On Hardy Spaces associated with the twisted Laplacian and sharp estimates for the corresponding wave operator

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Hardy spaces corresponding to the Laplacian

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2. The Hardy space $H^p(\mathbb{R}^n)$ is defined as the space of tempered distributions for which $\sup_{0<t<\infty} |e^{-t\Delta} f| \in L^p(\mathbb{R}^n), 0 < p < \infty$.
3. In the seminal work of C. Fefferman and E.M. Stein in 1970's, the real variable theory of $H^p(\mathbb{R}^n)$ was developed involving equivalent characterisations of $H^p(\mathbb{R}^n)$ in terms of maximal functions, boundary values of harmonic functions.

Theorem 1.

For every tempered distribution f on $\mathbb{R}^n$, the following conditions are equivalent.

1. $\sup_{0<t<\infty} |e^{-t\Delta} f| \in L^p(\mathbb{R}^n)$.
2. $\sup_{0<t<\infty} |f * \varphi_t| \in L^p(\mathbb{R}^n)$ *where* $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$.
3. $f = \sum_j c_j f_j$, *where* $f_j \subset Q(x_j, r_j)$

$$\|f_j\|_\infty \leq (2r_j)^{-n/p}$$

and

$$\int f_j(x) x^\alpha \, dx = 0,$$

for all $|\alpha| \leq \mathcal{N}$ *with* $\mathcal{N} \geq \lfloor n(1/p - 1) \rfloor$, *and* $\sum_j |c_j|^p < \infty$.

Moreover

$$\|f\|_{H^p} \sim \inf \left\{ \left(\sum_j |c_j|^p \right)^{1/p} : f = \sum_j c_j f_j, f_j's \text{ are as in (2) above} \right\}$$

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4. Hardy spaces corresponding to a non negative, self adjoint operator A on $L^2(\mathbb{R}^n)$ are defined via heat semigroup $e^{-tA}, t > 0$.
5. A suitable atomic characterisation is known if the heat kernel corresponding to e^{-tA} satisfies Gaussian estimates by L. Song and L. Yan in 2016.

Twisted Laplacian

We define the twisted Laplacian on $\mathbb{C}^n$ as

$$\mathcal{L} = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\bar{Z}_j + \bar{Z}_jZ_j\right)$$

where $Z_j = \frac{\partial}{\partial z_j} - \frac{1}{2}\bar{z}_j$, $\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j$, $j = 1, 2, \ldots, n$. Here $\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}$.

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$$\mathcal{L} = -\triangle + \frac{1}{4}|z|^2 + i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right),$$

where $z = (z_1, z_2, ...z_n)$ and $z_j = x_j + iy_j$. Here $\triangle$ is the standard Laplacian in $\mathbb{R}^{2n}$.

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6. In fact $\mathcal{L}_{\mathbb{H}^n}(f(z)e^{it}) = \mathcal{L}f(z)e^{it}$.

Twisted convolution

Let $\omega(z, w) := e^{\frac{i}{2} Im(z \cdot \bar{w})}$ on $\mathbb{C}^n \times \mathbb{C}^n$. The twisted convolution of two functions f and g is defined by

$$(f \times g)(z) = \int_{\mathbb{C}^n} f(w) g(z - w) \omega(z, w) \, dw.$$

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Using the connection on $\mathcal{L}$ and $\mathcal{L}_{\mathbb{H}^n}$ we have $\mathcal{L}(f \times g)(z) = f \times \mathcal{L}g(z)$ for all f, g in Schwartz class on $\mathbb{R}^{2n}$.

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Functional Calculus corresponding to $\mathcal{L}$

The spectral resolution of identity of $\mathcal{L}$ is given by $f = (2\pi)^{-n} \sum_{k \geq 0} f \times \varphi_k$ for $f \in L^2(\mathbb{C}^n)$. We also know that

$$\int_{\mathbb{C}^n} |f(z)|^2 \, dz = (2\pi)^{-2n} \sum_{k \geq 0} \int_{\mathbb{C}^n} |f \times \varphi_k(z)|^2 \, dz \tag{0.1}$$

where with abuse of notation dz is the usual Lebesgue measure on $\mathbb{R}^{2n}$.

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where with abuse of notation dz is the usual Lebesgue measure on $\mathbb{R}^{2n}$. Let m be a bounded measurable function on $[0, \infty)$. We define the corresponding multiplier operator $m(\mathcal{L})f = (2\pi)^{-n} \sum_{k \geq 0} m(2k + n) f \times \varphi_k$ for $f \in L^2(\mathbb{C}^n)$.

Definition

We define the Hardy space

$$\mathcal{H}^p(\mathbb{C}^n) = \left\{ f \in \mathcal{S}'(\mathbb{C}^n) : M_{\mathcal{L}} f(z) = \sup_{0<t<\infty} \left| e^{-t^2\mathcal{L}} f(z) \right| \in L^p(\mathbb{C}^n) \right\}.$$

$\mathcal{H}^p(\mathbb{C}^n)$ is equipped with the norm $\|f\|_{\mathcal{H}^p(\mathbb{C}^n)} = \|M_{\mathcal{L}} f\|_p$.

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When $p = 1$ such an atomic decomposition of $\mathcal{H}^1(\mathbb{C}^n)$ was done by Mauceri, Ricci and Michele.

Let us now define atoms associated to $\mathcal{L}$.

Definition 2.

Let $0 < p \leq 1$ and $0 < \sigma < \infty$. We call a measurable function f a (p, σ)-atom if there exists a cube $Q = Q(z_0, r)$ center at z_0 and length $2r$ such that

1. *supp* $f \subset Q$,
2. $\|f\|_\infty \leq (2r)^{-2n/p}$,
3. $\int f(z) z^\alpha \bar{z}^\beta \omega(z_0, z)\, dz = 0$ for all $|\alpha| + |\beta| \leq N$, with $N \geq \lfloor 2n(1/p - 1) \rfloor = \mathcal{N}_0$, whenever $r < \sigma$.

Atomic space

Let us define the atomic Hardy space $H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n)$ as follows

$$H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n) = \left\{ f = \sum_j c_j a_j : a_j' s\, are\, (p,\sigma) - \text{atom and} \sum_j |c_j|^p < \infty \right\}$$

and the "norm" in this space is defined by

$$\|f\|_{H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n)} = \inf \left\{ \left(\sum_j |c_j|^p \right)^{1/p} : f = \sum_j c_j a_j \in H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n) \right\}.$$

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Again when $\sigma = 1$, we just write the above space as $H^p_{\mathcal{L},at}(\mathbb{C}^n)$.

We prove the following theorem:

Theorem 3.

For any $0 < p < 1$, $f \in \mathcal{H}^p(\mathbb{C}^n)$ *if and only if* $f \in H^p_{\mathcal{L},at}(\mathbb{C}^n)$.
Moreover $\|f\|^p_{\mathcal{H}^p(\mathbb{C}^n)} \cong_p \|f\|_{H^p_{\mathcal{L},at}(\mathbb{C}^n)}$.

Oscillatory multipliers

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We prove the following sharp result concerning the boundedness of $m_\beta(\sqrt{\mathcal{L}})$ on $\mathcal{H}^p(\mathbb{C}^n), 0 < p \leq 1$ by using the above atomic charaterisation.

Theorem 4.

The operator $m_\beta(\sqrt{\mathcal{L}})$ is bounded on $\mathcal{H}^p(\mathbb{C}^n)$ for $0 < p \leq 1$ with $\beta \geq (2n-1)\,(1/p - 1/2)$.

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$\beta = 1$ is related to the solution of wave equation corresponding to $\mathcal{L}$.

Sketch of proof of Theorem 3

Let us consider a collection

$$\mathcal{S}_N = \{\varphi \in C_c^\infty(\mathbb{R}^{2n}) : \textit{supp}\ \varphi \in Q(0,1) \text{ and } |\partial^\alpha \varphi| \leq 1 \text{ for all } |\alpha| \leq N\}.$$

Here $Q(0,1) = [-1,1]^{2n}$.

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Here $Q(0,1) = [-1,1]^{2n}$.
Let $\varphi_t(z) = t^{-2n}\varphi(z/t)$. For given $0 < \sigma \leq \infty$, let us define a grand maximal function

$$\mathcal{M}_\sigma^N f(z) = \sup_{\varphi \in \mathcal{S}_N} \sup_{0<t<\sigma} |f \times \varphi_t(z)| .$$

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We take N sufficiently large (depends on n and p) and fix it.

For $0 < \sigma \leq \infty$ and $0 < p \leq 1$, we define the twisted Hardy space $H^p_{\mathcal{L}}(\mathbb{C}^n)$ as

$$H^p_{\mathcal{L},\sigma}(\mathbb{C}^n) := \left\{ f : f \in \mathcal{S}'(\mathbb{R}^{2n}), \|f\|_{H^p_{\mathcal{L},\sigma}(\mathbb{C}^n)} = \|\mathcal{M}_\sigma f\|_{L^p} < \infty \right\}.$$

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For $\sigma = 1$, we write the Hardy space $H^p_{\mathcal{L},1}(\mathbb{C}^n)$ as $H^p_{\mathcal{L}}(\mathbb{C}^n)$.

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For $\sigma = 1$, we write the Hardy space $H^p_{\mathcal{L},1}(\mathbb{C}^n)$ as $H^p_{\mathcal{L}}(\mathbb{C}^n)$.One of the first characterisation theorem is:

Theorem 5.

For $0 < p \leq 1$, the twisted Hardy space $H^p_{\mathcal{L}}(\mathbb{C}^n)$ coincides with the atomic Hardy space $H^p_{\mathcal{L},at}(\mathbb{C}^n)$ with norm equivalence.

For a fixed sufficiently large N, and $0 < \sigma \leq \infty$, we define the Euclidean grand maximal function using the usual Euclidean convolution as follows

$$\widetilde{\mathcal{M}}_\sigma f(z) = \sup_{\varphi \in \mathcal{S}_N} \sup_{0<t<\sigma} |f * \varphi_t(z)|.$$

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Let $h^p_\sigma(\mathbb{C}^n) := \{f \in \mathcal{S}'(\mathbb{C}^n) : \|\widetilde{\mathcal{M}}_\sigma f\|_p < \infty\}$.

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Define $\|f\|_{h^p_\sigma} = \|\widetilde{\mathcal{M}}_\sigma f\|_p$.

Goldberg proved the following characterization of local hardy spaces $h^p_\sigma(\mathbb{C}^n)$, $0 < p \leq 1$.

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Theorem 6.

For every tempered distribution f on $\mathbb{C}^n$ and fixed positive σ, the following conditions are equivalent.

1. $\widetilde{\mathcal{M}}_\sigma f \in L^p(\mathbb{C}^n)$
2. $f = \sum_j c_j f_j$, *where supp* $f_j \subset Q(z_j, r_j)$

$$\|f_j\|_\infty \leq (2r_j)^{-2n/p}$$

and

$$\int f_j(z) z^\alpha \bar{z}^\beta \, dz = 0,$$

for all $|\alpha| + |\beta| \leq \mathcal{N}$ *with* $\mathcal{N} \geq \lfloor 2n(1/p - 1) \rfloor$, *whenever* $r_j < \sigma$ *and* $\sum_j |c_j|^p < \infty$.

Moreover

$\|f\|_{h^p_\sigma} \sim \inf \left\{ \left(\sum_j |c_j|^p \right)^{1/p} : f = \sum_j c_j f_j, f_j's \text{ are as in (2) above} \right\}$

We first prove the following lemma.

Lemma 7.

Let f be a function such that $f \subset Q(z_0, \sigma)$. Then there is a positive constant $C(\sigma)$ depending on σ but independent of z_0 such that

$$C(\sigma)^{-1}\|\mathcal{M}_\sigma f\|_p \leq \|f(\cdot)\omega(z_0,\cdot)\|_{h^p_\sigma} \leq C(\sigma)\|\mathcal{M}_\sigma f\|_p$$

for every $0 < p \leq 1$.

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for every $0 < p \leq 1$.

Let us consider a partition of $\mathbb{C}^n$ into family of cubes $Q_j = Q(z_j, \sigma/2)$ and construct a C^∞ partition of unity $\{\zeta_j\}$ such that ζ_j is supported on $Q_j^* = Q(z_j, \sigma)$ and $|\partial_z^\alpha \partial_{\bar{z}}^\beta \zeta_j(z)| \lesssim_{\alpha,\beta} 2\sigma^{-|\alpha|-|\beta|}$ for all α, β.

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Lemma 8.

Let f be such that $\mathcal{M}_\sigma f \in L^p(\mathbb{C}^n)$. Then $g_j(z) = f(z)\zeta_j(z)\omega(z_j, z)$ is in h^p_σ and $\|\mathcal{M}_\sigma f\|_p^p \simeq \sum_j \|g_j\|_{h^p_\sigma}^p$.

By Lemma 8, given $f \in H^p_{\mathcal{L},\sigma}(\mathbb{C}^n)$ we first decompose $f = \sum_j g_j \bar{\omega}(z_j, z)$.

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$$\text{whenever } \ r_j < \sigma \text{, there exists } \ \theta_j \text{ such that } |\theta_j - w_j| < 2\sigma \text{ and} \qquad (0.4)$$
$$\int h_j(z) z^\alpha \bar{z}^\beta e^{\frac{i}{2} Im(\theta_j \cdot \bar{z})} \, dz = 0 \text{ for all } |\alpha| + |\beta| \leq N, \ \text{ where } N \geq \mathcal{N}_0.$$

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Also, conversely given a sequence $\{h_j\}$ of functions satisfying (0.3) and (0.4) and a sequence $\{\eta_j\}$ satisfying $\sum_j |\eta_j|^p < \infty$, the function $f(z) = \sum_j \eta_j h_j$ is in $H^p_{\mathcal{L},\sigma}(\mathbb{C}^n)$ and $\|f\|^p_{H^p_{\mathcal{L},\sigma}} \leq C(\sigma) \sum_j |\eta_j|^p$.

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Lemma 9.

Let f be a function supported on $Q = Q(z_0, r), r < \sigma$ such that

$$\|f\|_\infty \leq (2r)^{-2n/p}$$

and

$$\int f(w) w^\alpha \bar{w}^\beta e^{\frac{i}{2} Im(\theta \cdot \bar{w})}\, dw = 0, \; for\, all\, |\alpha| + |\beta| \leq N, \tag{0.5}$$

$$with\, N \geq 2\mathcal{N}_0 \;\; and \;\; for \;\; some \;\; \vartheta \;\; with \;\; |\theta - z_0| < 2\sigma.$$

Let σ be small enough (depending on n, p). Then f can be decomposed as $f = \sum_j \eta_j g_j$, where

1. $\sum_j |\eta_j|^p \leq C$,
2. $g_j \subseteq Q(z_j, r_j)$, $\|g_j\|_\infty \leq (2r_j)^{-2n/p}$,
3. $\int g_j(w) w^\alpha \bar{w}^\beta e^{\frac{i}{2} Im(z_j \cdot \bar{w})}\, dw = 0$, *for all* $|\alpha| + |\beta| \leq N$, *with* $N \geq \mathcal{N}_0$, *whenever* $r_j < \sigma$.

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Now, we write $f(z) = a^{(1)}(z) + b^{(1)}(z)$, where

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By the definition of the projection $P_{Q,N}$, it is clear that $\frac{1}{\mathcal{N}_1+1} a^{(1)}(z)$ satisfies the conditions (ii) and (iii).

For $b^{(1)}(z)$, using the estimate (0.6), we get

$$|b^{(1)}(z)| \leq \sum_{j=1}^{\mathcal{N}_1} \left| \frac{1}{|Q|} \int_Q f(w) \overline{e_j(w)} e^{\frac{i}{2} Im(z_0 \cdot \bar{w})} \, dw \right| \tag{0.7}$$

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If we choose $q = \frac{2n}{\frac{2n}{p} - \mathcal{N}_0 - 1}$, then $\|b^{(1)}\|_q \leq C\sigma^{\mathcal{N}_0+1}$. Recall that $\mathcal{N}_0 = \lfloor 2n(1/p - 1) \rfloor$ which further implies that $q > 1$.

Since $b^{(1)} \subset Q(z_0, \sigma)$, by Holder's inequality and boundedness of $\widetilde{\mathcal{M}}_\sigma$ on $L^q(\mathbb{C}^n)$, $q > 1$ we get

$$\begin{aligned} \|\omega(z_0,\cdot)b^{(1)}(\cdot)\|_{h^p_\sigma} =& \|\widetilde{\mathcal{M}}_\sigma(\omega(z_0,\cdot)b^{(1)}(\cdot))\|_p \\ \leq& C\sigma^{\frac{\mathcal{N}_0+1}{2n}} \|\omega(z_0,\cdot)b^{(1)}(\cdot)\|_q \\ \leq& C\sigma^{\frac{\mathcal{N}_0+1}{2n}} \sigma^{\mathcal{N}_0+1}. \end{aligned}$$

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Note that $a^{(2)}(z) = \sum_j \eta_j^{(2)} g_j^{(2)}(z)$ with $g_j^{(2)}$ satisfy (ii) and (iii) and $\sum_j |\eta_j^{(2)}|^p \lesssim 1$.

Using an iterative process we can write

$$h(z) = \sum_k a_k(z)$$

where $a_k \in H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n) and \sum_k a_k$ converges in $H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n)$ with

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This gives the required atomic decomposition of h.

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Let us assume $f \subset B = B(z_0, r)$ and $\tilde{B} = 4B$. We write

$$\begin{aligned}\int (\mathcal{M}_\sigma f(z))^p \; dz =& \int_{\tilde{B}} (\mathcal{M}_\sigma f(z))^p \; dz + \int_{\tilde{B}^c} (\mathcal{M}_\sigma f(z))^p \; dz \\ =& J_1 + J_2.\end{aligned}$$

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Since $|f| \lesssim r^{-2n/p}$, we have $\mathcal{M}_\sigma f(z) \lesssim r^{-2n/p}$. This implies

$$J_1 \lesssim r^{-2n} |\tilde{B}| \leq C.$$

Now, we consider J_2. Let $z \notin \tilde{B}$. When $r \geq \sigma$, for $0 < t < \sigma$, observe that $(f \times \varphi_t)$ is contained in $3B$. Therefore, $\mathcal{M}_\sigma f(z) = 0$, for $z \in \tilde{B}$ and the inequality (0.8) holds.

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Let us now consider the case when $r < \sigma$.We will use cancellation condition on f.Using the Taylor's series expansion for twisted translation for the function $\varphi_t(\cdot)$ at $z = z - z_0$ about $w = w - z_0$ we get

$$\varphi_t(z-w)e^{\frac{i}{2}Im(z_0 - w\cdot\overline{z-z_0})}$$

$$= \sum_{|\alpha|+|\beta| \leq \mathcal{N}_0} \frac{(-1)^{|\alpha|+|\beta|}}{(|\alpha|+|\beta|)!}(\mathrm{Re}(w-z_0))^{\alpha}(\mathrm{Im}(w-z_0))^{\beta}\tilde{X}^{\alpha}\tilde{Y}^{\beta}\varphi_t(z-z_0)$$

$$+\Phi(w)$$

Now, we consider J_2. Let $z \notin \tilde{B}$. When $r \geq \sigma$, for $0 < t < \sigma$, observe that $(f \times \varphi_t)$ is contained in $3B$. Therefore, $\mathcal{M}_\sigma f(z) = 0$, for $z \in \tilde{B}$ and the inequality (0.8) holds.
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where

$$\Phi(w) = \sum_{|\alpha|+|\beta|=\mathcal{N}_0+1} \frac{(-1)^{|\alpha|+|\beta|}}{\mathcal{N}_0+1!}\int_{s=0}^1 (1-s)^N \tilde{X}^\alpha \tilde{Y}^\beta \varphi_t(z-z_0+s(z_0-w))$$

$$(\mathrm{Re}(w-z_0))^\alpha(\mathrm{Im}(w-z_0))^\beta e^{\frac{i}{2}s\ Im(z_0-w\cdot\overline{z-z_0})}ds$$

Using the moment condition of the atom f_j and the above Taylor's expansion, we write

$$f \times \varphi_t(z) = \int_{\mathbb{R}^{2n}} \varphi_t(z-w) f(w) e^{\frac{i}{2} Im(z \cdot \bar{w})} \, dw$$
$$= e^{-\frac{i}{2} Im(z_0 \cdot \overline{z})} \int_{\mathbb{R}^{2n}} f(w) \Phi(w) e^{\frac{i}{2} Im(z_0 \cdot \overline{w})} \, dw$$

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When $0 < t < \sigma$ and $\varphi \in C_c^\infty(\mathbb{C}^n)$, one can check that

$$\left| \tilde{Z}^\alpha \bar{\tilde{Z}}^\beta \varphi_t(z - z_0 + s(z_0 - w)) \right| \lesssim_{\sigma,\mathcal{N}_0} t^{-2n-\mathcal{N}_0-1} \|\varphi\|_{\mathcal{N}_0+1}$$

. Using the above estimate we get

$$|\Phi(w)| \lesssim_\sigma \frac{r^{\mathcal{N}_0+1}}{t^{2n+\mathcal{N}_0+1}}.$$

We know that $z \notin \tilde{B}$, $w \in B$ and $z - w \in supp\ \varphi_t$. This implies that $|z - z_0| \leq ct$.

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Since $(2n + \mathcal{N}_0 + 1)p > 2n$, we get

$$J_2 \lesssim_\sigma r^{-2n} \int_{\tilde{B}^c} \left(\frac{r^{2n+\mathcal{N}_0+1}}{|z - z_0|^{2n+\mathcal{N}_0+1}} \right)^p dz \lesssim_\sigma r^{2n} r^{-2n} = C_\sigma.$$

This proves the estimate (0.8).

When $f \in H^p_{\mathcal{L},\sigma}(\mathbb{C}^n)$. It follows from the following observation,

$$\int (\mathcal{M}_\sigma f(z))^p \, dz \lesssim \sum_j |c_j|^p \int (\mathcal{M} f_j(z))^p \, dz \lesssim \sum_j |c_j|^p < \infty.$$

This proves $H^p_{\mathcal{L},at,\sigma}(\mathbb{C}^n) \subseteq H^p_{\mathcal{L},\sigma}(\mathbb{C}^n)$ for any $\sigma > 0$.

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With a slight modification we can also prove it for $\sigma = 1$.

Recall that

$$\mathcal{H}^p(\mathbb{C}^n) = \{f : M_{\mathcal{L}}f(z) = \sup_{0<t<\infty} \left| e^{-t^2\mathcal{L}} f(z) \right| \in L^p(\mathbb{C}^n)\}.$$

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We know that $e^{-t^2\mathcal{L}} f(z) = f \times p_{t^2}(z)$ where
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We also need to prove atomic decomposition for $f \in \mathcal{H}^p(\mathbb{C}^n)$ for a complete characterisation of $\mathcal{H}^p(\mathbb{C}^n)$.

Theorem 10.

For $0 < p < \infty$ $\mathcal{H}^p(\mathbb{C}^n) \subset H^p_{\mathcal{L}}(\mathbb{C}^n)$ *and there exists* $C > 0$ *such that*

$$\|f\|_{H^p_{\mathcal{L}}} \leq C\|f\|_{\mathcal{H}^p},$$

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One of the key things in the above theorem is to be able to realise a distribution on $\mathbb{C}^n$ as a distribution on $\mathbb{H}^n$ via the map $f \to f(z)e^{it}$.
After this realisation many techniques developed by Folland and Stein for Hardy spaces on homogeneous groups go through with appropriate modifications.

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Thanks for your attention!