Semiclassical L^p quasimode restriction estimates in two dimensions

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Outline

1. Quasimode restriction estimate
   - Laplace-Beltrami operator and Quasimode
   - Previous results
   - Main Theorem
2. Ideas of proof
   - Reduction
   - Main part

Main objects

- Δ_g: Laplace-Beltrami operator on a compact Riemannian manifold M
- $1_{[\lambda-1,\lambda+1]}(\sqrt{-\Delta_g})$: Spectral projection
- φ_λ: Eigenfunction of Δ_g such that $-\Delta_g \varphi_\lambda = \lambda^2 \varphi_\lambda$

Problem (Eigenfunction restriction estimate)

Let H be a submanifold of M. Determine $\delta(H,p)$ satisfying

$$\|\varphi_\lambda\|_{L^p(H)} \lesssim \lambda^{\delta(H,p)} \|\varphi_\lambda\|_{L^2(M)}$$

for $p \geqslant 2$.

- $P(h)$: semiclassical pseudodifferential operator with a symbol $p(x, \xi, h)$, in that,

$$P(h)f = p(x, hD, h)f(x) = (2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\xi} p(x, \xi, h) f(y) dy d\xi.$$

- We call a family of functions $f = f(h)$ an $O(h)$ quasimode if $f(h)$ is a L^2 normalized function such that

$$P(h)f = O_{L^2}(h).$$

- $1_{[h^{-1}-1, h^{-1}+1]}(\sqrt{-\Delta_g})f$ is a quasimode when $P(h) = -h^2\Delta_g - 1$.

Problem

Let H be a submanifold of M. For $p \geqslant 2$, determine $\delta(H, p)$ satisfying

$$\|f\|_{L^p(H)} \lesssim h^{-\delta(H,p)},$$

for a quasimode f.

When $H = M$

Theorem (Sogge)

Let $\dim M =: n \geqslant 2$*. Then*

$$\|1_{[\lambda-1,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(M)} \lesssim \lambda^{\delta(p)}$$

where

$$\delta(p) = \begin{cases} \frac{n-1}{4} - \frac{n-1}{2p}, & 2 \leqslant p \leqslant \frac{2(n+1)}{n-1}, \\ \frac{n-1}{2} - \frac{n}{p}, & \frac{2(n+1)}{n-1} \leqslant p \leqslant \infty. \end{cases}$$

Corollary

$$\|\varphi_\lambda\|_{L^p(M)} \lesssim \lambda^{\delta(p)} \|\varphi_\lambda\|_{L^2(M)}$$

When $dimH =: k < n$

$$\|1_{[\lambda-1,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(H)} \lesssim \lambda^{\delta(H,p)}$$

By Burq-Gerard-Tzvetkov and Hu, the above holds:

- When $k \leqslant n-2$, with $\delta(H,p) = \frac{n-1}{2} - \frac{k}{p}$ except for $k = n-2, p = 2$ (in this case, log loss remains)
- When $k = n-1$, with

$$\delta(H,p) = \begin{cases} \frac{n-1}{4} - \frac{n-2}{2p}, & 2 \leqslant p \leqslant \frac{2n}{n-1}, \\ \frac{n-1}{2} - \frac{n-1}{p}, & \frac{2n}{n-1} \leqslant p \leqslant \infty. \end{cases}$$

- When $k = n-1$ and H is well-curved w.r.t. geodesics, with

$$\delta(H,p) = \begin{cases} \frac{n-1}{3} - \frac{2n-3}{3p}, & 2 \leqslant p \leqslant \frac{2n}{n-1}, \\ \frac{n-1}{2} - \frac{n-1}{p}, & \frac{2n}{n-1} \leqslant p \leqslant \infty. \end{cases}$$

Curved w.r.t. geodesics

When $n = 2$, we say that H is well-curved w.r.t. geodesics, if a curve H has nonvanishing geodesic curvature.
Roughly speaking, for all s_0, there exists a geodesic z_{s_0} s.t.

$$|\gamma(s) - z_{s_0}(s)| \sim |s - s_0|^2$$

where H is given by a curve $s \mapsto \gamma(s)$.

Question

What happens if

$$|\gamma(s) - z_{s_0}(s)| \sim |s - s_0|^k$$

for some s_0 *and* $k \geqslant 3$*?*

When $M = \mathbb{T}$

Theorem (Hu)

Let $M = \mathbb{T}$ *and* $2 \leqslant p \leqslant 4$*. Assume that* $\gamma(s) = (s, s^k)$*. Then*

$$\|1_{[\lambda-1,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(\mathbb{T}) \to L^p(\gamma)} \lesssim \lambda^{\delta(k,p)},$$

where

$$\delta(k,p) = \frac{k}{2(2k-1)} - \frac{1}{(2k-1)p}.$$

Main result

- Set
$$s \mapsto z_{(\gamma(s_0),\dot\gamma(s_0))}(s)$$
is the geodesic satisfying $z_{(\gamma(s_0),\dot\gamma(s_0))}(0) = \gamma(s_0)$ and $\dot z_{(\gamma(s_0),\dot\gamma(s_0))}(0) = \dot\gamma(s_0)$.

We say that γ and the geodesics have the maximal order of contact $k-1$ if for all s_0
$$\sum_{j=2}^{k} |\gamma^{(j)}(s_0) - z^{(j)}_{(\gamma(s_0),\dot\gamma(s_0))}(0)| \neq 0$$
and there exist s_1 s.t. for $j \in \{0, \cdots, k-1\}$,
$$\gamma^{(j)}(s_1) = z^{(j)}_{(\gamma(s_1),\dot\gamma(s_1))}(0) \text{ and } \gamma^{(k)}(s_1) \neq z^{(k)}_{(\gamma(s_1),\dot\gamma(s_1))}(0).$$

- $|\gamma(s) - z_{s_0}(s)| \gtrsim |s-s_0|^k$ and $|\gamma(s) - z_{s_1}(s)| \sim |s-s_1|^k$.

Main result

Theorem (O.-Ryu)

Let $n = 2$ and $2 \leqslant p \leqslant 4$. Assume that γ and the geodesics have the maximal order of contact $k - 1$. Then

$$\|1_{[\lambda-1,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^p(\gamma)} \lesssim \lambda^{\delta(k,p)},$$

where

$$\delta(k,p) = \frac{k}{2(2k-1)} - \frac{1}{(2k-1)p}.$$

Corollary

$$\|\varphi_\lambda\|_{L^p(\gamma)} \lesssim \lambda^{\delta(k,p)} \|\varphi_\lambda\|_{L^2(M)}$$

Idea of proof

- Reduction to oscillatory integrals
- Dyadic decomposition
- Maximal order of contact

Basic reduction

It is enough to show ...

- By interpolation with the known L^4 estimate,

$$\|1_{[\lambda-1,\lambda+1]}(\sqrt{-\Delta_g})\|_{L^2(M)\to L^2(\gamma)} \lesssim \lambda^{\frac{k-1}{2(2k-1)}} = \lambda^{\delta(k,2)}.$$

- Using resolvent $(\lambda P + i)^{-1}$ with $P = -\lambda^{-2}\Delta_g - 1$,

$$\|\check{\chi}(\lambda P)f\|_{L^2(\gamma)} \lesssim \lambda^{\delta(k,2)}\|f\|_2.$$

for every $\chi \in C^\infty(\mathbb{R}\backslash\{0\})$ such that $|\chi^{(m)}(s)| \lesssim (1+|s|)^{-N}$.

- By TT^*-argument,

$$\|\check{\chi}(\lambda P)\|_{L^2(\gamma)\to L^2(\gamma)} \lesssim \lambda^{2\delta(k,2)}.$$

- Up to small error,

$$\check{\chi}(\lambda P)f(u) = \lambda^2 \int \chi(s) e^{i\lambda(\phi(s,\gamma(u),\eta)-\gamma(v)\cdot\eta)} b(s,\gamma(u),\eta) f(v)\, d\eta\, ds\, dv,$$

where ϕ is a solution to

$$\partial_s \phi(s,x,\eta) + p(x, \partial_x \phi(s,x,\eta)) = 0, \quad \phi(0,x,\eta) = x\cdot\eta,$$

$b \in C_c^\infty(\mathbb{R} \times T^*\mathbb{R}^2)$, and p is a symbol satisfying $p(x,\xi) = |\xi|_g^2 - 1$.

- $$z_{(x,\partial_x \phi)}(-s) = \partial_\eta \phi$$

- Main part: $|s| \sim 2^{-j} \gtrsim \lambda^{-1}$ and $|u-v| \sim 2^{-j}$
- Using stationary phase method w.r.t. s, η, it is enough to show that

$$\sum_j \|T_j\|_{L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})} \lesssim \lambda^{2\delta(k,p)}$$

where

$$T_j f(u) := \lambda^{\frac{1}{2}} 2^{\frac{j}{2}} \int B_j(u,v) e^{i\lambda\Phi(u,v)} f(v) dv,$$

$$\Phi(u,v) = \phi(s(u,v), \gamma(u), \eta(u,v)) - \gamma(v) \cdot \eta(u,v),$$

$(s(u,v), \eta(u,v))$ is a critical point of
$(s,\eta) \mapsto \phi(s, \gamma(u), \eta) - \gamma(v) \cdot \eta$, and $|\operatorname{supp} B_j(u,\cdot)| \lesssim 2^{-j}$.

Estimates on Φ

By the L^2-method, the matter is reduced to

Proposition

Assume that $|u_0| \lesssim 2^{-j}$, $|u - u_0|, |v - v_0|, |w - v_0| \leqslant \varepsilon 2^{-j}$ *for small* $\varepsilon > 0$. *Then the following estimates hold.*

i) *Upper bounds of the phase difference. For* $m \geqslant 1$,

$$\left|\partial_u^m(\Phi(u,v) - \Phi(u,w))\right| \lesssim 2^{-(2k-2-m)j}|v - w|.$$

ii) *Lower bounds of the phase difference.*

$$2^j\left|\partial_u(\Phi(u,v)-\Phi(u,w))\right|+\left|\partial_u^2(\Phi(u,v)-\Phi(u,w))\right| \gtrsim 2^{-(2k-4)j}|v-w|.$$

Relation between $\partial_u \partial_v \Phi$ and geodesic

- Recall that $\Phi(u,v) = \phi(s(u,v), \gamma(u), \eta(u,v)) - \gamma(v)\cdot\eta(u,v)$, $s \mapsto z_{(x,\xi)}(s)$ is geodesic, and

$$\partial_s \phi(s,x,\eta) + p(x, \partial_x \phi(s,x,\eta)) = 0, \quad z_{(x,\partial_x\phi)}(-s) = \partial_\eta \phi,$$

- $\partial_u \Phi = \dot\gamma(u)\cdot k(u,v)$ where $k(u,v) = \partial_x \phi(s(u,v), \gamma(u), \eta(u,v))$.
  - $|k(u,\cdot)| \equiv 1$
  - $|\partial_u\partial_v\Phi| = |\dot\gamma\cdot\partial_v k| \sim |(\dot\gamma - k)\cdot\partial_v k| \sim |\partial_v|\dot\gamma - k|^2|$
- $\partial_\eta \phi(s(u,v), \gamma(u), \eta(u,v)) = \gamma(v)$.
  - $\gamma(v) = z_{(\gamma(u),k(u,v))}(-s(u,v))$
  - $|z_{(\gamma(u),\dot\gamma(u))}(v-u) - \gamma(v)| \sim |u-v||\dot\gamma(u) - k(u,v)|$

$$\partial_u\partial_v\Phi(u,v) \sim \left|\partial_v\right|\frac{z_{(\gamma(u),\dot\gamma(u))}(v-u) - \gamma(v)}{v-u}\left|^2\right| = O(2^{-(2k-3)j}).$$

Thank you for your attention!