# Two parameter maxiaml average over tori

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# Maximal averages over circles and spheres

- Let $d\sigma$ be the normalized Lebesgue measure on the unit circle or the unit sphere centered at the origin ($\mathbb{S}^1$ or $\mathbb{S}^2$).
- The circular (or spherical) average of $f$ at $x \in \mathbb{R}^2$ (or $\mathbb{R}^3$) is defined by
$$A_t f(x) = \int f(x - ty) d\sigma(y) = f * d\sigma_t(x).$$
- The circular (or spherical) maximal function is defined by
$$Mf(x) = \sup_{t>0} |A_t f(x)|.$$

Theorem

1. *(Stein '76)*
*The spherical maximal operator in $\mathbb{R}^3$ is bounded on $L^p$ if and only if $p > \frac{3}{2}$.*
2. *(Bourgain '86)*
*The circular maximal operator is bounded on $L^p$ if and only if $p > 2$.*

# Some remarks on classical results

- There are a lot of general results.
  1. There are higher dimensional analogs (easier).
  2. We can replace circles and spheres with curves and surfaces with non-vanishing curvature.
  3. Curves and surfaces may depend on the location $x$.

- Local smoothing type estimates are important in the maximal estimates. It comes from the curvature of geometric objects.

- Nevertheless, some results consider degenerate surfaces under certain conditions. The degeneracy makes the range of $p$ which makes the maximal operator bounded smaller.

- Recently there have been several results considering multi-parameter maximal functions.

# Multi parameter maximal functions

Theorem (Marletta, Ricci,'98)

*Given a hypersurface $\Gamma$ with homogeneous degree $d$ and non-vanishing Gaussian curvature, the maximal operator defined by*

$$Mf(x) = \sup_{a,b>0} \int_{y \in \mathbb{R}^{n-1}} f(x - (ay, b\Gamma(y)))dy$$

*is bounded in $L^p$ if and only if $p > \frac{d}{d-1}$.*

Theorem (Cho '98, Heo '16)

*If $|\widehat{d\mu}(\xi)| \lesssim \prod_{i=1}^{d}(1+|\xi_i|)^{-a_i}$ for $a_i > \frac{1}{2}$, then the maximal operator defined by*

$$Mf = \sup_{t_i>0 \text{ for all } i} \widehat{d\mu}(t_1 D_1, \cdots t_d D_d)f$$

*is bounded in $L^p$ for a "suitable" range of $p$.*

# Torus $\mathbb{T}_t^s$ in $\mathbb{R}^3$

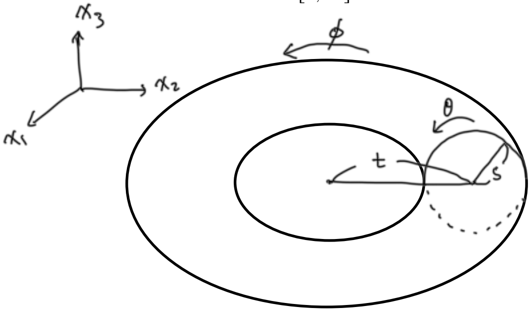
We define

$$\Phi_t^s(\theta,\phi) = ((t + s\cos\theta)\cos\phi, (t + s\cos\theta)\sin\phi, s\sin\theta)$$

for $\theta, \phi \in [0, 2\pi)$.

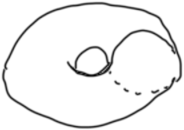
Define a measure on $\mathbb{T}_t^s$ by

$$\langle f, d\sigma_t^s \rangle = \int_{[0,2\pi]^2} f(\Phi_t^s(\theta,\phi))\, d\theta d\phi.$$

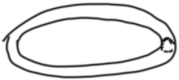


# Geometric observations of torus

The "shape" of the torus depends on the ratio $s/t$.



$s/t$ large

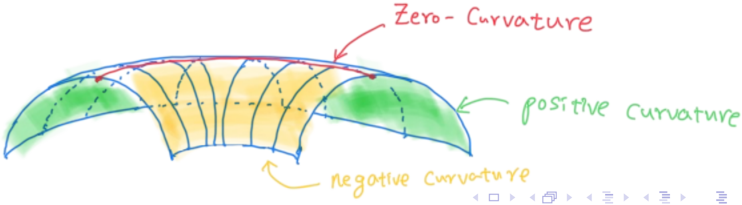


$s/t$ small

If $s/t$ is too large, then the torus becomes singular.
Thus, we need to bound $s/t$,

$$0 < s/t < C.$$

The torus is flat in some part. $\Rightarrow$ degenerate.



# Two types of maximal averages over tori

- Define an average $A_t^s f = f * d\sigma_t^s$.
- As we have seen in the geometric structure of the torus, there are two natural types of maximal functions.

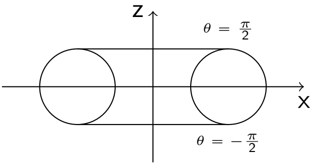
$$M_{\mathbb{T}_1^{c_0}} : f \mapsto \sup_{t>0} |A_t^{c_0 t} f|,$$

$$M : f \mapsto \sup_{0<s<c_0 t} |A_t^s f|.$$

- Since $\mathbb{T}_1^{c_0}$ has a part where curvature vanishes, the range of $p$ which makes $M_{\mathbb{T}_1^{c_0}}$ bounded on $L^p$ is smaller than that of the spherical maximal function in $\mathbb{R}^3$.
- Meanwhile, $\mathbb{T}_1^{c_0}$ always has at least one non-vanishing principal curvature. This makes the range of $p$ same with that of the circular maximal function, $p > 2$.

# One parameter maximal average over tori

The Gaussian curvature of $\mathbb{T}_1^{c_0}$ is $K(\theta, \phi) = \frac{\cos\theta}{c_0(1+c_0\cos\theta)}$. It vanishes only if $\theta = \pm\frac{\pi}{2}$. Thus, if $||\theta| - \frac{\pi}{2}| > \epsilon > 0$, then the operator is bounded in $L^p$ for $p > \frac{d}{d-1} = \frac{3}{2}$.




**Theorem**

*$M_{\mathbb{T}_1^{c_0}}$ is bounded in $L^p$ if and only if $p > 2$.*

**Proof.**

We can apply the result of [Ikromov, Kempe, Müller, '10], or directly control the operator using a slicing argument. □

# Two parameter maximal average over tori

Theorem (L, S.Lee '22)

*Define an operator*

$$M : f \mapsto \sup_{0<s<c_0 t} |A_t^s f|.$$

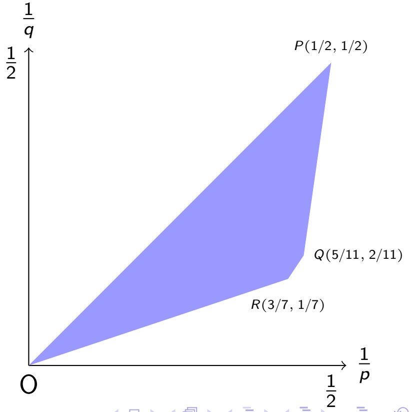
*Then, $M$ is bounded in $L^p$ if and only if $p > 2$.*

Theorem (L, S.Lee '22)

*The localized maximal operator*

$$M_c : f \mapsto \sup_{\substack{t,s\sim 1,\\ s<c_0 t}} |A_t^s f|$$

*is bounded from $L^p$ to $L^q$ when $(\frac{1}{p}, \frac{1}{q})$ is contained in the interior of the quadrangle $\mathcal{Q}$ on the right. Also, if $(\frac{1}{p}, \frac{1}{q})$ is not contained in $\overline{\mathcal{Q}}$, then $M_c$ is not bounded.*



# Local smoothing estimate

- (Miyachi, 1980)The estimate for a fixed $t > 0$ gives

$$\|A_t f\|_{L^{p,\frac{1}{p}}(dx)} \lesssim \|f\|_{L^p(dx)}.$$

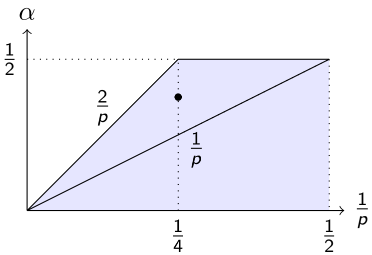
- However, we need

$$\|A_t f\|_{L^{p,\frac{1}{p}+\epsilon}(dxdt)} \lesssim \|f\|_{L^p(dx)}$$

for an $\epsilon > 0$ to prove the theorem.

**Theorem (Guth, Wang, Zhang, 2020)**

*$A_t$ is bounded from $L^p(\mathbb{R}^2)$ to $L^{p,\alpha}(\mathbb{R}^2 \times [1,2])$ when $\alpha < \max\{\frac{2}{p}, \frac{1}{2}\}$.*

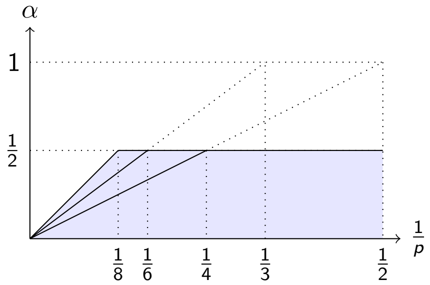


# Local smoothing estimates

Theorem (L, S.Lee '22)

*For $p \geqslant 2$ and smooth cutoff functions $\psi_1(t), \psi_2(t,s)$ which have supports near $t, \sim 1$ and $t, s \sim 1$ respectively, we have the following.*

1. $\|A_t^s f\|_{L_x^{p,\alpha}} \lesssim \|f\|_{L^p}$ *if* $\alpha < \max\{\frac{1}{2}, \frac{2}{p}\}$ *for any fixed* $t, s \sim 1$.
2. $\|\psi_1(t) A_t^{c_0 t} f\|_{L_{x,t}^{p,\alpha}} \lesssim \|f\|_{L^p}$ *if* $\alpha < \max\{\frac{1}{2}, \frac{3}{p}\}$.
3. $\|\psi_2(t,s) A_t^s f\|_{L_{x,t,s}^{p,\alpha}} \lesssim \|f\|_{L^p}$ *if* $\alpha < \max\{\frac{1}{2}, \frac{4}{p}\}$.




- We obtained that $|\widehat{d\sigma_t^s}(\xi)| \lesssim |(\xi_1, \xi_2)|^{-\frac{1}{2}} |\xi|^{-\frac{1}{2}}$.
- We remark that one additional integrating variable gives additional smoothing of order $\frac{1}{p}$.

# Comparison with the case of sphere

- After the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \mathcal{P}_j f$, we have

$$\widehat{d\sigma_t}(\xi) \approx a e^{it|\xi|}|\xi|^{-1} + b e^{-it|\xi|}|\xi|^{-1} + \text{error}.$$

- Thus, we have

$$A_t f(x) \approx \sum_{j=0}^{\infty} \int e^{i(x\cdot\xi + t|\xi|)}|\xi|^{-1}\widehat{\mathcal{P}_j f}(\xi) d\xi.$$

  Now we can apply various theories associated to the extension operator for the cone $(\xi, |\xi|)$.

- In the case of torus, we have

$$\widehat{d\sigma_t^s}(\xi) \approx e^{i(t|\xi'|+s|\xi|)}|\xi'|^{-\frac{1}{2}}|\xi|^{-\frac{1}{2}}$$

  where $\xi' = (\xi_1, \xi_2)$.

- Thus, we need to analyze the extension operator for the conic surface $(\xi, |\xi'|, |\xi|)$ which is 3-dimensional surface in $\mathbb{R}^5$.

Thank you for your attention.