Resurgence

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Overview

Bibliography

- David Sauzin. *Introduction to 1-summability and resurgence*. arXiv: 1405.0356, 2014.
- Inês Aniceto, Gökçe Başar, Ricardo Schiappa. *A Primer on Resurgent Transseries and Their Asymptotics*. arXiv: 1802.10441, 2018.

Overview

Birth of Resurgence

The Rainbow:

Almost 200 years ago George Biddell Airy tried to describe a rainbow. He came up with the so called Airy function to describe the light distribution:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_0^\infty \cos\left(\frac{t^3}{3} + z\,t\right) \mathrm{d}t$$

- Hard to compute values
- George Stokes tried to approximate

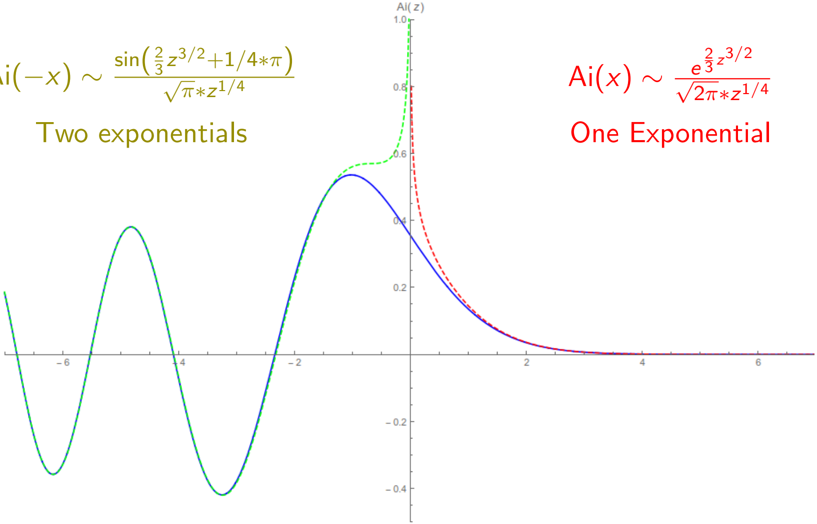
The Airy Function

$$\mathrm{Ai}(-x) \sim \frac{\sin\left(\frac{2}{3}z^{3/2}+1/4*\pi\right)}{\sqrt{\pi}*z^{1/4}}$$

Two exponentials

$$\mathrm{Ai}(x) \sim \frac{e^{\frac{2}{3}z^{3/2}}}{\sqrt{2\pi}*z^{1/4}}$$

One Exponential



⇒ Where does the second exponential come from?

The Stokes Phenomenon

Stokes Phenomenon:

The asymptotic behavior of complex functions can differ depending on the region in the complex plane. These regions are separated by Stokes lines.

- This phenomenon can be seen as a consequence of an underlying structure called resurgence framework
- It is intricately related to asymptotic series and re-summations of them

Even though the former discussion was in $\mathbb{R}$ it is straightforward to take it to $\mathbb{C}$ by replacing the cos in the integral of the Airy function by an exponential.

Resurgence: A Short Story

- Setting: Let us solve a differential equation via a power series ansatz Example: Euler Equation
- Problem: The Series might be divergent ⇒ Improve convergence by Borel transform
- Solution?: Perform analytic continuation of the underlying series
  ⇒ Transform back via Laplace transformation (not unique because of singularities!!)
- Not the full story: Observe that we need to include more ´sectors´ to obtain a full solution
- A bigger picture: the mechanism that relates these sectors is called Resurgence

What to expect:

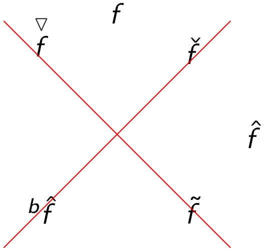
f

$\overset{\triangledown}{f}$ $\check{f}$

$\hat{f}$

${}^{b}\hat{f}$ $\tilde{f}$

What to expect:



$\Rightarrow$ Pictures!!

Overview

Resurgence and Nonlinear Differential Equations

Linear Case:

Solution is a superposition of linearly independent solutions:

$$f(x) = A \cdot f_1(x) + B \cdot f_2(x) + \dots$$

Integration parameters are independent of the coordinates.

Non-linear Case:

We can still write a now infinite superposition of sectors (that are not necessarily solutions of the equation):

$$g(x) = A(x) \cdot g_1(x) + B(x) \cdot g_2(x) + \dots$$

The combination of the sectors that solves the equation is coordinate dependent. Resurgence is a mechanism to determine the sectors and to fix how the parameters A, B, ... depend on the coordinates.

This series structure is called trans-series. This is due to the fact that the sectors contain different exponential contributions. In some sense this is a series in exponential contributions.

A Glimpse of Resurgence

Non linear differential equation (example):

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z,\sigma) - f(z,\sigma)^2 = 0$$

Ansatz:

$$f(z,\sigma) = \sum_{n=0}^{\infty} \sigma^n f^{(n)}(z), \quad \longleftarrow \text{ transseries structure}$$

$$\Rightarrow f(z,\sigma)^2 = \sum_{n=0}^{\infty} \sigma^n \sum_{j=0}^{n} f^{(n-j)}(z) f^{(j)}(z)$$

Then the following recursion emerges:

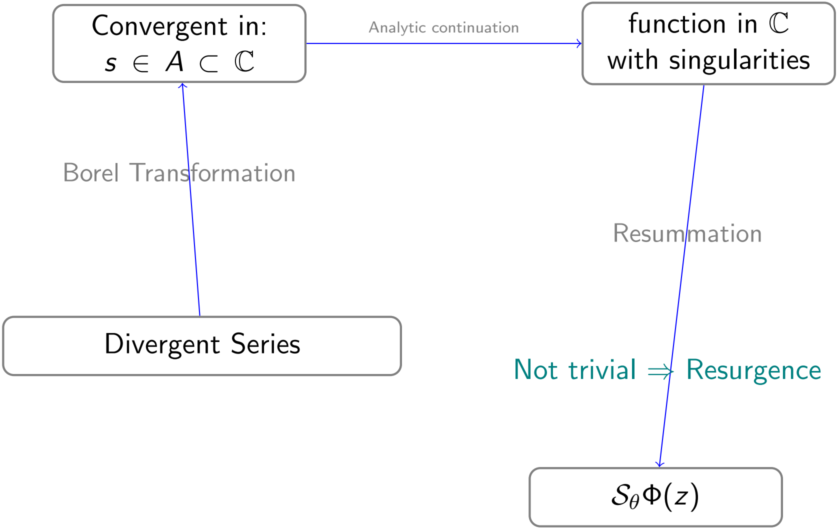
$$f^{(n)\prime}(z) = \sum_{j=0}^{n} f^{(n-j)}(z) f^{(j)}(z) \tag{1}$$

- Sectors are related
- This can be seen as a consequence of a more general structure called Alien Calculus

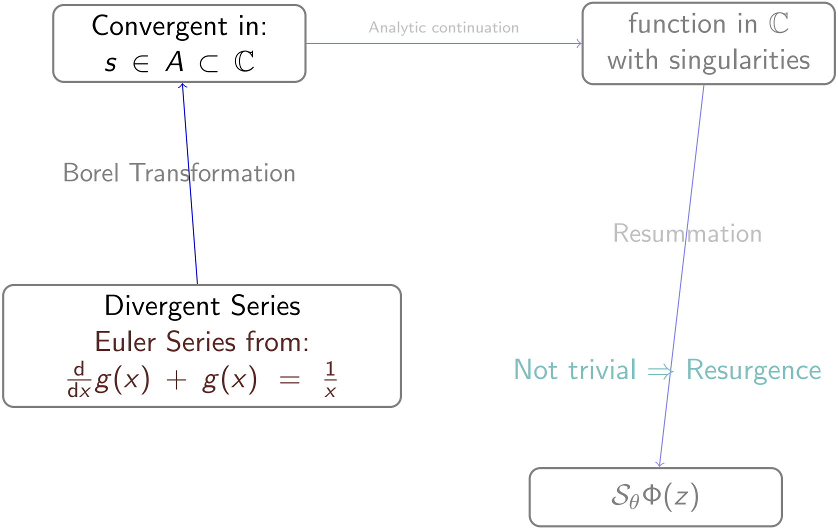
Too good to be true?

Overview

Overview: Resumation



Overview:


Convergent in:
$s \in A \subset \mathbb{C}$
Analytic continuation
function in $\mathbb{C}$
with singularities
Borel Transformation
Resummation
Divergent Series
Euler Series from:
$\frac{\mathrm{d}}{\mathrm{d}x} g(x) + g(x) = \frac{1}{x}$
Not trivial $\Rightarrow$ Resurgence
$\mathcal{S}_\theta \Phi(z)$

Divergent Series

The Euler Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) + g(x) = \frac{1}{x}, \quad \text{Series Ansatz: } g(z) = \sum_{n=1}^{\infty} n! \left(\frac{1}{x}\right)^n$$

This Series is divergent!

Series:

$$\mathbb{C}[[z^{-1}]] = \left\{ \phi(z) = \sum_{n \geq 0} a_n z^{-n} \middle| a_0, a_1, \cdots \in \mathbb{C} \right\}$$

Gevrey-s Divergent Series

$\phi(z) = \sum_{n \geq 0} a_n z^{-n} \in \mathbb{C}[[z^{-1}]]$ for which there exist $A, B > 0$ such that $|a_n| \leq A B^n (n!)^s$. We denote this vector space by $\mathbb{C}[[z^{-1}]]_s$

Convergent Series

$$\mathbb{C}\{\eta\} = \left\{ \text{Series with positive radius of convergence} \right\}$$

Divergent Series

The Euler Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x) + g(x) = \frac{1}{x}, \quad \text{Series Ansatz: } g(z) = \sum_{n=1}^{\infty} n! \left(\frac{1}{x}\right)^n$$

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Series:

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Gevrey-1 Divergent Series

$\phi(z) = \sum_{n\geq 0} a_n z^{-n} \in \mathbb{C}[[z^{-1}]]$ for which there exist $A, B > 0$ such that $|a_n| \leq A B^n (n!)^1$. We denote this vector space by $\mathbb{C}[[z^{-1}]]_1$

Convergent Series

$$\mathbb{C}\{\eta\} = \left\{ \text{Series with positive radius of convergence} \right\}$$

The Borel Plane

Borel Transformation

$$\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]]_1 \longrightarrow \mathbb{C}\{\zeta\} \quad (2)$$

$$\sum_{n=0}^{\infty} a_n z^{-n-1} \longrightarrow \sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!} \quad (3)$$

takes Gevrey-1 to convergent Series!

$$g(z) = \sum_{n=1}^{\infty} n!\,(-1)^n\,(1/z)^n \longrightarrow \sum_{n=1}^{\infty} (-\zeta)^n$$

A Tecnicality

On the Borel Plane the product is a convolution. We need an identity element that we call δ. Therefore:

$$\mathcal{B} : \mathbb{C}[[z^{-1}]]_1 \longrightarrow \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \quad (4)$$

Overview:

Convergent in:
$s \in A \subset \mathbb{C}$
$$\sum_{n=1}^{\infty} (-\zeta)^n$$

Analytic continuation →

function in $\mathbb{C}$ with singularities

Borel Transformation ↑

Divergent Series
$$g(z) = \sum_{n=1}^{\infty} n! \left(\frac{1}{x}\right)^n$$

Resummation ↓

Not trivial $\Rightarrow$ Resurgence

$\mathcal{S}_\theta \Phi(z)$

Germs and Analytic Continuation along a Path

Germ

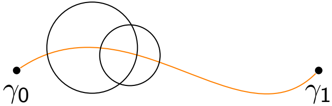
On the space of functions holomorphic in a neighborhood of a point p we can put the equivalence relation:

$$f \sim g \Leftrightarrow \exists \, \text{neighborhood} \, U \ni p \, : \, f|_U = g|_U$$

The set of germs at p is defined the quotient set.

Analytic continuation along a path

Collection of Pairs (f_t, U_t) such that:



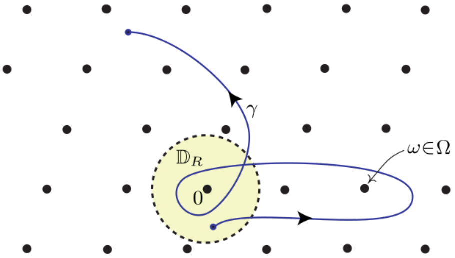
Let f be holomorphic in the domain U. If an analytic continuation of f to a connected open set $V \supset U$ exists, then the analytic continuation is unique.

Omega Continuability

Omega Continuable Germ

Ω non-empty closed discrete subset of $\mathbb{C}$ and $\hat{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ be holomorphic germ at the origin. We say $\hat{\phi}$ is an Ω-continuable germ if there exists $R > 0$ of $\hat{\phi}$ such that the disk $\mathbb{D}_R^* \cap \Omega = \emptyset$ and $\hat{\phi}$ admits analytic continuation along any path in $\mathbb{C}\backslash\Omega$ originating from $\mathbb{D}_R^*$. We will call:

$$\hat{\mathcal{R}}_\Omega := \{\text{all } \Omega\text{-continuable germs}\} \subset \mathbb{C}\{\zeta\} \tag{5}$$



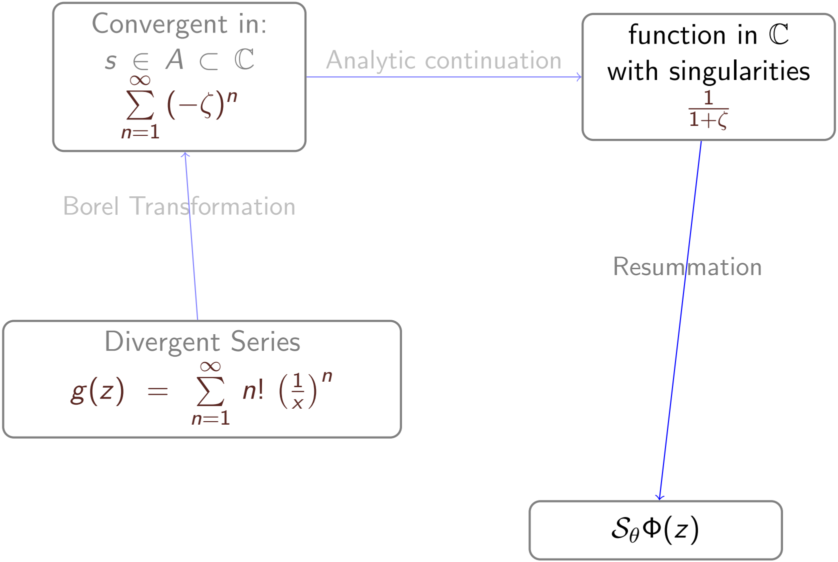
[1]

The Euler Example:

$$\Omega = \{-1\}$$

$$\sum_{n=1}^{\infty} (-\zeta)^n = \frac{1}{1+\zeta}$$

Overview: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$g(z) = \sum_{n=1}^{\infty} n! \left(\frac{1}{x}\right)^n$
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Inverse of the Borel transformation

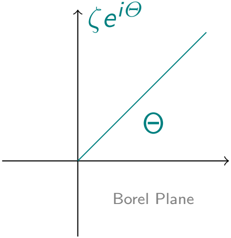
Resummation

Laplace Transformation for $\hat{\psi} \in \hat{\mathcal{R}}_{\Omega}$:

$$\left(\mathcal{L}^{\Theta}\hat{\psi}\right)(z) = \int_0^{\infty} e^{-z\,\zeta e^{i\Theta}}\,\hat{\psi}\left(\zeta e^{i\Theta}\right)\, e^{i\Theta} d\zeta \tag{6}$$

The re-summed now convergent expression:

$$\mathcal{S}^{\Theta}\psi\,(z) = \left(\mathcal{L}^{\Theta}\mathcal{B}\left[\psi\right]\right)(z) \tag{7}$$



Convergence condition:
$\arg(z) = -\Theta$

The Example of the Euler Series

As an example consider $z = 1$. Convergence implies $\Theta = 0$:

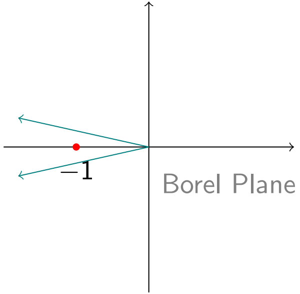
$$\mathcal{S}^0\left(\sum_{n=1}^{\infty} n!\left(\frac{1}{z}\right)^n\right)(1) = \mathcal{L}^0\left(\frac{1}{1+\zeta}\right)(1) = \int_0^{\infty} e^{-\zeta}\frac{1}{1+\zeta}\,\mathrm{d}\zeta \quad (8)$$
$$= 0.596347$$

Can we re-sum at every point?

Singularity at -1

$\Rightarrow$ No re-sum on $\mathbb{R}^-$

Can we cross the singularity?



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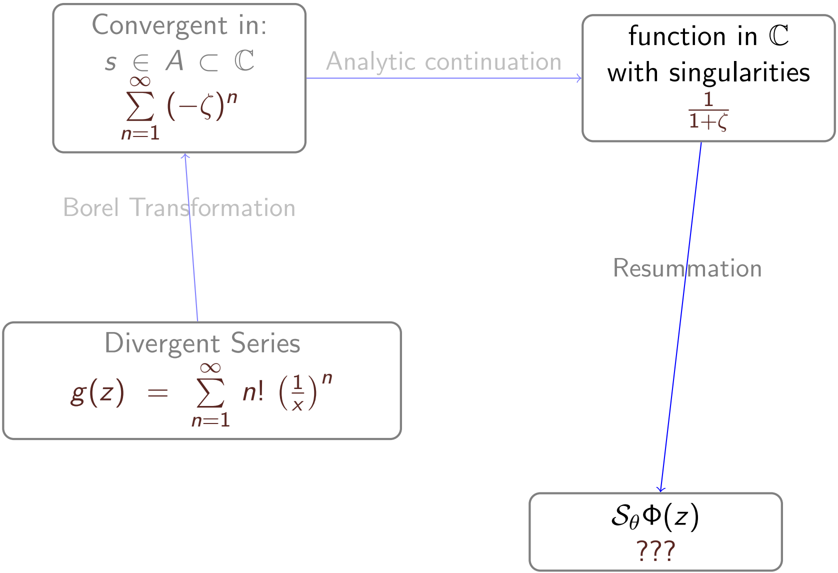
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$g(z) = \sum_{n=1}^{\infty} n! \left(\frac{1}{x}\right)^n$
$\mathcal{S}_{\theta}\Phi(z)$
???

Overview

Central Question:

Suppose we have an omega continuable germ holomorphic around the origin. Can we access information about the singularities of the underlying function?

Euler Example:

$$\sum_{n=1}^{\infty} (-\zeta)^n$$

Here it is easy because we know the analytic continuation of the series. This is not the normal case!!!

We want to understand the singularity at -1 purely from this holomorphic germ.

We need to enlarge our dictionary:

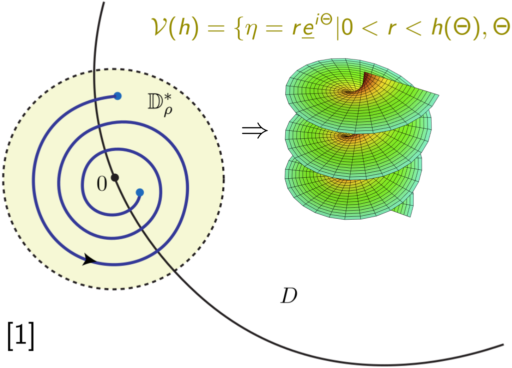
1. Singularities
2. Simple Singularities
3. Alien operators

Singularities: Spiral Continuation

Spiral Continuation

Let f be a function holomorphic in the Disk D to which the (locally only) singular point 0 is adherent. f has spiral contiuation around 0 if for every $L > 0$ there exists $\rho > 0$ such that f can be analytically continued along any path of length L starting in $D \cap \mathbb{D}_\rho^*$ and staying in $\mathbb{D}_\rho^*$.

$$\mathcal{V}(h) = \{\eta = r\underline{e}^{i\Theta} | 0 < r < h(\Theta), \Theta \in \mathbb{R}\}$$



[1]

Projection:

$\tilde{D} \subset \tilde{\mathbb{C}} \xrightarrow{\pi} D \subset \mathbb{C}^*$

On $\mathcal{V}(h)$ we then have:

$\check{f} := f \circ \pi$

Singularities: The Space of Singularities

The space of singular germs (ANA)

On the set of all $\check{f} : \mathcal{V}(h) \to \mathbb{C}$ we put:

$$\check{f}_1 \sim \check{f}_2 \Leftrightarrow \check{f}_1 \equiv \check{f}_2 \text{ on } \mathcal{V}(h_1) \cap \mathcal{V}(h_2)$$

and define ANA as the quotient set.

Let $f : \mathbb{C}^* \to \mathbb{C}$. The corresponding elements of ANA are:

$$S(1/\eta) + R(\eta), \quad \text{with: } R(\eta) = \sum_{n \geq 0}^{\infty} a_n \eta^n, \quad S(1/\eta) = \sum_{n > 0}^{\infty} a_{-n} (1/\eta)^n$$

The space of singularities (SING)

Clearly the convergent series are contained in ANA: $\mathbb{C}\{\eta\} \hookrightarrow$ ANA. We define the space of singularities as the quotient SING := ANA $/\mathbb{C}\{\eta\}$. We have the natural projection:

$$\text{sing}_0 : \text{ANA} \to \text{SING}$$

$$\text{sing}_0 \left(S(1/\eta) + R(\eta) \right) = \left[S(1/\eta) \right]_{\sim_{\mathbb{C}\{\eta\}}} \cong S(1/\eta) + \underline{\text{any holomorphic part}}$$

Simple Singularities

Simple Singularities

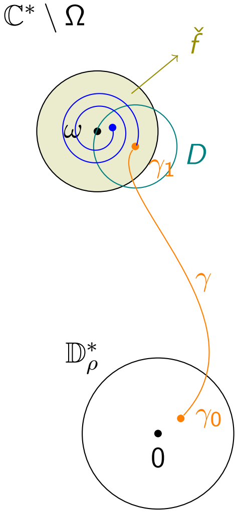
We say a function f has a simple singularity at 0 if locally around 0 it can be written as:

$$f(\zeta) \approx \frac{a}{\zeta} + \hat{\varphi}(\zeta)\log(\zeta) + R(\zeta), \; a \in \mathbb{C}, \, \hat{\varphi}, \, R \in \mathbb{C}\{\zeta\}$$

Space of Simple Singularities

We call $\mathrm{SING}^{\mathrm{simp}} \subset \mathrm{SING}$ the space of simple singularities

Alien Operators: What they do



Alien Operators

Let $\omega \in \Omega$, γ is a path in $\mathbb{C}\backslash\Omega$ starting at $\gamma_0 \in \mathbb{D}^*_\rho$ and ending at γ_1 such that $D \subset \mathbb{C}\backslash\Omega$ is an open disk centered at γ_1 to which ω is adherent. Then we define an alien Operator as:

$$A^\gamma_\omega : \mathbb{C}\delta \oplus \hat{\mathcal{R}}_\Omega \longrightarrow \mathrm{SING}$$

$$c\delta + \hat{\varphi} \longrightarrow \mathrm{sing}_0\left(\check{f}\left(\zeta\right)\right), \ \zeta \in \pi^{-1}(-\omega + D)$$

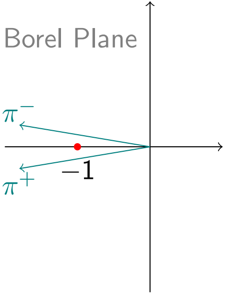
with: $\eta \rightarrow f(\eta) := \mathrm{cont}_\gamma \hat{\varphi}\left(\omega + \eta\right)$

Basically the alien operator A^γ_ω zooms in on the singularity at ω of the function underlying the holomorphic germ $\hat{\varphi}$.

Aliens to Cross Singularities

Euler example to concretely see alien operators at work:

$$\sum_{n=1}^{\infty}(-\zeta)^n \to \frac{1}{1+\zeta} \Rightarrow A_\omega^\gamma\left(\sum_{n=1}^{\infty}(-\zeta)^n\right) = \frac{1}{1+\zeta} \leftarrow \text{ Linear Eq.}$$



We remember the Laplace Transformation for $\hat{\psi} \in \hat{\mathcal{R}}_\Omega$:

$$\left(\mathcal{L}^\Theta \hat{\psi}\right)(z) = \int_0^\infty e^{-z\,\zeta e^{i\Theta}}\,\hat{\psi}\left(\zeta e^{i\Theta}\right)\, e^{i\Theta}\mathrm{d}\zeta$$

By closing the integration contour at infinity we can conclude:

$$\left(\mathcal{L}^{\pi^+} - \mathcal{L}^{\pi^-}\right)\frac{1}{1+\zeta} = 2\pi i e^z \qquad (9)$$

Concluding the Euler Example

Again consider $z = 1$:

$$\mathcal{S}^0 \left(\sum_{n=1}^{\infty} n! \left(\frac{1}{z} \right)^n \right) (1) = \int_0^{\infty} e^{-\zeta} \frac{1}{1+\zeta} \, \mathrm{d}\zeta = 0.596347$$

Crossing the singularity $\Leftrightarrow 2\pi i$
Therefore the result is not unique:

$$\mathcal{S}^0 \left(\sum_{n=1}^{\infty} n! \left(\frac{1}{z} \right)^n \right) (1) = 0.596347 + 2\pi i e \, n, \;\; n \in \mathbb{Z}$$

This is the full solution.

Overview

A Bigger Picture

Having concluded the simple example of the Euler series we want to understand the most important equation of resurgence.
Therefore we need to again enlarge our dictionary: We need:

- introduce more general alien operators
- simple resurgent functions
- introduce the bridge equation

In the following we want to give an overview, because a better treatment would be very time intensive.

More Alien Operators

It turns out that for more complicated situations it makes sense to define more powerful operators:

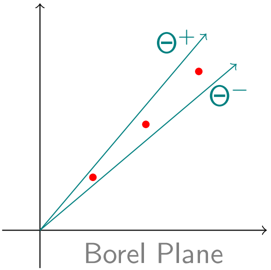
Stokes Automorphism

In order to cross a ray of singularities we use the Stokes automorphism:

$$\mathcal{S}^{\Theta^+} = \mathcal{S}^{\Theta^-} \circ \underline{\mathfrak{S}}_{\Theta}$$

$$\exp\left(\sum_{\omega_\Theta} \dot{\Delta}_{\omega_\Theta}\right)$$

Combinations of A_ω^γ



Alien Operators: Simple Resurgent Functions

Simple Singularities

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$$f(\zeta) \approx \frac{a}{\zeta} + \hat{\varphi}(\zeta)\log(\zeta) + R(\zeta),\ a \in \mathbb{C},\ \hat{\varphi},\ R \in \mathbb{C}\{\zeta\}$$

Resurgent Functions

We call a $c\delta + \hat{\varphi} \in \mathbb{C}\delta \oplus \hat{\mathcal{R}}_\Omega$ an Ω-resurgent function with $\hat{\varphi}$ an Ω-continuable germ and c a complex number.

Simple Resurgent Function

A simple Ω-resurgent function is a resurgent function $\overset{\triangledown}{\varphi}$ such that $\forall\,\omega,\ \gamma,\ A^{\gamma}_{\omega}\overset{\triangledown}{\varphi}$ is a simple singularity. The space of simple resurgent functions is called $\mathbb{C}\delta \oplus \hat{\mathcal{R}}^{\text{simp}}_\Omega$

The action of alien operators on simple resurgent functions is:

$$\overset{\triangledown}{\varphi} \in \mathbb{C}\delta \oplus \hat{\mathcal{R}}^{\text{simp}}_\Omega \ \Rightarrow\ A^{\gamma}_{\omega}\overset{\triangledown}{\varphi} \in \mathbb{C}\delta \oplus \hat{\mathcal{R}}^{\text{simp}}_{-\omega+\Omega}$$

The Name Resurgence

- Imagine the set Ω is closed, meaning $\omega \in \Omega \Rightarrow \omega + \Omega = \Omega$
- Then alien operators take simple Ω-resurgent functions to itself
- The structure of simple resurgent functions is as follows:

$$\phi(\zeta) \approx \frac{a}{\zeta - A} + \hat{\varphi}(\zeta - A)\log(\zeta - A) + R(\zeta), \; a \in \mathbb{C}, \; \hat{\varphi}, \; R \in \mathbb{C}\{\zeta\},$$

where ϕ and φ are trans-series sectors.

We say, that φ resurges at A. This can be written for all the sectors, giving rise to the name Resurgence.

The Bridge Equation

Consider a differential equation for $\Phi(z)$. Trans-series Ansatz:

$$\Phi(z) = \sum_n \sigma^n e^{-\frac{A\cdot n}{z}} \Phi_n(z)$$

Then the alien derivative fullfills:

- $\dot{\Delta}_\omega$ are derivations!
- $\left[\dot{\Delta}_\omega, \frac{\partial}{\partial z}\right] = 0$ and $\left[\frac{\partial}{\partial \sigma}, \frac{\partial}{\partial z}\right] = 0$

$\Rightarrow$ If there is a differential equation for $\frac{\partial \Phi}{\partial \sigma}$ also $\dot{\Delta}_\omega \Phi$ fullfills it!
$\Rightarrow$ They are related by a function in σ (constant in z):

$$\dot{\Delta}_\omega \Phi(z,\sigma) = S_\omega(\sigma) \frac{\partial \Phi}{\partial \sigma}(z,\sigma)$$

- relates alien operators with traditional derivatives
- The function $S_\omega(\sigma)$ encodes information about the differential equation

Overview

Conclusion

- Resurgence is a tool to deal with divergent series
- It can be used to solve differential equations
- It is based on alien calculus
- As such it provides information about how the so called trans-series sectors are related.
- Bridge equation relates to ´normal´ analysis