\mathfrak{t}) \in \mathbb{T} \times \mathbb{R}_+^2 : t_1/t_2 \in \mathbb{J}} |f * \sigma_{\mathfrak{t}}^\theta(x)|.$$

- Any nontrivial L^p fails if $\mathbb{J}$ is not compact.
- Compared with the circular maximal function, L^p bound on $\mathfrak{M}$ becomes highly nontrival. Erdoğan ('03) proved $L^{4,\frac{1}{6}}$–L^4 estimate but no L^p-estimate was known until recently.

Possible range of p for the maximal bounds

- (Erdoǵan, '03) $\mathfrak{M}$ is bounded on L^p only if $p > 4$.

- Let C_δ be a δ neighborhood of the unit circle $\mathbb{S}^1$. For every $x \in B(0, 1/2)$, thanks to three free parameters, there is a rotated ellipse $(\mathbb{E}_t^\theta + x)$ centered at x which meets $\mathbb{S}^1$ with contact order 3. Thus,

$$\text{length}\Big[(\mathbb{E}_t^\theta + x) \cap C_\delta\Big] \sim \delta^{\frac{1}{4}},$$

and

$$\mathfrak{M}\chi_{C_\delta}(x) \gtrsim \delta^{\frac{1}{4}}, \quad \forall x \in B(0, 1/2).$$

- L^p maximal bound implies

$$\delta^{\frac{1}{4}} \lesssim \delta^{\frac{1}{p}},$$

thus, letting $\delta \to 0$ gives $p \geq 4$. Further elaboration gives failure for $p = 4$.

- Similar argument shows the strong circular (2-parameter) maximal function $\mathcal{M}$ can be bounded on L^p only if $p > 3$.

Theorem (Lee–L.–Oh, '23)

If $p > 12$, then

$$\|\mathfrak{M}f\|_{L^p} \lesssim \|f\|_{L^p}.$$

If $p > 4$, then

$$\|\mathcal{M}f\|_{L^p} \lesssim \|f\|_{L^p}.$$

- Let R_θ^* denote the transpose of R_θ and

$$\Phi_\pm^\theta(x,\mathfrak{t},\xi) = x\cdot\xi \pm |(R_\theta^*\xi)_\mathfrak{t}|, \quad \xi\in\mathbb{R}^2.$$

- As before, we consider the operator

$$\mathcal{U}_\pm^\theta f(x,\mathfrak{t}) = a(x,\mathfrak{t})\int_{\mathbb{R}^2} e^{i\Phi_\pm^\theta(x,\mathfrak{t},\xi)}\widehat{f}(\xi)d\xi.$$

Theorem

Let us set $\Delta = \{\mathfrak{t}\in(2^{-1},2^2)^2 : t_1 = t_2\}$. *Suppose* $\operatorname{supp} a(x,\cdot)\cap\Delta = \emptyset$ *for all* $x\in B(2,0)$. *Then, if* $p\ge 20$, *we have*

$$\|\mathcal{U}_\pm^\theta f\|_{L^p_{x,\mathfrak{t},\theta}} \le C\|f\|_{L^p_\alpha}, \quad \alpha > 1/2 - 4/p.$$

For $p\ge 12$, *we have*

$$\|\mathcal{U}_\pm^0 f\|_{L^p_{x,\mathfrak{t}}} \le C\|f\|_{L^p_\alpha}, \quad \alpha > 1/2 - 3/p.$$

- The smoothing orders are sharp (but the range of p is not sharp).
  - $\frac{1}{2} - \frac{2}{p}$ (1-parameter smoothing)
  - $\frac{1}{2} - \frac{3}{p}$ (2-parameter smoothing)
  - $\frac{1}{2} - \frac{4}{p}$ (3-parameter smoothing)

- Chen–Guo–Yang (A multi-parameter cinematic curvature, arXiv:2306.01606) obtained multiparameter smoothing estimate which generalizes the smoothing estimates.

- Combining this and Sobolev imbedding gives

$$\|\mathfrak{M}_{loc} f\|_{L^p} \lesssim 2^{(\epsilon - \frac{1}{p})j} \|f\|_{L^p}, \quad p > 20$$

$$\|\mathcal{M}_{loc} f\|_{L^p} \lesssim 2^{(\epsilon - \frac{1}{p})j} \|f\|_{L^p}, \quad p > 12$$

whenever $\operatorname{supp} \widehat{f} \subset \mathbb{A}_j$. Further interpolation with

$$\|\mathfrak{M}_{loc} f\|_{L^4} \lesssim 2^{(\epsilon + \frac{1}{4})j} \|f\|_{L^4}, \qquad \|\mathcal{M}_{loc} f\|_{L^4} \lesssim 2^{\epsilon j} \|f\|_{L^4},$$

which is consequence of the L^4 local smoothing estimate Guth–Wang–Zhang ('20) extends the ranges.

3-parameter smoothing estimate

- Note that

$$\nabla_{x,\mathfrak{t},\theta}\Phi_+^\theta(x,\mathfrak{t},\xi) = \Big(\xi, \frac{t_1(R_\theta^*\xi)_1^2}{|(R_\theta^*\xi)_\mathfrak{t}|}, \frac{t_2(R_\theta^*\xi)_2^2}{|(R_\theta^*\xi)_\mathfrak{t}|}, \frac{2(t_1^2-t_2^2)(R_\theta^*\xi)_1(R_\theta^*\xi)_2}{|(R_\theta^*\xi)_\mathfrak{t}|}\Big) \in \mathbb{R}^5.$$

- By rotational symmetry, we may set $\theta = 0$. Letting $(\xi_1, \xi_2) = (r, ru)$ (near e_1), we have

$$\nabla_{x,\mathfrak{t},\theta}\Phi_+^0(x,\mathfrak{t},r,ru) = r(1,\Upsilon(u)), \quad u \in (-2,2).$$

- If $t_1 \neq t_2$, then Υ is nondegenerate, i.e.,

$$\det(\Upsilon'(u), \Upsilon''(u), \Upsilon'''(u), \Upsilon''''(u)) \neq 0.$$

This allows us to use decoupling inequality for the nondegenerate curve (Bourgain–Demeter–Guth, '16), with which we follows the standard strategy conical extension, variable coefficient generalization.

Further developments in $\mathbb{R}^2$

- Maximal averages over δ-neighborhood $\mathbb{E}^{\theta}_{t}(\delta)$ of ellipses $\mathbb{E}^{\theta}_{t}$:

$$M_\delta f(x) = \sup_{t\in[1,2]^2} \fint_{\mathbb{E}^{0}_{t}(\delta)} f(x-y)dy, \quad \mathfrak{M}_\delta f(x) = \sup_{(\theta,t)\in\mathbb{T}\times[1,2]^2} \fint_{\mathbb{E}^{\theta}_{t}(\delta)} f(x-y)dy.$$

- Pramanik–Yang–Zahl(*Furstenberg-type problem for circles, and a Kaufman-type restricted projection theorem in $\mathbb{R}^3$, arXiv:2207.02259*)

$$\|M_\delta f\|_3 \lesssim \delta^{-\epsilon}\|f\|_3, \quad \forall\epsilon > 0.$$

- Zahl(*On Maximal Functions Associated to Families of Curves in the Plane, arXiv:2306.01606*)

$$\|\mathfrak{M}_\delta f\|_4 \lesssim \delta^{-\epsilon}\|f\|_4, \quad \forall\epsilon > 0.$$

- These estimates was obtained by technique in discrete incidence geometry. Local smoothing estimates allows to remove the ϵ-loss.

Theorem ($d = 2$)

$\mathfrak{M}$ is bounded on L^p if and only if $p > 4$. $\mathcal{M}$ is bounded on L^p if and only if $p > 3$.

- Those maximal bounds were further generalized by Zahl (for the curves in $\mathbb{R}^2$) to m parameter maximal functions under the multi-parameter cinematic curvature condition.

Conclusion

• For $d \geq 3$, find optimal range of boundedness of the strong spherical maximal function

$$\mathcal{M}f(x) = \sup_{\mathfrak{t} \in \mathbb{R}_+^d} \left| \int_{\mathbb{S}^{d-1}} f(x - (t_1 y_1, \ldots, t_d y_d)) d\sigma(y) \right|$$

Discrete incidence geometric estimates in higher dimensions?

• Multiparameter local smoothing estimate for

$$\mathcal{U}_\pm f(x, \mathfrak{t}) = a(x, \mathfrak{t}) \int e^{i(x \cdot \xi \pm |\xi_{\mathfrak{t}}|)} \widehat{f}(\xi) d\xi,$$

where $a \in C_c^\infty(B(0,2) \times [2^{-1}, 2^2]^d)$. Sharp smoothing order and optimal range of p? More generally, are there underlining principles of multi-parametric local smoothing?

• Considering what is known about the maximal functions given by hypersurfaces over the last several decades, there are many natural, possible multi-parametric generalizations of the known results, such as $L^p - L^q$ bounds and bounds depending on (multi-parameter) dilation sets, etc.

Thank you very much !