Topological invariants of gapped states and cosheaves on sites

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based on work with Nikita Sopenko (IAS, Princeton), J. Math. Phys. 63 (2022) 091903, and work in progress with Adam Artymowicz (Caltech) and Bowen Yang (CMSA, Harvard).

Basics, I

In this talk d denotes the dimension of space (so space-time has dimension $d+1$).

- Hall conductance σ_H is the skew-symmetric part of the conductivity tensor σ_{ij} which determines the electric current $\mathbf{J}$ as a function of the electric field $\mathbf{E}$:

$$\mathbf{J} = \sigma \mathbf{E}.$$

- At low temperatures, σ_H of insulating 2d systems becomes insensitive to variations of magnetic field and other parameters. In fact, $2\pi\sigma_H$ appears to be rational number.
- If QFT applies, then the leading term in the effective action describing the response is the Chern-Simons action

$$S_{CS}^{(2)} = \frac{k}{4\pi} \int_{\mathbb{R}^3} AdA, \quad A \in \Omega^1(\mathbb{R}^3),$$

where $k = 2\pi\sigma_H$. "Quantization" of k leads to quantization of σ_H.

Basics, II

- Analogous terms in the effective action exist for any $d = 2n$:

$$S_{CS}^{(2n)} = \frac{k}{(2\pi)^n} \int_{\mathbb{R}^{2n+1}} A(dA)^n, \quad A \in \Omega^1(\mathbb{R}^{2n+1}).$$

  For $n > 1$ they describe nonlinear response of a system to external electric and magnetic fields. The coefficient k is believed to be a rational number.

- There are also non-abelian generalizations where $U(1)$ is replaced with a compact semi-simple Lie group G. Instead of $k \in \mathbb{R}$, one has an invariant symmetric polynomial on the Lie algebra of G. "Topological" reponse to external gauge fields can again be described by the Chern-Simons action.

Some questions

The above was good enough explanation for a physicist, but not for a mathematician.

- Why is it OK to use Quantum Field Theory, and more specifically Topological Quantum Field Theory, to describe response?
- Why is $k = 2\pi\sigma_H$ robust under variations of parameters?
- Is k rational?

New developments

- Zero-temperature Hall conductance and its higher-dimensional and non-abelian analogs are examples of homotopy invariants of ground states of "insulating" 2d system
- To give a rigorous definition, it is crucial to work in infinite volume, hence need to use operator-algebraic techniques
- No appeal to QFT is needed, but QFT serves as a motivation

Part I: Construction of invariants

Lattice spin systems: standard setup

The "lattice" $\Lambda \subset \mathbb{R}^d$ is either $\mathbb{Z}^d$ or a more general Delone subset. To each site $j \in \Lambda$ we attach a f.d. Hilbert space V_j, let $\mathscr{A}_j = End(\mathcal{V}_j)$.

Let $\mathscr{A}$ be the norm-completion of $\otimes_{j\in\Lambda}\mathscr{A}_j$. It is a C^*-algebra. Its elements are quasilocal observables.

A state $\omega : \mathscr{A} \to \mathbb{C}$ is a positive linear functional of unit norm. Dynamics is specified by a skew-adjoint derivation of $\mathscr{A}$:

$$a \mapsto \mathsf{H}(a) = \sum_{j\in\Lambda}[\mathsf{h}_j, a], \quad \mathsf{h}_j \in \mathscr{A}, \quad \mathsf{h}_j^* = -\mathsf{h}_j.$$

H is unbounded and is only defined on a dense domain $\mathscr{D} \subset \mathscr{A}$.

ω is a gapped ground state if $\exists \Delta > 0$ such that

$$-i\omega(a^*\mathsf{H}(a)) \geq \Delta(\omega(a^*a) - |\omega(a)|^2), \quad \forall a \in \mathscr{D} \subset \mathscr{A}.$$

Lattice spin systems: rigging

- There is a natural $*$-subalgebra $\mathscr{A}_{al} \subset \mathscr{A}$ of well-localized observables ($\mathsf{b} \in \mathscr{A}_{al}$ means that b can be approximated by a local observable on a ball of radius r with $O(r^{-\infty})$ accuracy). $\mathscr{A}_{al}$ is a Fréchet algebra and dense in $\mathscr{A}$.
- Work with derivations which preserve $\mathscr{A}_{al}$:

$$\mathsf{b} \mapsto \mathsf{H}(\mathsf{b}) = \sum_j [\mathsf{h}_j, \mathsf{b}],$$

where $\mathsf{h}_j \in \mathscr{A}_{al}$ is well-localized near j and uniformly bounded. Get a Fréchet-Lie algebra of "nice" derivations $\mathfrak{D}_{al}$. They have a common dense domain $\mathscr{D} = \mathscr{A}_{al} \subset \mathscr{A}$.

All Hamiltonians and generators of symmetries ("charges") are assumed to lie in $\mathfrak{D}_{al}$.

Locally generated automorphisms

Since $\mathfrak{D}_{al}$ is a Fréchet space, it makes sense to talk about continuous and smooth functions with values in $\mathfrak{D}_{al}$.

Theorem 1. Any $\mathsf{F} \in C^0([0,1], \mathfrak{D}_{al})$ can be "exponentiated" to a family of automorphisms $\alpha(s) : \mathscr{A}_{al} \to \mathscr{A}_{al}$.

This follows from a version of Lieb-Robinson bounds proved by Nachtergaele, Ogata, and Sims.

Definition. An automorphism $\alpha : \mathscr{A} \to \mathscr{A}$ is called locally-generated (LGA) if it arises from such a family of derivations F. LGAs form a group.

Definition. Two states ω, ω' are "in the same phase" if $\omega' = \omega \circ \alpha$ for some LGA α.

A state is in a trivial phase if it is related by an LGA to an unentangled pure state ω_0.

Symmetries of systems and states

A symmetry of a system is a homomorphism α from an abstract group G to the group of LGAs. G can be discrete or a Lie group. Here we focus on Lie groups. Then one needs to require smooth dependence on parameters of G. This makes sense because LGAs form an "infinite-dimensional Lie group" (more precisely, diffeological group).

A Hamiltonian $\mathsf{H} \in \mathfrak{D}_{al}$ is G-invariant if $\alpha(g) \circ \mathsf{H} \circ \alpha(g)^{-1} = \mathsf{H}$ for all $g \in G$.

A state is G-invariant if $\omega \circ \alpha(g) = \omega$ for all g.

Definition. $\mathsf{F} \in \mathfrak{D}_{al}$ does not excite ω if $\omega(\mathsf{F}(a)) = 0$ for all $a \in \mathscr{A}_{al}$. Such F form a Lie subalgebra $\mathfrak{D}^{\omega}_{al} \subset \mathfrak{D}_{al}$.

$U(1)$ symmetry

In this talk I focus on $G = U(1)$, for simplicity.

For $G = U(1)$, a symmetry is determined by a "charge" $\mathsf{Q} \in \mathfrak{D}_{al}$ such that $\exp(2\pi\mathsf{Q}) = \mathrm{Id}$.

A G-invariant state ω satisfies $\omega(\mathsf{Q}(a)) = 0$ for all $a \in \mathscr{A}_{al}$. Thus $\mathsf{Q} \in \mathfrak{D}^{\omega}_{al}$.

Main object of interest: derivations which do not excite ω and are $U(1)$-invariant. These form a Fréchet-Lie algebra $\mathfrak{D}^{\omega,U(1)}_{al}$:

$$\mathsf{F} \in \mathfrak{D}^{\omega,U(1)}_{al} \text{ iff } [\mathsf{Q},\mathsf{F}] = 0 \text{ and } \omega(\mathsf{F}(a)) = 0 \ \forall a \in \mathscr{A}_{al}.$$

Obviously, $\mathsf{Q} \in \mathfrak{D}^{\omega,U(1)}_{al}$.

Approximate localization at a point

Every $b \in \mathscr{A}_{al}$, $b = -b^*$, defines an element $\mathrm{ad}_b \in \mathfrak{D}_{al}$. We call such derivations inner.

The kernel of the map $b \mapsto \mathrm{ad}_b$ consists of scalars, $b = \lambda \cdot 1_{\mathscr{A}}$, $\lambda \in i\mathbb{R}$. We can make this map injective by restricting to the "traceless" subspace $\mathfrak{d}_{al} \subset \mathscr{A}_{al}$.

Proposition

F *is inner iff* $\mathsf{F}(c) = \|c\| \times O(r^{-\infty})$, *where* c *is a local observable and* r *is a distance from* $0 \in \mathbb{R}^d$ *to the support of* c.

Thus an inner derivation is approximately localized at 0 (or any other point of $\mathbb{R}^d$).

Since the map $\mathfrak{d}_{al} \to \mathfrak{D}_{al}$ is injective, any inner derivation has a well-defined average in any state. In particular, an inner derivation in $\mathfrak{D}^{\omega}_{al}$ or $\mathfrak{D}^{\omega,U(1)}_{al}$ can be averaged over ω.

Approximate localization on regions

Proposition (A. Kitaev; N. Sopenko and AK)

Suppose $\Lambda = A \cup B$. Any $\mathsf{F} \in \mathfrak{D}_{al}$ can be written as

$$\mathsf{F} = \mathsf{F}_A + \mathsf{F}_B,$$

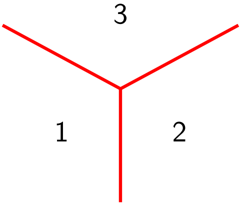
where F_A and F_B are approximately localized on A and B respectively. If ω is gapped and $\mathsf{F} \in \mathfrak{D}^{\omega}_{al}$, one can choose F_A and F_B to be in $\mathfrak{D}^{\omega}_{al}$. If in addition ω and F are $U(1)$-invariant, one can choose F_A and F_B to be in $\mathfrak{D}^{\omega,U(1)}_{al}$.

This is the key technical result which enables the construction of Hall conductance and other invariants of gapped states.

From now on, I will use a shorthand $\mathfrak{D} \equiv \mathfrak{D}^{\omega,U(1)}_{al}$.

Hall conductance

Let $\mathsf{Q} \in \mathfrak{D}$ and $\Lambda \subset \mathbb{R}^2$. Partition $\mathbb{R}^2$ into three sectors.



Write $\mathsf{Q} = Q_1 + Q_2 + Q_3$ where $Q_i \in \mathfrak{D}$ is approximately localized in region i. Let $\mathsf{Q}_{ij} = [Q_i, Q_j]$.

Q_{12} is localized near the ray 12. But

$$[\mathsf{Q}_1, \mathsf{Q}_2] = [\mathsf{Q} - \mathsf{Q}_2 - \mathsf{Q}_3, \mathsf{Q}_2] = -[\mathsf{Q}_3, \mathsf{Q}_2],$$

so Q_{12} is also localized near the ray 23.

Therefore Q_{12} is localized near the vertex 123, so it is inner and $\sigma^{\omega}_{d=2} = -i\omega(\mathsf{Q}_{12}) \in \mathbb{R}$ is well-defined.

Main theorem

Theorem (N. Sopenko and AK)

$\sigma^{\omega}_{d=2}$ does not depend on any choices made (except orientation of $\mathbb{R}^2$). It is also unchanged under $\omega \mapsto \omega \circ \alpha$ where α is an LGA. It is proportional to the zero-temperature Hall conductance of the state ω.

One can generalizate to arbitrary Lie groups. Then $\sigma^{\omega}_{d=2}$ takes values in G-invariant quadratic polynomials on the Lie algebra of G.

For special classes of states (such as states of the form $\omega_0 \circ \alpha$ where ω_0 is factorized pure state and α is any LGA) $\sigma^{\omega}_{d=2}$ is quantized (integral with suitable normalization).

Quantization of Hall conductance was first proved by M. Hastings and S. Michalakis by a very different method.

Part II: Formal developments

Questions

- How do we generalize to $d > 2$?
- Why conical regions?
- Can one get more invariants using other configurations of regions?

Approximate locality of nice derivations

- For any $U \subset \Lambda$ can define $\mathfrak{D}(U) \subset \mathfrak{D} = \mathfrak{D}_{al}^{\omega, U(1)}$, a Lie subalgebra of derivations approximately localized on U.
- If $\Lambda = \cup_i U_i$ and $\mathrm{Q} = \mathfrak{D}(\Lambda) = \mathfrak{D}$, can write $\mathrm{Q} = \sum_i \mathrm{Q}_i$, where $\mathrm{Q}_i \in \mathfrak{D}(U_i)$.
- The decomposition $\mathrm{Q} = \sum_i \mathrm{Q}_i$ is unique up to derivations "living" on $U_i \wedge U_j$ which is a "generalized intersection" of U_i and U_j.

Assignment of vector spaces to regions is reminiscent of presheaves of vector spaces.

The appearance of unions and intersections of regions is reminiscent of sheaves.

Actually, a closer analogy is with pre-cosheaves and cosheaves.

Pre-cosheaves of vector spaces

Let X be a topological space.

Definition

A pre-cosheaf $\mathcal{F}$ of vector spaces on X is an assignment $U \mapsto \mathcal{F}(U) \in VECT$ to every open $U \subset X$ and a linear map $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$ to every inclusion of opens $U \subset V$ such that $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$ for all $U \subset V \subset W$.

The map ρ_{VU} is called co-restriction. For pre-sheaves, the map is in the opposite direction and is called restriction.

Example: take $\mathcal{F}(U)$ to be the space of continuous functions on X with support in U.

In this example ρ_{VU} is injective for all $U \subset V$. Such pre-cosheaves are called coflasque. Let's restrict to those, then can suppress the maps ρ_{VU} from notation.

Cosheaves of vector spaces

Definition

A (coflasque) cosheaf is a pre-cosheaf with the "Mayer-Vietoris property": $\forall U_1, U_2$ and $\forall Q \in \mathcal{F}(U_1 \cup U_2)$ one can write $Q = Q_1 + Q_2$ for some $Q_i \in \mathcal{F}(U_i)$. Q_i are unambiguous up to

$$Q_1 \mapsto Q_1 + P_{12}, \ Q_2 \mapsto Q_2 - P_{12}, \quad P_{12} \in \mathcal{F}(U_1 \cap U_2).$$

This can be expressed as the exactness of the sequence

$$0 \leftarrow \mathcal{F}(U_1 \cup U_2) \leftarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \leftarrow \mathcal{F}(U_1 \cap U_2)$$

Actually, in the coflasque case we get a short exact sequence

$$0 \leftarrow \mathcal{F}(U_1 \cup U_2) \leftarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \leftarrow \mathcal{F}(U_1 \cap U_2) \leftarrow 0$$

Sheaves and cosheaves on sites

To use these notions in our case, we need to deal with two issues:

- Λ is a very boring (discrete) topological space
- Localization in our case is only approximate, so U_i are "fuzzy". Then what do we mean by intersection, union, and cover?

M. Kashiwara and P. Schapira (2001): one can organize functions with growth conditions into a sheaf if we replace the category $Open(X)$ with a different category equipped with a "Grothendieck topology". A category equipped with a Grothendieck topology is called a site.

Similarly, functions with decay conditions (test functions) can be organized into a cosheaf on a suitable site.

Semilinear sets

Definition

A semilinear set is a subset of $\mathbb{R}^d$ defined by a finite system of linear equations and inequalities.

A closed semilinear set is the same as a finite union of polyhedra. A polyhedron is the same as a closed convex semilinear set.

Let's use the metric $d(x, y) = \|x - y\|_\infty$.

Proposition

For any $U \subset \mathbb{R}^d$ and $r \geq 0$ let U^r be the r-thickening of U. If U is semilinear (polyhedron), then U^r is semilinear (polyhedron).

Definition

Let CS_d be the set of non-empty closed semilinear sets with a pre-order: $U \leq V$ iff $U \subseteq V^r$ for some $r \geq 0$.

A category of closed semilinear sets

- A pre-ordered set can be thought of as a category: $Mor(U, V)$ is a singleton if $U \leq V$ and empty otherwise.
- A pre-ordered set is like a poset, except $U \leq V$ and $V \leq U$ does not imply $U = V$. But it implies U and V are isomorphic objects.
- In CS_d, all non-empty bounded U are isomorphic. But cones with the same vertex and different bases are not isomorphic.
- In fact, CS_d is equivalent to the category of closed polyhedral subsets of S^{d-1} (sphere at infinity).
- An initial object in CS_d is any bounded closed semilinear subset (for example, $0 \in \mathbb{R}^d$).

Coherent topology on CS_d

- CS_d has all finite coproducts (joins). One can take the join $U \vee V$ to be $U \cup V$.
- CS_d has all finite products (meets, generalized intersections). For polyhedra, one can take the meet $U \wedge V$ to be $U^{d(U,V)} \cap V^{d(U,V)}$, where $d(U, V)$ is the distance between U and V.

Products and coproducts are unique up to isomorphism.

Definition

A cover of U is a finite nonempty collection $\{U_i\}_{i \in I}$ such that $U_i \leq U$ $\forall i \in I$ and $U \leq \bigvee_{i \in I} U_i$.

This makes CS_d into a site.

Spherical polyhedra

Definition

A polyhedral subset of S^{d-1} is a finite union of spherical polyhedra.

Theorem

The category CS_d is equivalent to the category of polyhedral subsets of S^{d-1} (with inclusions as morphisms).

This is because every polyhedral subset of $\mathbb{R}^d$ is isomorphic to a polyhedral cone with apex at 0.

Thus we can view sheaves and cosheaves on the site CS_d as sheaves and cosheaves on the sphere "at infinity".

A subtlety

The empty subset of S^{d-1} corresponds to the isomorphism class of a bounded polyhedron in $\mathbb{R}^d$ (for example, the point 0). The latter is an initial object in the category of closed semilinear sets.

Normally, one allows empty covers of the empty set.

Here it is more natural not to allow empty covers, because in $\mathbb{R}^d$ the initial object is not "empty".

The cosheaf $\mathfrak{D}$ and its Čech complex

Proposition

The assignment $U \mapsto \mathfrak{D}(U)$ is a coflasque cosheaf of vector spaces on the site CS_d.

Note that for U bounded semilinear (the analog of empty set), $\mathfrak{D}(U) \neq 0$. This is because we do not allow the empty cover of the initial object.

For any cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of $\mathbb{R}^d$, and any pre-cosheaf $\mathcal{F}$ one can define the homological Čech complex $C_\bullet(\mathfrak{U}, \mathcal{F})$:

$$0 \leftarrow \mathcal{F}(\Lambda) \leftarrow \bigoplus_i \mathcal{F}(U_i) \leftarrow \bigoplus_{i<j} \mathcal{F}(U_i \wedge U_j) \leftarrow \bigoplus_{i<j<k} \mathcal{F}(U_i \wedge U_j \wedge U_k) \leftarrow \dots$$

For a coflasque cosheaf, the homology of this complex is trivial.

Lie algebra structure

$\mathfrak{D}$ is a pre-cosheaf of Lie algebras, i.e. for any $U \leq V$ the inclusion $\mathfrak{D}(U) \to \mathfrak{D}(V)$ is a Lie algebra homomorphism.

$\mathfrak{D}$ is not a cosheaf of Lie algebras. But we have the following "Property I": $\forall U, U'$, we have $[\mathfrak{D}(U), \mathfrak{D}(U')] \subset \mathfrak{D}(U \wedge U')$. This property encodes locality.

Equivalently, the co-restriction maps $\mathfrak{D}_{al}(U) \to \mathfrak{D}_{al}(V)$ are inclusions of Lie ideals. We call such a structure a "local Lie algebra" over a site.

Next step: turn a local Lie algebra into a Differential Graded Lie Algebra (DGLA).

Reminder on DGLAs

A DGLA is a chain complex of vector spaces $(\mathcal{L}, \partial)$ equipped with a bilinear operation $[\cdot, \cdot]$ of degree 0 such that

- $[a, b] = -(-1)^{|a|\cdot|b|}[b, a]$
- $[\cdot, \cdot]$ obeys a graded version of the Jacobi identity
- $\partial[a, b] = [\partial a, b] + (-1)^{|a|}[a, \partial b]$ (super-Leibniz)

A DGLA $\mathcal{L}$ is called acyclic if its homology (as a chain complex) is trivial.

Let $[\mathcal{L}, \mathcal{L}]$ be the vector space whose elements are finite sums of commutators of elements of $\mathcal{L}$. It is also a DGLA (the commutator DGLA). $[\mathcal{L}, \mathcal{L}]$ need not be acyclic even if $\mathcal{L}$ is acyclic.

DGLA attached to a cover

Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a cover of a site. Let $\mathcal{F}$ be a local Lie algebra over the same site.

Proposition

The Čech complex $C_{\bullet+1}(\mathfrak{U}, \mathcal{F})$ is a Differential Graded Lie Algebra.

The DGLA grading is shifted w.r. to the standard Čech grading so that $\oplus_i \mathcal{F}(U_i)$ is in degree 1. I will use the standard Čech grading, then $[\cdot, \cdot]$ has degree $+1$.

The graded Lie bracket is the skew-symmetrization of the usual commutator. For example, let $\mathsf{a} = \oplus_i \mathsf{a}_i$ and $\mathsf{b} = \oplus_i \mathsf{b}_i$ be two 0-chains. Their bracket is a 1-chain with components

$$[\mathsf{a}_i, \mathsf{b}_j] - [\mathsf{a}_j, \mathsf{b}_i] \in \oplus_{i<j} \mathcal{F}(U_i \wedge U_j).$$

The case $d = 2$: Hall conductance

We have $\mathsf{Q} \in \mathfrak{D}(\mathbb{R}^2)$ which is central: $[\mathsf{Q}, \cdot] = 0$. Pick a cover $\mathfrak{U}$ of $\mathbb{R}^2$. There exists a 0-chain $\mathsf{q} = \bigoplus_{i \in I} \mathsf{q}_i$ such that $\mathsf{Q} = \sum_i \mathsf{q}_i$.

Consider a 1-chain $\mathsf{p} = \frac{1}{2}[\mathsf{q}, \mathsf{q}]$. It is a cycle.

$\mathfrak{U}$ induces a cover $\tilde{\mathfrak{U}}$ of S^1 by closed intervals. Let α be a 1-cochain of the Čech complex $C^\bullet(\tilde{\mathfrak{U}}, \mathbb{R})$. Let $\mathsf{p}(\alpha) = \sum_{i<j} \alpha_{ij} \mathsf{p}_{ij} \in \mathfrak{D}(\Lambda)$.

Theorem

If α is a cocycle, then $\mathsf{p}(\alpha)$ is inner, and thus $\omega(\mathsf{p}(\alpha)) \in \sqrt{-1}\mathbb{R}$ is well-defined. It does not depend on the choice of q. It depends on $\mathfrak{U}$ and α only through the class of α in $\varinjlim \check{H}^1(\tilde{\mathfrak{U}}, S^1) \simeq H^1(S^1)$.

Since $H^1(S^1)$ is one-dimensional, the Hall conductance invariant is the unique invariant one can construct from the 1-cycle $[\mathsf{q}, \mathsf{q}]$.

Geometry at infinity

One may consider lattice systems which "live" on an r-neighborhood of a polyhedral subset $W \subset \mathbb{R}^d$. Each such W defines a polyhedral subset $\tilde{W} \subset S^{d-1}$.

The Hall conductance invariants are labeled by $H^1(\tilde{W})$. This space can have dimension larger than 1.

For example, take any graph Γ on S^{d-1} made of geodesics and take W be a cone whose base is Γ. A lattice system "living" in a neighborhood of W is quasi-2d.

Its Hall conductance invariants take values in $H^1(\Gamma)$.

Higher Hall conductances

In general, to construct numerical invariants, need non-trivial cycles of the "commutator" DGLA $[C_\bullet(\mathfrak{U},\mathfrak{D}), C_\bullet(\mathfrak{U},\mathfrak{D})]$.

Can do this systematically using a Maurer-Cartan equation in $\mathfrak{D}[[t]]$. Let t be a formal variable of degree -2, let

$$\mathrm{g}(t) = t\mathrm{q} + t^2\mathrm{q}^{(2)} + t^3\mathrm{q}^{(4)} + \ldots, \quad \mathrm{q}^{(n)} \in C_n(\mathfrak{U},\mathfrak{D}),$$

and look for a solution of $\partial \mathrm{g}(t) + \frac{1}{2}[\mathrm{g}(t),\mathrm{g}(t)] = t\mathrm{Q}$. Then $\mathrm{p}^{(n-1)} = \partial \mathrm{q}^{(n)}$ is an $(n-1)$-cycle which lies in $[C_\bullet(\mathfrak{U},\mathfrak{D}), C_\bullet(\mathfrak{U},\mathfrak{D})]$.

To get an inner derivation, contract $\mathrm{p}^{(n-1)}$ with a Čech $(n-1)$-cocycle β of S^{d-1}. The higher Hall conductance is $\omega(\mathrm{p}^{(n-1)}(\beta))$. It depends only on the class of β in $H^{n-1}(S^{d-1})$.

Thus we get one invariant for even d and no invariants for odd d.

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