An introduction to $\mathcal{PT}$-Symmetric Quantum Theory

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Introduction

This presentation is based on the work of Carl Bender

Ref. Carl M. Bender, *Introduction to $\mathcal{PT}$-Symmetric Quantum Theory*, arXiv:quant-ph/0501052v1

Ref. Carl M. Bender, Mariagiovanna Gianfreda, Sahin K. Ozdemir, Bo Peng, and Lan Yang, *Twofold transition in $\mathcal{PT}$-symmetric coupled oscillators*, Phys. Rev. A **88**, 062111 (2013)

Ref. Carl M. Bender, Bjorn K. Berntson, David Parker, E. Samuel, *Observation of PT phase transition in a simple mechanical system*, https://arxiv.org/abs/1206.4972

Introduction

The central idea of $\mathcal{PT}$-symmetric Quantum Theory

Traditional QM

Hamiltonian has to be Hermitian

⬇

$\mathcal{PT}$-symmetric QM

Hamiltonian has space-time symmetry

Introduction

IF

$\mathcal{PT}$-symmetry of the Hamiltonian is not broken.

THEN

The Hamiltonian will exhibit all the features of a quantum theory described by a Hermitian Hamiltonian.

A little background on quantum mechanics

A quantum theory is specified by the Hamiltonian operator, H, that acts on a Hilbert space.

The Hamiltonian does 3 things!

A little background on quantum mechanics

First

The Hamiltonian determines the energy eigenstates E_n of the system (energy levels)

$$H\Psi_n = E_n\Psi_n$$

Since E_n is a physical measurable quantity it must be real.

A little background on quantum mechanics

Second

The Hamiltonian determines the time evolution of the theory

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = H\Psi(t)$$

Whose formal solution is $\Psi(t) = e^{iHt}\Psi(0)$.

It is important that the Hamiltonian preserves probability. The probability must not change over time (unitarity).

A little background on quantum mechanics

Third

The Hamiltonian incorporates the symmetries of the theory

Continuous Symmetries:
Lorentz invariance

Discrete Symmetries:
Space reflection or time reversal

A quantum theory is symmetric under a transformation represented by an operator A if:

$$[A, H] = 0$$

If we are going to build a new quantum theory we need to guarantee that:

The energy levels, E_n, must be real.

and

The time evolution must be unitary.

Real and Symmetric Hamiltonians

Finite dimensional systems

$$H = \begin{pmatrix} a & b & c & \dots \\ b & d & e & \dots \\ c & e & f & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Infinite dimensional systems

$$H \equiv H(\hat{x}, \hat{p})$$

$\hat{x}$ and $\hat{p}$ are operators that act on coordinate space

Where:

$\hat{x}$ is a *real* and *symmetric* diagonal matrix.

$\hat{p} = -i\dfrac{d}{dx}$ is *imaginary* and *anti-symmetric*.

Hermitian Hamiltonians

These are complex Hamiltonians that include the real and symmetric Hamiltonians as a special case.

Property of Hermiticity

$$H = H^\dagger$$

$\dagger \equiv$ Dirac Hermitian conjugation (transpose $+$ complex conjugation)

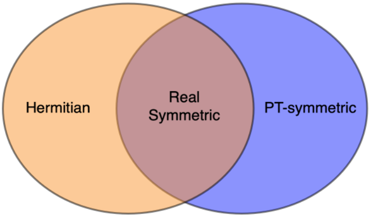
While Hermiticity is sufficient it is not necessary

PT-Symmetric Hamiltonians

An alternative type of complex Hamiltonians

This Hamiltonians also include real, symmetric Hamiltonians as a special case.
They have the property that they commute with the operator $\mathcal{PT}$.

$$[H, \mathcal{PT}] = 0$$



Two important discrete symmetries

Parity symmetry: $\mathcal{P}$

$$\hat{x} \to -\hat{x}$$
$$\hat{p} \to -\hat{p}$$

Time reversal symmetry: $\mathcal{T}$

$$\hat{x} \to \hat{x}$$
$$\hat{p} \to -\hat{p}$$
$$i \to -i$$

Why $\mathcal{T}$ changes the sign of i?

$$[\hat{x}, \hat{p}] = i\hbar$$

$$i\hbar\frac{\partial \Psi}{\partial t} = H\Psi$$

Quantum Hamiltonian

We modify the Harmonic oscillator

$$\hat{H} = \hat{p}^2 + \hat{x}^2$$

$$\Downarrow$$

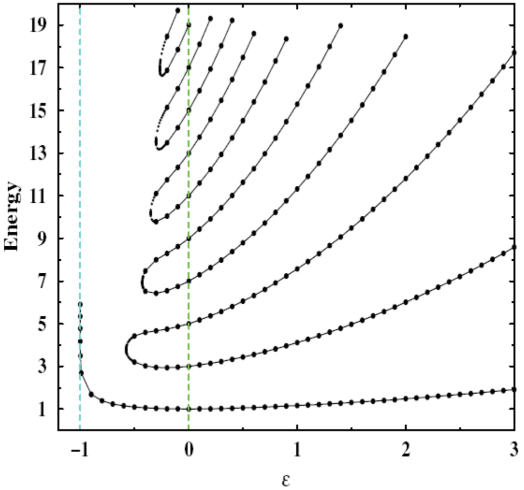
$$\hat{H} = \hat{p}^2 + \hat{x}^2(ix)^\epsilon$$

Where the parameter ϵ is real

The eigenvalues vs parameter ϵ

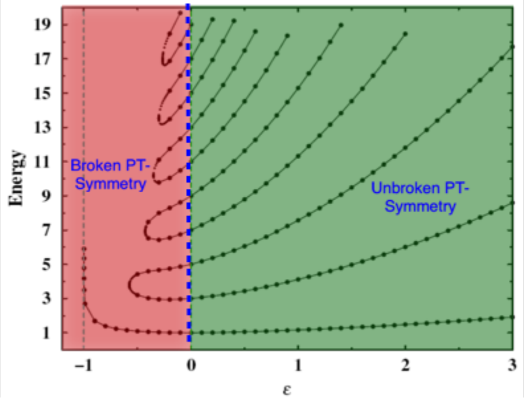
$$\hat{H} = \hat{p}^2 + \hat{x}^2(ix)^\epsilon$$

Ref. Carl M. Bender and S. Boettcher Physical Review Letters **80**, 5243 (1998)



PT-Symmetric Hamiltonians

$$\hat{H} = \hat{p}^2 + \hat{x}^2(ix)^\epsilon$$



We have a $\mathcal{PT}$-phase transition at $\epsilon = 0$ (The Harmonic Oscillator)

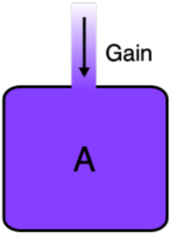
PT-Symmetric Hamiltonians

The $\mathcal{PT}$ symmetry of a Hamiltonian H is unbroken if all of the eigenfunctions of H are simultaneously eigenfunctions of $\mathcal{PT}$.

Lets try to understand what is a $\mathcal{PT}$ transition

$$H = [a + ib]$$

This Hamiltonian is not Hermitian

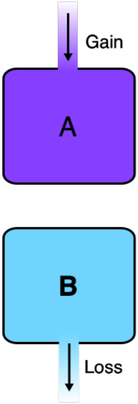


Lets try to understand what is a $\mathcal{PT}$ transition

$$H = \begin{bmatrix} a + ib & 0 \\ 0 & a - ib \end{bmatrix}$$

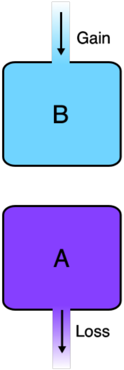
This Hamiltonian is $\mathcal{PT}$-symmetric

The system is not in equilibrium

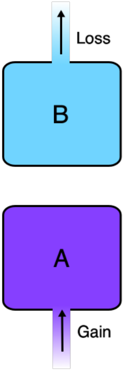


PT-Symmetric Hamiltonians

Lets try to understand what is a $\mathcal{PT}$ transition



Under parity operation $\mathcal{P}$

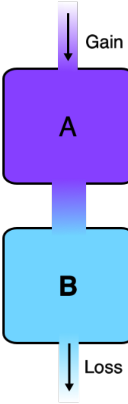


Under time reversal $\mathcal{T}$

Lets try to understand what is a $\mathcal{PT}$ transition

$$H = \begin{bmatrix} a+ib & g \\ g & a-ib \end{bmatrix}$$

This Hamiltonian is $\mathcal{PT}$-symmetric

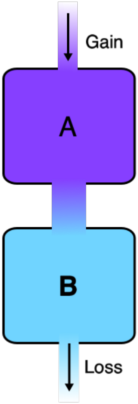


Lets try to understand what is a $\mathcal{PT}$ transition

$$det(H-EI) = a^2+b^2-2aE+E^2-g^2$$

$$E_\pm = a \pm \sqrt{g^2-b^2}$$

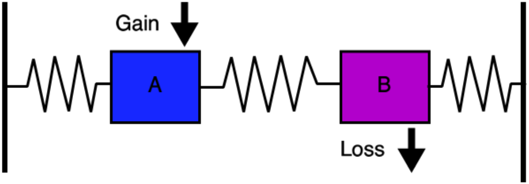
If $g^2 < b^2$ we have a PT-phase transition



PT-Symmetric Hamiltonians

Couple Harmonic Oscillators

$$\begin{cases} \ddot{x} + \omega^2 x + \mu \dot{x} = -\epsilon y \\ \ddot{y} + \omega^2 y - \nu \dot{y} = -\epsilon x \end{cases}$$



PT-Symmetric Hamiltonians

Couple Harmonic Oscillators

$$\begin{cases} \ddot{x} + \omega^2 x + \mu \dot{x} = -\epsilon y \\ \ddot{y} + \omega^2 y - \nu \dot{y} = -\epsilon x \end{cases}$$

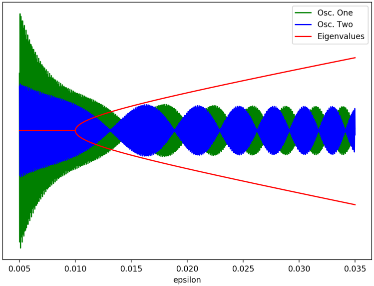
We look for solutions of the form $e^{i\lambda t}$

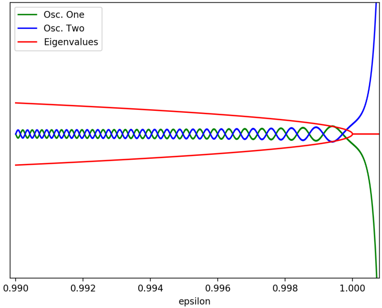
If we set $\mu = \nu = 2\gamma$

The frequencies λ are given by:

$$\lambda^2 = \omega^2 - 2\gamma^2 \pm \sqrt{\epsilon^2 - 4\gamma^2\omega^2 + 4\gamma^4}$$

$\mathcal{PT}$-transitions on couple harmonic oscillator

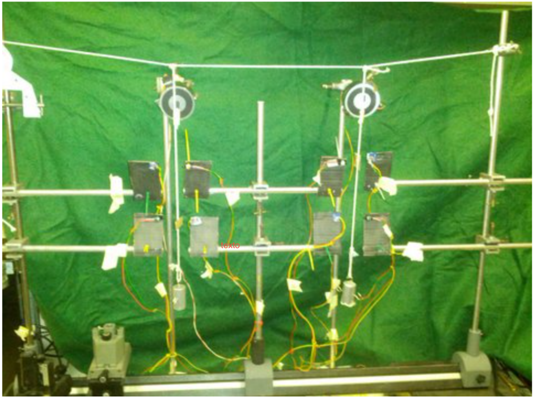




$\epsilon_1 \simeq 0.0099$

$\epsilon_2 = 1.0$

PT-Symmetric Hamiltonian Quantum Mechanics



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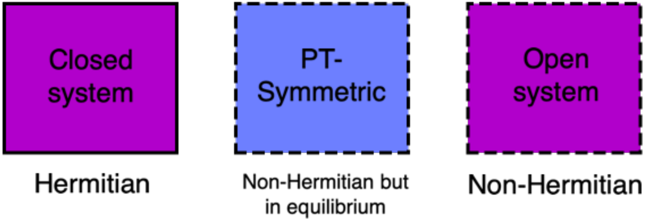
PT-Symmetric Hamiltonians

$\mathcal{PT}$-symmetric systems are interesting



Every time that you identify a $\mathcal{PT}$-phase transition you should contact immediately the nearest physics department!

PT-Symmetric Hamiltonian Quantum Mechanics

$\mathcal{PT}$-Symmetric Quantum Theory

(i) Must possess a Hilbert space of state vectors having a inner product with a positive norm.

(ii) The time evolution of the theory must be unitary. The norm must be preserved in time.

PT-Symmetric Hamiltonians

We need to construct a inner product for our Hilbert space associated with our $\mathcal{PT}$-symmetric Hamiltonian

In conventional Hermitian quantum mechanics the Hilbert space inner product is specified even before we begin to look at the eigenstates.

For our non-Hermitian Hamiltonian we have to guess the inner product

PT-Symmetric Hamiltonians

A reasonable guess for the inner product of two functions $f(x)$ and $g(x)$ might be

$$\langle f, g \rangle \equiv \int dx f(x)^{\mathcal{PT}} g(x)$$

Where:

$$f(x)^{PT} = f(-x)^*$$

PT-Symmetric Hamiltonians

With respect to this inner product the eigenfunctions ϕ_n and ϕ_m are orthogonal for $n \neq m$

$$\langle \phi_n, \phi_m \rangle = (-1)^n \, \delta_{nm}$$

However, when $n = m$ we see that the $\mathcal{PT}$ norms of eigenfunctions are not positive...

Bad News!! The norm associated with PT-Symmetry is not positive definite

We need to construct a inner product for our Hilbert space

We will identify a new symmetry that all $\mathcal{PT}$-symmetric Hamiltonians having an unbroken $\mathcal{PT}$-symmetry possess.

The linear operator: $\mathcal{C}$

The properties of this operator resemble those of the charge conjugation operator in particle physics

This will allow us to introduce an inner product structure associated with $\mathcal{CPT}$ conjugation for which the norms of quantum states are positive definite.

$$\langle \Psi, \Phi \rangle^{\mathcal{CPT}} = \int dx \Psi^{\mathcal{CPT}} \Phi$$

Where:

$$\Psi^{\mathcal{CPT}}(x) = \int dx \mathcal{C}(x,y) \Psi^* (-y) \qquad \mathcal{C}(x,y) = \sum_{n=0}^{\infty} \Psi_n(x) \Psi_m(y)$$

This inner product satisfies the requirements for the quantum theory