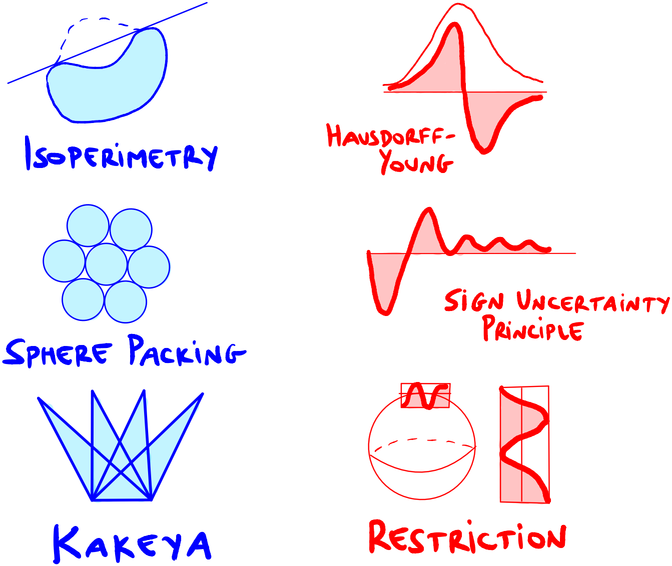
Sharp extension inequalities on finite fields

Diogo Oliveira e Silva
(joint with C. González-Riquelme: 2405.16647)

Geometria em Lisboa
16 July 2024

Positive vs. Oscillatory



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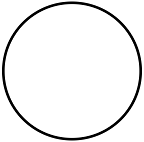

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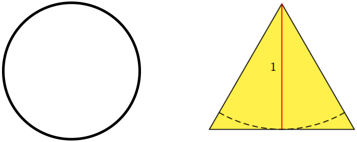

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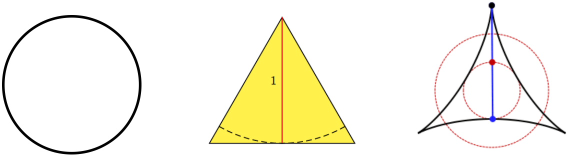

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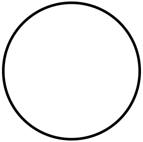

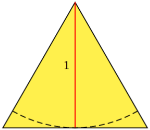

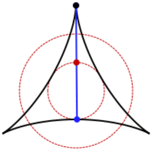

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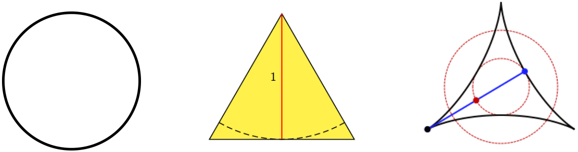

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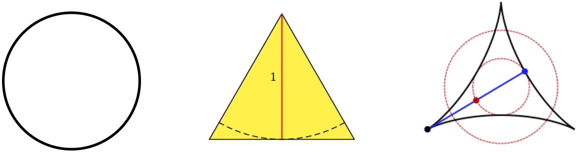

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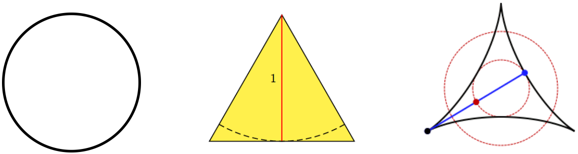

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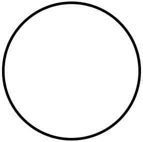

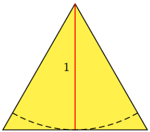

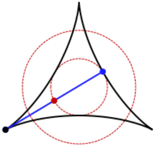

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Kakeya Set Conjecture on $\mathbb{R}^d$

If $K \subset \mathbb{R}^d$ is a Kakeya set, then $\dim_H(K) = d$.

Fourier transform on $\mathbb{R}^d$

$$\boxed{\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x}$$

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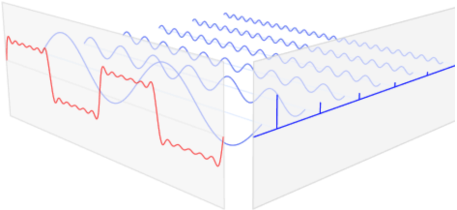

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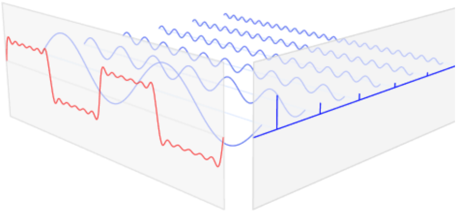

- This defines a contraction from L^1 to L^∞
- It extends to a unitary operator on L^2
- It extends to contraction from L^p to $L^{p'}$, if $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$

Restriction Problem on $\mathbb{R}^d$

Given $1 \leq p \leq 2$, for which exponents $1 \leq q \leq \infty$ does

$$\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \, \mathrm{d}\sigma(\omega) \lesssim \|f\|^q_{L^p(\mathbb{R}^d)} \quad \text{hold?}$$

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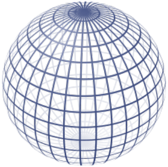
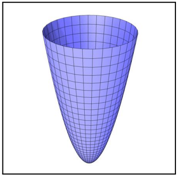
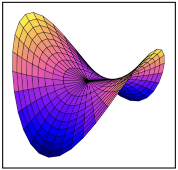
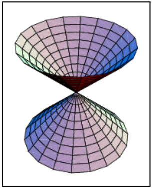

Bochner–Riesz Conjecture on $\mathbb{R}^d$

In what sense do Fourier series/integrals converge?

$$\widehat{S_R^\delta f} = S_R^\delta \widehat{f}, \ \textit{where } S_R^\delta(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta$$

- $\|S_1^\delta f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ *iff* $\delta > 0$ *and* $d|\frac{1}{2} - \frac{1}{p}| < \frac{1}{2} + \delta$?

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$$f_\delta^*(\omega) = \sup_{\boldsymbol{a} \in \mathbb{R}^d} \frac{1}{|T^\delta|} \int_{T_\omega^\delta(\boldsymbol{a})} |f|$$

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More recently:

- ℓ^2-decoupling (2015)
- Vinogradov's Mean-Value Theorem (2016)
- Local Smoothing Conjecture (solved in $\mathbb{R}^{2+1}$ only in 2020)

Why should *any* nontrivial restriction inequality hold?

The adjoint of the restriction operator, $\mathcal{R}f = \widehat{f}|_{\mathbb{S}^{d-1}}$, is the extension operator, $\mathcal{E}f = \widehat{f\sigma}$, given by

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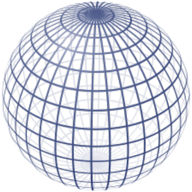
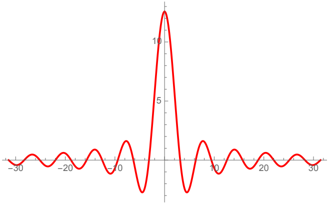
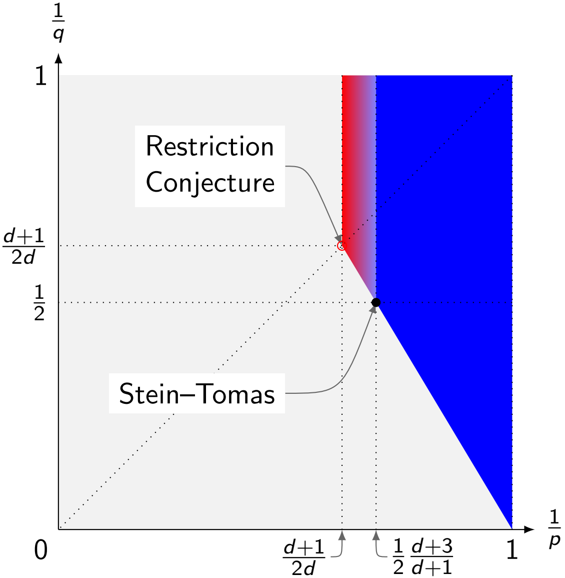




Riesz diagram for the restriction operator to $\mathbb{S}^{d-1}$



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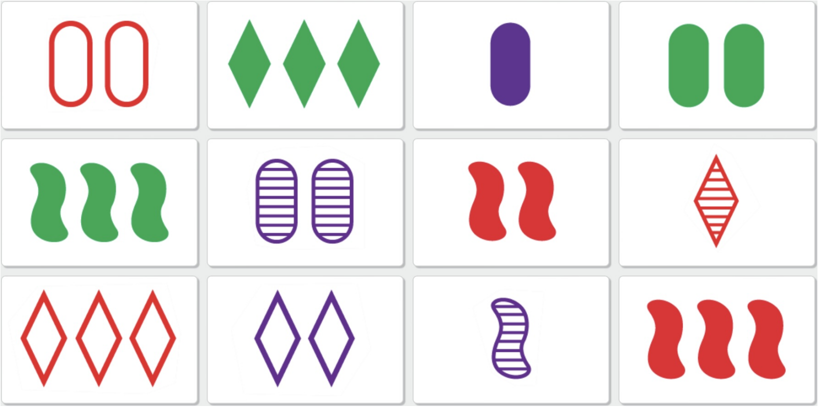

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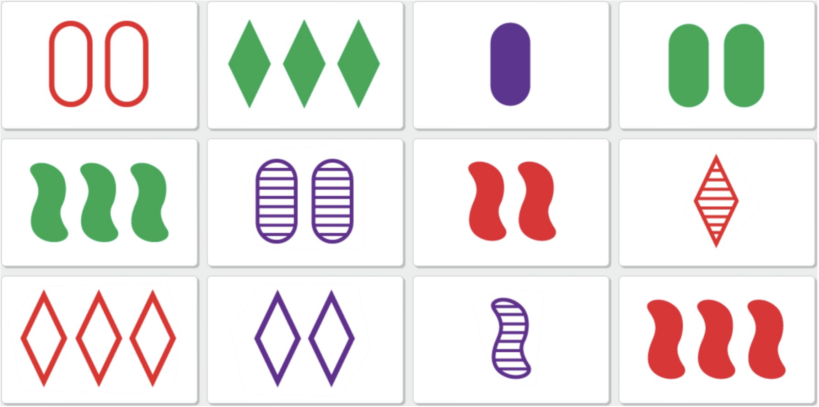

Applications to error correcting codes, cryptographic algorithms. . .

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Other discrete analogues: Hickman–Wright (2018), Dhar–Dvir (2021), Arsovski (2024), Dhar (2024), Salvatore (2022)

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- Mockenhaupt–Tao: $\mathbf{R}^*_\Sigma(2 \to 4)$ holds if $\Sigma \in \{\mathbb{P}^1, \mathbb{P}^2, \Gamma^2\}$

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Sharp restriction theory

Sharp parabolic restriction on $\mathbb{R}^d$

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Let $1 < p < 2 + \frac{2}{d}$ and $q = \frac{d+2}{d} p'$. Gaussians are critical points for $L^p \to L^q$ extension from the paraboloid if and only if $p = 2$.

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Carneiro–Oliveira–Sousa (2022): "Gaussians never extremize Strichartz inequalities for **hyperbolic paraboloids**" (Existence ✓)

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Let $d \geq 2$ and $1 < p < \frac{2d}{d-1}$ and $q = \frac{d+1}{d-1}p'$. Foschians are critical points for the $L^p \to L^q$ extension from the 1-cone if and only if $p = 2$. Existence of maximizers $\checkmark$.

What about sharp discrete restriction inequalities?

Sharp restriction theory on $\mathbb{F}^d$: new results I

Theorem 1 (González-Riquelme–OS, 2024)

It holds that $\mathbf{R}^*_{\mathbb{P}^2}(2 \to 4) = (1 + q^{-1} - q^{-2})^{\frac{1}{4}}$, *i.e.*

$$\|(f\sigma)^\vee\|^4_{L^4(\mathbb{F}_q^3, \mathrm{d}\boldsymbol{x})} \leq \left(1 + \frac{1}{q} - \frac{1}{q^2}\right) \|f\|^4_{L^2(\mathbb{P}^2, \mathrm{d}\sigma)} \tag{2}$$

is sharp, and equality holds if $f : \mathbb{P}^2 \to \mathbb{C}$ *is a constant function. Moreover, any maximizer of* (2) *has constant modulus.*

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Theorem 2 (González-Riquelme–OS, 2024)

Let $q = p^n$ *and* $w \in \mathbb{F}_q$ *be such that* $q \equiv 1 \mod 4$ *and* $w^2 = -1$. *Then* $f : \mathbb{P}^2 \to \mathbb{C}$ *is a maximizer of* (2) **if and only if** *there exist* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $a, b, c \in \mathbb{F}_q$, *such that*

$$f(\eta(1, w) + \zeta(1, -w)) = \lambda \exp \frac{2\pi i \mathrm{Tr}_n(a\eta + b\zeta + c\eta\zeta)}{p}$$

Theorem 3 (González-Riquelme–OS, 2024)

Let $p > 3$. *It holds that* $\mathbf{R}^*_{\mathbb{P}^1}(2 \to 6) = (1 + q^{-1} - q^{-2})^{\frac{1}{6}}$, *i.e.*

$$\|(f\sigma)^\vee\|^6_{L^6(\mathbb{F}_q^2, \mathrm{d}\boldsymbol{x})} \leq \left(1 + \frac{1}{q} - \frac{1}{q^2}\right) \|f\|^6_{L^2(\mathbb{P}^1, \mathrm{d}\sigma)} \tag{3}$$

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Sharp restriction theory on $\mathbb{F}^d$: new results II

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Theorem 4 (González-Riquelme–OS, 2024)

It holds that $\mathbf{R}^*_{\mathbb{H}^2}(2 \to 4) = (1 + q^{-1} - q^{-2})^{\frac{1}{4}}$, and $f : \mathbb{H}^2 \to \mathbb{C}$ is a maximizer **if and only if** *there exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $a, b, c \in \mathbb{F}_q$, such that*

$$f(\eta(1,1) + \zeta(1,-1)) = \lambda \exp \frac{2\pi i \mathrm{Tr}_n(a\eta + b\zeta + c\eta\zeta)}{p}$$

Sharp restriction theory on $\mathbb{F}^d$: new results III

Theorem 5 (González-Riquelme–OS, 2024)

Let $q \equiv -1 \mod 4$. *The extension inequality*

$$\|(f\nu)^\vee\|^4_{L^4(\mathbb{F}_q^4, \mathrm{d}\boldsymbol{x})} \leq \frac{q^4(q^5 - 2q^4 + 2q^3 - 3q + 3)}{(q-1)^3(q^2+1)^3} \|f\|^4_{L^2(\Gamma^3, \mathrm{d}\nu)} \quad (4)$$

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Theorem 6 (González-Riquelme–OS, 2024)

Constants are **not** *critical points for the $L^2(\Sigma, \mathrm{d}\nu) \to L^4(\mathbb{F}_p^4, \mathrm{d}\mathbf{x})$ extension inequality from $\Sigma \in \{\Gamma_0^3, \Upsilon_0^3\}$.*

Algebraic extension $\Leftrightarrow$ Counting problem

Algebraic extension ⇔ Counting problem

Proposition

The extension inequality

$$\|(f\sigma)^\vee\|_{L^{2k}(\mathbb{F}_q^d, \mathrm{d}\boldsymbol{x})} \leq \mathbf{R}^*_\Sigma(2 \to 2k)\|f\|_{L^2(\Sigma, \mathrm{d}\sigma)}$$

is equivalent to the combinatorial inequality

$$\sum_{\boldsymbol{\xi} \in \mathbb{F}_q^d} \left| \sum_{\substack{\boldsymbol{\xi}_1 + \ldots + \boldsymbol{\xi}_k = \boldsymbol{\xi} \\ \boldsymbol{\xi}_i \in \Sigma}} \prod_{i=1}^k f(\boldsymbol{\xi}_i) \right|^2 \leq \mathbf{C}^*_\Sigma(2 \to 2k) \left(\sum_{\boldsymbol{\xi} \in \Sigma} |f(\boldsymbol{\xi})|^2 \right)^k$$

in the sense that they have the same set of maximizers, and the corresponding best constants are related via

$$\mathbf{C}^*_\Sigma(2 \to 2k) = q^{-d}|\Sigma|^k \mathbf{R}^*_\Sigma(2 \to 2k)^{2k}$$

Elementary number theory over finite fields

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$$S(a) := \sum_{x \in \mathbb{F}_p} e(ax^2) = \left(\frac{a}{p}\right) S(1), \text{ where}$$

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- **Quadratic reciprocity.** Given arbitrary odd positive coprime integers p and r,

$$\left(\frac{p}{r}\right)\left(\frac{r}{p}\right) = (-1)^{\frac{(p-1)(r-1)}{4}}$$

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  Consider the quadratic field extension $\mathbb{F}_q(\alpha)/\mathbb{F}_q$, where $\alpha \in \mathbb{F}_q^{\text{alg}}$ satisfies $\alpha^2 = c$:

$$\begin{aligned}|\mathcal{Q}(c,r)| &= \left|\{(x,y) \in \mathbb{F}_q^2 : \ (x+\alpha y)(x-\alpha y) = r\}\right| \\ &= |\{(x,y) \in \mathbb{F}_q^2 : \ (x+\alpha y)^{q+1} = r\}| \\ &= |\{a \in \mathbb{F}_q(\alpha) : \ a^{q+1} = r\}| = q+1\end{aligned}$$

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Orthogonality in the last step. Alt. five conic sizes (also $q = p^n$).

Proposition (Paraboloids)

Let $d, k \geq 2$ and $p > k$ be an odd prime. Let $\sigma = \sigma_{\mathbb{P}^d}$ denote the normalized surface measure on the paraboloid $\mathbb{P}^d \subset \mathbb{F}_p^{d+1}$. Then

$$\sigma^{*k}(\boldsymbol{\xi}, \tau) = 1 + \varepsilon_p^{d(k+1)} p^{\frac{d(1-k)}{2}} \varphi(\boldsymbol{\xi}, \tau), \quad (\boldsymbol{\xi}, \tau) \in \mathbb{F}_p^{d+1}$$

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$$\varphi(\boldsymbol{\xi}, \tau) = \begin{cases} p\mathbf{1}_{\{\tau = \boldsymbol{\xi}^2/k\}} - 1 & 2 | d \\ (-1)^{\frac{(p-1)(k+1)}{4}} \left(\frac{p}{k}\right) \left(p\mathbf{1}_{\{\tau = \boldsymbol{\xi}^2/k\}} - 1\right) & 2 \nmid d, k \\ \varepsilon_p \sqrt{p} (-1)^{\frac{(p-1)(\ell+1)}{4} + \frac{p^2-1}{8}\nu_2(k)} \left(\frac{p}{\ell}\right) \left(\frac{\boldsymbol{\xi}^2/k - \tau}{p}\right) & 2 \nmid d, 2 | k \end{cases}$$

Convolution measures on $\mathbb{F}^d$

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Proof: Fourier inversion, orthogonality, Gauss sums, quadratic reciprocity

Muito obrigado