Skeins on Tori

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UC Davis

July 17, 2024

Topological Quantum Field Theory Seminar
University of Lisbon

Outline

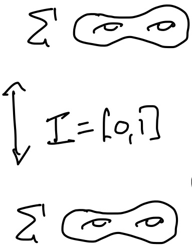
- SL_2 skeins: skein algebra, skein module, skein category
- SL_N skeins
- $\dim \mathrm{SkMod}_{\mathrm{SL}_N}(T^3)$
- Hochschild homology HH_0 of skein category for T^2, Morita equivalences
- role of DAHA and elliptic Schur-Weyl duality

Skeins

where skeins arise:
quantum topology
Witten-Reshetikhin-Turaev TQFT
deformation of G-character stacks, coordinate ring of character variety
HOMFLYPT invariant of knots and links

$\mathcal{A}$-Skein algebra of (oriented) surface Σ

$M = \Sigma \times I = \Sigma \times [0,1]$



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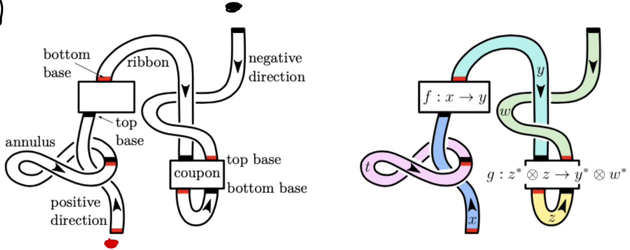

FIGURE 2. An example of a ribbon graph and its colouring. Image from [Coo19, Section 4.2]

ribbons labeled by objects $x, y, z, w, t, \ldots \in \mathcal{A}$

$\boxed{\text{coupon}}$ morphisms

skein relations. ...

skein picture

need not have ribbons end here or on $\partial M = \Sigma_i$ maybe

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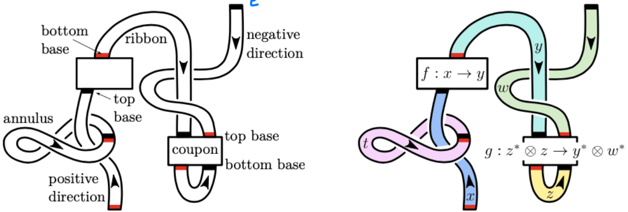

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FIGURE 2. An example of a ribbon graph and its colouring. Image from [Coo19, Section 4.2]

V = 2-dimensional irrep of SL_2 labels all ribbons
Recall $V \simeq V^*$ and any fin. dim. irrep is a summand of some $V^{\otimes d}$

SL_2 skein relations

(local)

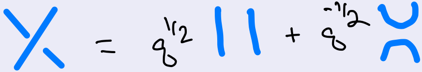
$q^{1/2}\, T - q^{-1/2}\, T^{-1} = (q - q^{-1})\, \mathrm{id}_{V \otimes V}$

sign idempotent

projection

$e^-(2) = \frac{1}{[2]}\left(q \cdot \mathrm{id} - q^{1/2}\, T\right)$ $\qquad V \otimes V \longrightarrow \Lambda^2 V \longrightarrow V \otimes V$

$= \frac{-1}{[2]}$ 



crossingless

V = N-dimensional irrep of SL_N
Recall any fin. dim. irrep is a summand of some $V^{\otimes d}$

SL_N skein relations

$q^{1/N}$  $-$ $q^{-1/N}$  $=$ $(q - q^{-1})$ 

$q^{1/N}\, T \; - \; q^{-1/N}\, T^{-1} \; = \; (q - q^{-1})\, \mathrm{id}_{V \otimes V}$

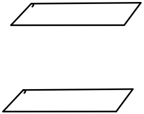
sign idempotent

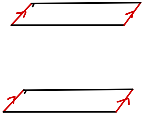
$e^-(N) = \pm \frac{1}{[N]}$ 

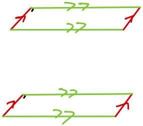
$V^{\otimes N} \longrightarrow \Lambda^N V \longrightarrow V^{\otimes N}$

$\mathrm{SkAlg}_{\mathrm{SL}_2}(\Sigma)$

Σ | $\mathrm{I} \times \mathrm{I}$ | $S^1 \times \mathrm{I}$ | $S^1 \times S^1$



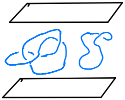




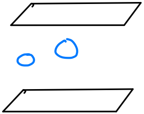
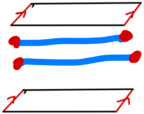
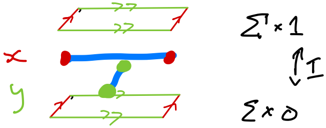
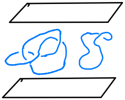
$\Sigma \times 1$

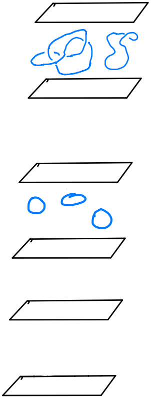
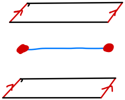
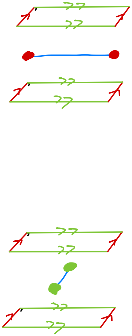
$\updownarrow \mathrm{I}$

$\Sigma \times 0$



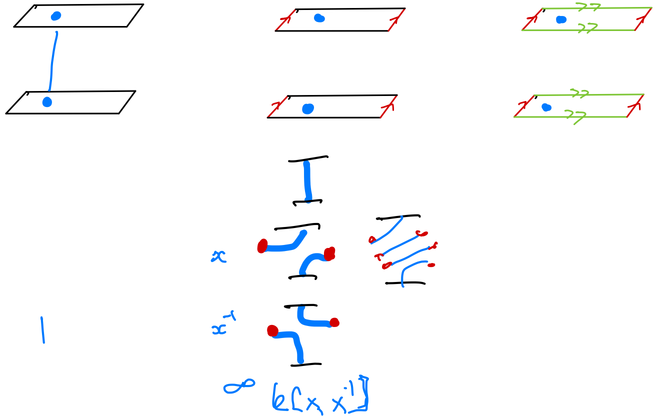
dim

Σ	$I \times I$	$S^1 \times I$	$S^1 \times S^1$
			
			
dim	1	∞ $k[x]$	∞

Σ	$\mathrm{I} \times \mathrm{I}$	$S^1 \times \mathrm{I}$	$S^1 \times S^1$
crossingless const *			
dim	1	∞	∞
algebra	k	$k[x]$	anti spherical DAHA

d points/disks in $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ that ribbons (all labeled V) can end on.

Σ	$\mathrm{I} \times \mathrm{I}$	$S^1 \times \mathrm{I}$	$S^1 \times S^1$



dim

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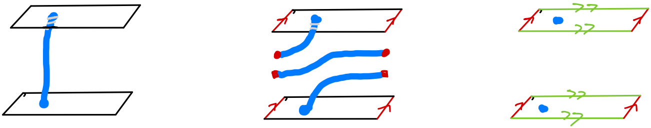

dim	1	∞	?
algebra	$\Bbbk$	$\Bbbk[x, x^{-1}]$	

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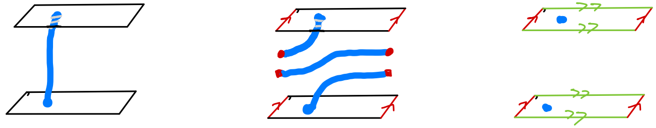
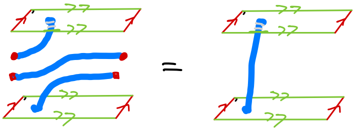
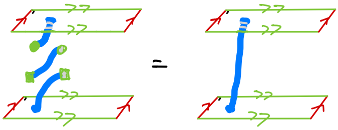


Σ	$\mathrm{I} \times \mathrm{I}$	$S^1 \times \mathrm{I}$	$S^1 \times S^1$
dim	1	∞	4
algebra	$\Bbbk$	$\Bbbk[x, x^{-1}]$	$\Bbbk[x,y]/(x^2-1, y^2-1)$

skein relation

surprising skein relation for $d = 1$, $\Sigma = T^2 = S^1 \times S^1$





$$\mathrm{SkAlg}_{\mathsf{SL}_2,1}(\Sigma) \simeq \mathcal{K}[x,y]/(x^2-1, y^2-1)$$

References, citations, related work

Frohman Gelca Kauffmann bracket skein algebra, SL_2
Terwilliger
Koornwinder
Morton Samuelson elliptic Hall algebra and spherical DAHA
Bullock Przytycki
Bullock Frohman Kania-Bartoszynska
Bittman Chandler Mellit Novarini
Ben-Zvi Brochier Jordan
Cooke
Gunningham Jordan Safronov

the G-skein module of a closed oriented 3-manifold M is finite-dimensional

Conjecture of Witten:

Theorem (Gunningham-Jordan-Safronov)

Let q be generic, M be a closed oriented 3-manifold, and G a connected reductive algebraic group. Then

$$\dim \mathrm{SkMod}_G(M) < \infty.$$

Let's compute these dimensions for $G = \mathrm{SL}_N$, $M = S^1 \times S^1 \times S^1$.

[Carrega-Gilmer '17]

For $N = 2, 3, 4, \ldots$ we compute
$\dim \mathrm{SkMod}_{\mathrm{SL}_N}(T^3) =$ 9, 29, 75, 131, 266, 357, 617, 810, 1207, 1386, 2272, ...

Theorem (Gunningham-Jordan-V-Yang)

The SL_N-skein module of the 3-torus T^3 has dimension

$$\dim \mathrm{SkMod}_{\mathrm{SL}_N}(T^3) = \sum_{\lambda \vdash N} \gcd(\lambda)^3 = \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^{\oplus 3}} \mathcal{P}(\gcd(v, N))$$

where $\mathcal{P}(d) =$ the number of partitions of d

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HH_0 for 2-torus, i.e. $M = T^2 \times \mathrm{I}$

For $N = 2, 3, 4, \ldots$ we compute
$\dim \mathrm{HH}_0(\mathrm{SkAlg}_{\mathrm{SL}_N}(T^2)) =$
$5, 11, 23, 31, 60, 63, 109, 126, 183, 176, 330, 269, \ldots$

Theorem (Gunningham-Jordan-V-Yang)

The dimensions of the zeroeth Hochschild homology of the skein algebras of the 2-torus T^2 for $G = \mathrm{SL}_N$ is given by

$$\dim \mathrm{HH}_0(\mathrm{SkAlg}_{\mathrm{SL}_N}(T^2)) = \sum_{\lambda \vdash N} \gcd(\lambda)^2 = \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}} \mathcal{P}(\gcd(v, N))$$

Theorem (Gunningham-Jordan-**V**-Yang)

(1) The SL_N*-skein module of the 3-torus* T^3 *has dimension*

$$\begin{aligned}\dim \mathrm{SkMod}_{\mathrm{SL}_N}(T^3) &= \sum_{\lambda \vdash N} \gcd(\lambda)^3 \\ &= \sum_{(v_1,v_2,v_3)\in(\mathbb{Z}/N\mathbb{Z})^{\oplus 3}} \mathcal{P}(\gcd(v_1,v_2,v_3,N))\end{aligned}$$

(2) The dimensions of the zeroeth Hochschild homology of the skein algebras of the 2-torus T^2 *for* $G = \mathrm{SL}_N$ *is given by*

$$\begin{aligned}\dim \mathrm{HH}_0(\mathrm{SkAlg}_{\mathrm{SL}_N}(T^2)) &= \sum_{\lambda \vdash N} \gcd(\lambda)^2 \\ &= \sum_{(v_1,v_2)\in(\mathbb{Z}/N\mathbb{Z})^{\oplus 2}} \mathcal{P}(\gcd(v_1,v_2,N))\end{aligned}$$

$N = 2$

$\mathcal{P}(\gcd(\mathbf{v_1}, \mathbf{v_2}, 2))$

$P(1)=1$
$P(2)=2$

$2, 1+1$

	1	2
1	1	1
2	1	2

$\dim \mathrm{HH}_0(\mathrm{SkAlg}(T^2)) = 5$

$\mathcal{P}(\gcd(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, 2))$

$\mathbf{v}_3 = 1$	1	2
1	1	1
2	1	1

$\mathbf{v}_3 = 2$	1	2
1	1	1
2	1	2

$\dim \mathrm{SkMod}(T^3) = 9$

4 + 5

same 4 as before

$4 = \dim k[x,y]/(x^2-1, y^2-1)$

$N = 6$

$\gcd(\mathbf{v}, 6)$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	2	1	2
5	1	1	1	1	1	1
6	1	2	3	2	1	6

$\mathcal{P}(\gcd(\mathbf{v}, 6))$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	2	1	2
5	1	1	1	1	1	1
6	1	2	3	2	1	11

$k = 2, \sum = 60$ $\qquad$ $k = 3, \sum = 266$

T^2 $\qquad$ T^3

$P(6) = 11$

$\mathrm{HH}_0(\mathrm{SkAlg}(T^2))$ versus $\mathrm{SkMod}(T^3)$

grading by $H_1(T^2; \mathbb{Z}/N\mathbb{Z}) = (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}$

grading by $H_1(T^3; \mathbb{Z}/N\mathbb{Z}) = (\mathbb{Z}/N\mathbb{Z})^{\oplus 3}$

contribution from $\mathrm{HH}_0(\mathrm{SkAlg}(T^2))$ lives in degree $(*, *, 0)$ piece

elliptic Schur-Weyl duality

K(Braid group)

$\mathrm{SkAlg}_{SL_N, N}(T^2) \longleftarrow\!\!-\,-\,-\,-\; \text{DAHA } \mathbb{H}_N$

at specialized parameters

$\mathrm{SkAlg}_{\mathsf{SL}_N}(T^2)$ Morita equivalent to antispherical DAHA $e^- \mathbb{H} e^-$

shift isomorphism $\implies$ Morita equivalent to $e^+(\mathcal{K}_\omega[\Lambda \oplus \Lambda] \# \mathcal{K}[\mathcal{S}_N])e^+$
$= \mathcal{K}_\omega[\Lambda \oplus \Lambda]^{\mathcal{S}_N}$

Morita equivalent to $\mathcal{K}_\omega[\Lambda \oplus \Lambda] \# \mathcal{K}[\mathcal{S}_N]$

compute $\mathsf{HH}_0(\mathcal{K}_\omega[\Lambda \oplus \Lambda] \# \mathcal{K}[\mathcal{S}_N])$ by hand.

dim $= \sum_{\lambda \vdash N} \mathsf{gcd}(\lambda)^2$

$G = SL_N, GL_N$

$D_q(G) - \mathrm{mod}^G$

strongly equivariant quantum D-modules

Ben-Zvi, Brochier, Jordan
Cooke
GJS

$\mathrm{SkCat}_G(T^2)$-mod

Jordan
Schur Weyl functor

DAHA-mod

elliptic braid group

$G = SL_N, GL_N$

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$HK^{univ} = D_q(G) / \text{moment map ideal}$

GJV

empty skein

$\mathbb{H}_N e^-(N)$

at $q, q^{1/N}$ parameters

$$HK^{univ} = D_q(G) \big/ \text{moment map ideal}$$

$\mathrm{End}_{D_q(G)\text{-mod}}(HK^{univ})$

empty skein

$\mathbb{H}_N e^-(N)$ at $q, q^{1/N}$ parameters

Take End

Take End

$\mathrm{SkAlg}_{SL_N,0}(T^2)$

$e^- \mathbb{H} e^{-\,op}$ (anti-spherical)

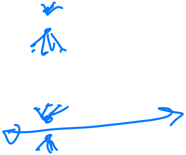
Morita

Morita

$e\mathrm{Sk}_N e$

$\mathrm{SkAlg}_{SL_N,N}(T^2)$

$\mathbb{H}$



SL_2 : Frohman - Gelca

Koornwinder, Terwilliger

Kauffmann bracket skein algebra $T^2 \longleftrightarrow sH_{q,q}$

$T^2 \setminus D^2 \longleftrightarrow sH_{q,t}$

GL_N :

[Morton - Samuelson '21]

$$\begin{array}{ccccc}
\mathrm{BrSkAlg}_{GL_N}(T^2 \setminus D^2) & & \longrightarrow & & \text{spherical } H_N \text{ type } GL \\
\downarrow \text{fill hole} & & & & \downarrow {\scriptstyle N \to \infty \text{ [Schiffmann Vasserot]}} \\
\mathrm{SkAlg}_{GL_N}(T^2) & \xrightarrow{\text{[MS '17]}} & EHA_{q,q^{-1}} & \xleftarrow{t=q^{-1}} & EHA_{q,t} \text{ elliptic Hall algebra} \\
\downarrow & & & & \\
\mathrm{SkAlg}_{GL_N}(S^1 \times I) & & & & \\
\downarrow & & & & \\
\mathrm{SkAlg}_{GL_N}(I \times I) & & & &
\end{array}$$

to get Morita equivalences

Proposition (GJV)

- $e^- \in \mathbb{H}$ is a full idempotent
- $e^- \in \mathrm{SkAlg}_{\mathrm{SL}_N,N}(T^2)$ is a full idempotent

Corollary

$\mathbb{H}_N \simeq \mathrm{SkAlg}_{\mathrm{SL}_N,N}(T^2)$

$e^- \mathrm{SkAlg}_N e^-$
$\shortparallel$
Morita + $e^- \mathbb{H} e^- \simeq \mathrm{SkAlg}_0$

Corollary

can use braids instead of tangles

THANK YOU

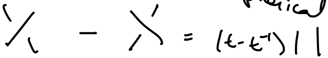
$k[y_1^{\pm} \dots y_n^{\pm}]\, H_n^{fin}(t) \otimes k[x_1^{\pm} \dots x_n^{\pm}] \underset{q_0}{\quad} \cong$ vector sp DAHA GL

$e^- \mathbb{H}(q,t) e^- \xrightarrow{\text{shift}} H_n^{fin}(q,t)$

$k[y_1^{\pm} \dots y_n^{\pm}]\, H_n^{fin} \otimes k[x_1^{\pm} \dots x_n^{\pm}] \Big/ x_1 \cdots x_n = 1$

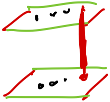
$\Big/ y_1 \cdots y_n = 1$

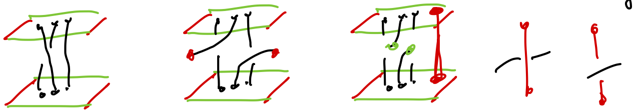
$\underline{GL}$ $\quad e^+ \mathbb{H}(q, qt) e^+$ spherical



$x_1 \cdots x_n y_i = q y_i x_1 \cdots x_n$

$x_j y_1 \cdots y_n = q^{-1} y_1 \cdots y_n x_j$





H_n^{fin} $\qquad x_2$ $\qquad y_2$ $\qquad q$