L^p improving properties and maximal estimates for certain multilinear averaging operators

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Based on joint works with Jin Bon Lee (SNU) and Kalachand Shuin (Indian Institute of Science)

- The spherical (circular) averaging operator

$$A^t_{\mathbb{S}^{d-1}} f(x) := \int_{\mathbb{S}^{d-1}} f(x - ty) \, \mathrm{d}\sigma(y).$$

- The spherical maximal function

$$M^*_{\mathbb{S}^{d-1}} f(x) = \sup_{t>0} |A^t_{\mathbb{S}^{d-1}} f(x)| := \sup_{t>0} \left| \int_{\mathbb{S}^{d-1}} f(x - ty) \, \mathrm{d}\sigma(y) \right|$$

  with $d\sigma$ is the normalized surface measure on the sphere $\mathbb{S}^{d-1}$.
- It was known that the maximal operator $M^*_{\mathbb{S}^{d-1}}$ is bounded in L^p if and only if $p > \frac{d}{d-1}$. (Stein, $d \geq 3$, Bourgain, $d = 2$)

- The lacunary spherical maximal operator

$$M_{\mathbb{S}^{d-1}}f(x) := \sup_{j\in\mathbb{Z}} |A^{2^j}_{\mathbb{S}^{d-1}}f(x)|$$

- Calderón proved L^p estimates of the operator $M_{\mathbb{S}^{d-1}}$ for $1 < p \leq \infty$ and $d \geq 2$.
- Seeger and Wright showed L^p estimates of general lacunary maximal operators $M_{\mathcal{S}}$ for $1 < p \leq \infty$, when the Fourier transform of the surface measure σ of $\mathcal{S}$ satisfies

$$|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\epsilon} \text{ for any } \epsilon > 0.$$

- Lacey used the L^p-improving estimates ($L^p - L^q$ estimates with $p \leq q$) of spherical averages to prove sparse domination of the corresponding lacunary and full spherical maximal functions.
- The idea of Lacey together with L^p-improving estimates of certain bilinear averaging operators, can be used to study sparse domination of maximal operators associated with the bilinear operators.

Multilinear averaging operator I

- Let $\mathcal{S}$ be a compact and smooth hypersurface contained in a unit ball $\mathbb{B}^d(0,1)$ with κ non-vanishing principal curvatures.
- Let Θ_j be rotation matrices in $\mathbf{M}_{d,d}(\mathbb{R})$ for $j = 1, 2, \cdots, m$.
- Assume that $\Theta = \{\Theta_j\}_{j=1}^m$ are mutually linearly independent.

Definition

For $F = (f_1, f_2, \cdots, f_m)$ with $f_1, f_2, \ldots, f_m \in \mathscr{S}(\mathbb{R}^d)$, we define

$$\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})(x) := \int_{\mathcal{S}} \prod_{j=1}^{m} f_j(x + \Theta_j y)\, \mathrm{d}\sigma_{\mathcal{S}}(y)$$

where $d\sigma_{\mathcal{S}}$ is the normalized surface measure on $\mathcal{S}$.

- Note that κ satisfies $1 \leq \kappa \leq d - 1$.

- The bilinear Hilbert transform

$$BHT_\alpha(f,g)(x) := p.v. \int_{-\infty}^{\infty} f(x-t)g(x-\alpha t)\frac{\mathrm{d}t}{t}, \quad \alpha \neq 0, 1.$$

- The bilinear maximal operator

$$M_\alpha(f,g)(x) := \sup_{t>0} \frac{1}{2t}\int_{-t}^{t} |f(x-y)g(x-\alpha y)|\, \mathrm{d}y, \quad \alpha \neq 0, 1.$$

- One may regard averaging operators $\mathcal{A}_{\mathcal{S}}^{\Theta}$ as a generalization of bilinear maximal operator M_α without the supremum because the condition $\alpha \neq 0, 1$.
- Greenleaf, Iosevich, Krause and Liu considered the operator

$$B_\theta(f,g)(x) = \int_{\mathbb{S}^1} f(x-y)g(x-\theta y)\, \mathrm{d}\sigma(y),$$

where θ denotes a counter-clockwise rotation.

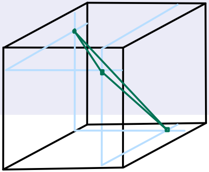
L^p-improving estimates ($p \le 1$)

Proposition (C. Lee and Shuin, 2024)

Let $\mathcal{S}$ be a compact smooth hypersurface contained in $\mathbb{B}^d(0,1)$ with $\kappa \le d-1$ nonvanishing principal curvatures. Assume that $\{\Theta_j\}_{j=1}^m$ is a family of mutually linearly independent rotation matrices. Let $\mathcal{V}_\kappa^{ij} = \{z = (z_1, \ldots, z_m) \in [0,1]^m : z_i = z_j = \frac{\kappa+1}{\kappa+2}, z_l = 0, l \ne i, j\}$ and $\mathrm{conv}(\mathcal{V}_\kappa)$ be its convex hull. Then for $(\frac{1}{p_1}, \ldots, \frac{1}{p_m}) \in \mathrm{conv}(\mathcal{V}_\kappa)$ we have the following inequalities:

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathcal{F})\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

whenever $1 \le \frac{1}{p} \le \frac{2(\kappa+1)}{\kappa+2} = \sum_{j=1}^m \frac{1}{p_j}$.



- When $p > 1$, one can obtain different L^p-improving estimates for $\mathcal{A}_{\mathcal{S}}^{\Theta}$ under specific choice of $\{\Theta_j\}$ and $\mathcal{S}$.
- In this case, we do not need any curvature condition on $\mathcal{S}$ and only the dimension of surfaces matters.
- Let $\mathcal{S}^k$ be a k-dimensional C^2 surface in $\mathbb{R}^d$. Choose mutually linearly independent $\{\Theta_j\}$. Moreover, we assume that for any choice of $\{j_i\}_{i=1}^{l}$ with $2 \le l \le k+1 \le m$, the family $\{\Theta_j\}$ satisfies

$$\dim\left(\mathrm{span}_{1\le i\le l}\big(\{\Theta_{j_i}(y',0)\in\mathbb{R}^d : y'\in\mathbb{R}^k\}\big)\right) \ge \min\{k-1+l, d\},$$

$$\dim\left(\bigcap_{i=1}^{l}\{\Theta_{j_i}(y',0)\in\mathbb{R}^d : y'\in\mathbb{R}^k\}\right) \le k+1-l.$$

- The second assumption yields that dimension of intersection of any subset $\{\Theta_{j_i}\}_{i=1}^{k+1}$ of $\{\Theta_j\}_{j=1}^{m}$ equals to zero.

L^p-improving estimates $(p > 1)$

Theorem (C. Lee and Shuin, 2024)

Let $m \geq d \geq 2$ and $\mathcal{S}^k$ be a k-dimensional C^2 surface in $\mathbb{B}^d(0,1)$. Assume that $\{\Theta_j\}$ satisfies above two conditions and k is given by such that

$$\frac{m-d+k}{m} \geq \frac{d-k-1}{d}k, \quad \frac{m-1}{m} \geq \frac{(d-k)k}{d}.$$

Then $\mathcal{A}^{\Theta}_{\mathcal{S}^k}$ is of strong type $(m, \ldots, m, \frac{d}{d-k})$. That is, we have

$$\|\mathcal{A}^{\Theta}_{\mathcal{S}^k}(F)\|_{L^{\frac{d}{d-k}}(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^m(\mathbb{R}^d)}.$$

- To prove the theorem, we mainly use the nonlinear Brascamp-Lieb inequality (Bennett, Bez, Buschenhenke, Cowling and Flock, 2020).
- In the theorem, one can use $m \geq d$ to check that the conditions for k are equivalent when $d = 2k + 1$.
- Moreover, if we assume $k = d - 1$, then we only need the upper bound for dimension to guarantee the following result:

Corollary

Let $m \geq d \geq 2$ and let $\mathcal{S}^{d-1}$ be a C^2 hypersurface. Assume that $\{\Theta_j\}$ is chosen to be mutually linearly independent and satisfy

$$\dim\Big(\bigcap_{i=1}^{l}\{\Theta_{j_i}(y',0) \in \mathbb{R}^d : y' \in \mathbb{R}^k\}\Big) \leq k + 1 - l.$$

Then $\mathcal{A}^{\Theta}_{\mathcal{S}^{d-1}}$ is of strong-type $(m, \dots, m, d)$.

- One can find similar results in [Iosevich, Palsson and Sovine, 2022]: restricted strong-type $(m, \ldots, m, m)$ and $\left(m\frac{d+1}{d}, \ldots, m\frac{d+1}{d}, d+1\right)$ estimates for $\mathcal{A}^{\Theta}_{\mathcal{S}^{d-1}}$ when $\mathcal{S}^{d-1}$ is a sphere.
- Note that the authors consider $m \leq d$ cases with linearly independent $\{\Theta_j\}$, so it can not be directly compared to the corollary in which $m \geq d$ and the above condition are considered.
- When $m = d$, the corollary for $\mathbb{S}^{d-1}$ gives strong-type $(m, \ldots, m, m)$ estimates.

lacunary maximal operator associated with $\mathcal{A}_{\mathcal{S}}^{\Theta}$

$$\mathcal{M}_{\mathcal{S}}^{\Theta}(\mathrm{F})(x) = \sup_{l\in\mathbb{Z}} \Big| \int_{\mathcal{S}} \prod_{j=1}^{m} f_j(x - 2^l \Theta_j y) \ \ \mathrm{d}\sigma_{\mathcal{S}}(y) \Big|$$

Theorem (C. Lee and Shuin, 2024)

Let $1 \le p_i^\circ \le \infty$, $\sum_{i=1}^{m} \frac{1}{p_i^\circ} = \frac{1}{p^\circ}$ *with* $p^\circ \ge 1$ *for* $d \ge 2$. *Suppose that* $\mathcal{A}_{\mathcal{S}}^{\Theta}$ *satisfies the following Sobolev regularity estimates:*

$$\left\| \mathcal{A}_{\mathcal{S}}^{\Theta}(F) \right\|_{L^{p^\circ}(\mathbb{R}^d)} \lesssim 2^{-\varepsilon|\mathbf{n}|} \prod_{j=1}^{m} \|f_j\|_{L^{p_j^\circ}(\mathbb{R}^d)},$$

where f_j *with* $supp(\widehat{f_j}) \subset \mathbb{A}_{n_j} = \{\xi_j \in \mathbb{R}^d : 2^{n_j-1} \le |\xi_j| \le 2^{n_j+1}\}$, j, *and* $\varepsilon > 0$. *Then* $\mathcal{M}_{\mathcal{S}}^{\Theta}$ *maps* $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ *for* $(\frac{1}{p_1}, \ldots, \frac{1}{p_m}) \in \mathrm{conv}(\mathcal{V}_\kappa^\circ) \cup \{(0, \ldots, 0)\}$ *and* $1/p = 1/p_1 + \cdots + 1/p_m$, *where* $\mathrm{conv}(\mathcal{V}_\kappa^\circ)$ *denotes an interior of the convex hull of* $\mathrm{conv}(\mathcal{V}_\kappa)$ *and the origin.*

Multilinear averaging operator II

- Christ and Zhou studied $L^{p_1} \times L^{p_2} \to L^p$ with $1/p_1 + 1/p_2 = 1/p$ boundedness of bilinear lacunary maximal functions defined on a class of singular curves

$$\mathcal{M}(f_1, f_2)(x) := \sup_{l \in \mathbb{Z}} \Big| \int_{\mathbb{R}^1} \prod_{j=1}^{2} f_j(x - 2^l \gamma_j(t)) \eta(t) \, \mathrm{d}t \Big|,$$

where $\gamma = (\gamma_1, \gamma_2) : (-1, 1) \to \mathbb{R}^2$ and $\eta \in C_0^\infty((-1, 1))$.

- Thus, they have proved $L^{p_1} \times L^{p_2} \to L^p$ estimates for $1 < p_1, p_2 \leq \infty$, $1/p_1 + 1/p_2 = 1/p$ of the bilinear lacunary spherical maximal operator $\mathfrak{M}_{\mathbb{S}^{2d-1}}$ for dimension $d = 1$ where

$$\mathfrak{M}_{\mathbb{S}^1}(f_1, f_2)(x) = \sup_{l \in \mathbb{Z}} \Big| \int_{\mathbb{S}^1} \prod_{j=1}^{2} f_j(x - 2^l y_j) \, \mathrm{d}\sigma(y) \Big|,$$

with $d\sigma(y)$ is the normalized surface measure on the circle $\mathbb{S}^1$.

- For $d \geq 2$, the complete $(L^{p_1} \times L^{p_2} \to L^p)$-estimate of the operator $\mathfrak{M}_{\mathbb{S}^{2d-1}}$ was not known.
- However, there are some partial results of the operator $\mathfrak{M}_{\mathbb{S}^{2d-1}}$ (Palsson and Sovine, Borges, Foster, Ou, Pipher and Zhou)
- Very recently Borges and Foster have obtained almost sharp results including some endpoint estimates.
- We give a different proof of the same estimate for $\mathfrak{M}_{\mathbb{S}^{2d-1}}$.

Definition

For $F = (f_1, f_2, \cdots, f_m)$, we define

$$A_\Sigma(F)(x) := \int_\Sigma \prod_{j=1}^{m} f_j(x + y_j)\, d\sigma_\Sigma(y), \;\; y = (y_1, \ldots, y_m) \in \mathbb{R}^{md}$$

where Σ is a compact $(md-1)$ dimensional smooth hypersurface contained in a unit ball $\mathbb{B}^{md}(0,1)$ with κ non-vanishing principal curvatures.

- Note that $1 \le \kappa \le md - 1$.
- A_Σ is a direct analogue of a spherical averages $A^t_{\mathbb{S}^{d-1}} f(x)$ for $t = 1$.
- Note $A_{\mathbb{S}^{md-1}}(F)(x) = A^1_{\mathbb{S}^{md-1}}(f_1 \otimes \cdots \otimes f_m)(x, \ldots, x)$.

L^p-improving estimates

Proposition (C. Lee and Shuin, 2024)

Let $d \geq 2$ and let Σ be a compact smooth hypersurface Σ with κ nonvanishing principal curvatures with $(m-1)d < \kappa \leq md - 1$. Then for $1 \leq p_j \leq 2$, $j = 1, 2, \ldots, m$ and $\frac{m+1}{2} \leq \sum_{j=1}^{m} \frac{1}{p_j} < \frac{2d+\kappa}{2d}$,

$$\|A_\Sigma(F)\|_{L^1(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)}.$$

Moreover, we have for $\frac{m+1}{2} \leq \frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j} < \frac{2d+\kappa}{2d}$

$$\|A_\Sigma(F)\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)}.$$

lacunary maximal estimates associated with A_Σ

Proposition (C. Lee and Shuin, 2024)

Let $1 \leq p, p_1, \ldots, p_m < \infty$ *and* $1/p = 1/p_1 + \cdots + 1/p_m$. *Then for* f_j *with* $supp(\widehat{f_j}) \subset \mathbb{A}_{n_j} := \{\xi_j \in \mathbb{R}^d : 2^{n_j - 1} \leq |\xi_j| \leq 2^{n_j + 1}\}$,

$$\|A_\Sigma(F)\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

where $\delta = \delta(p, \kappa, m, d) > 0$ *and* $|n| = \sqrt{\sum_{j=1}^m n_j^2}$.

Theorem (C. Lee and Shuin, 2024)

Let $\frac{m+1}{2} \leq \frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j} < \frac{2d+\kappa}{2d}$ *for* $1 \leq p_j \leq 2$ *and* $\kappa > (m-1)d$. *The operator* $\mathfrak{M}_\Sigma(\mathrm{F})(x) = \sup_{l \in} \left| \int_\Sigma \prod_{j=1}^m f_j(x - 2^l y_j) \ \ d\sigma_\Sigma(y) \right|$ *maps* $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.

Thank you for your attention