Feynman graph integrals from topological-holomorphic field theories and their applications

Minghao Wang

Boston University

This is a joint work with Brian Williams.
For purely holomorphic case, see my previous work: arxiv 2401.08113

1. Introduction

- Motivated by Chern-Simons theory, Kontsevich, Axelrod and Singer considered Feynman graph integrals from topological field theories, and proved the finiteness of these integrals.

- Later on, People proved a brunch of mathematical results using these integrals. These results include formality of little disk operads, construction of universal finite type knot invariants, quantization of Poisson manifolds, etc.

2. Topological case.

Spacetime: $M = \mathbb{R}^{d'}$

Propagator:

$$\tilde{P}(x-y) = \frac{\Gamma\left(\frac{d'}{2}\right)}{\pi^{\frac{d'}{2}}} \cdot \frac{1}{|x-y|^{d'}} \cdot \left(\sum_{i=1}^{d'} (-1)^{i-1}(x_i - y_i)\left(\prod_{j\neq i} d(x_j - y_j)\right)\right)$$

The propagator is obtained by solving the following equation:

$$d\tilde{P}(x-y) = \delta(x-y)d^{d'}(x-y)$$

↓ de Rham differential

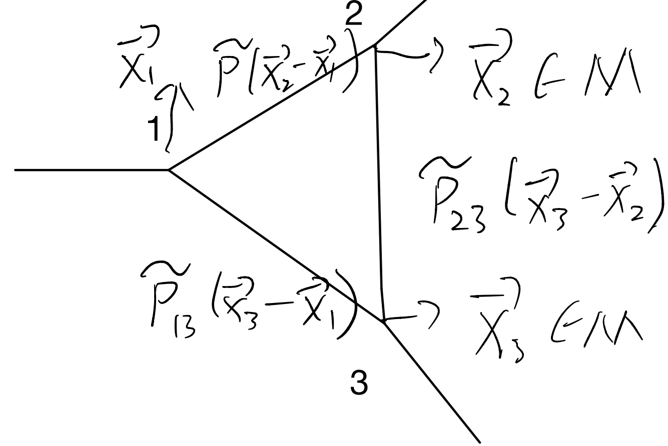
$\mathrm{conf}_n(M) = \{(x_1, x_2, \cdots, x_n) \in M^n \mid x_i \neq x_j,$

for $i \neq j\}$

$\tilde{p} \in C^\infty(\mathrm{conf}_2(M))$

δ is poincare dual of $\Delta \in (M)^2$

Feynman graph integrals: $\Gamma =$



For a differential form with compact support: $\Phi \in \Omega_c^*((\mathbb{R}^{d'})^3)$

$$W(\Gamma, \Phi) = \int_{Conf_3(\mathbb{R}^{d'})} \tilde{P}_{12}\tilde{P}_{13}\tilde{P}_{23}\Phi$$

More generally, we consider:

1. A directed graph: Γ

2. A Differential form with compact support: $\Phi \in \Omega_c^*((\mathbb{R}^{d'})^{|\Gamma_0|})$

$$W(\Gamma, \Phi) = \int_{Conf_{|\Gamma_0|}(\mathbb{R}^{d'})} (\prod_e \text{[illegible]} \tilde{P}_e) \wedge \Phi$$

Theorem “finiteness” (Kontsevich, Axelrod and Singer)

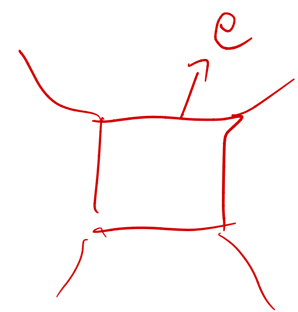
$$W(\Gamma, \Phi) < +\infty.$$

$W(\Gamma, -) \in D^*(M^{|\Gamma_0|})$

$\Phi \in \Omega^*_c \longrightarrow W(\Gamma, \Phi)$

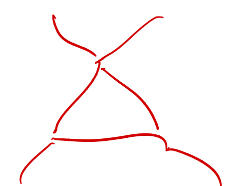
Theorem “topological anomalies vanishes” (Kontsevich) When d’>1, we have:

$$dW(\Gamma, -) = \sum_{edge\ e} \pm W(\Gamma/e, -)$$

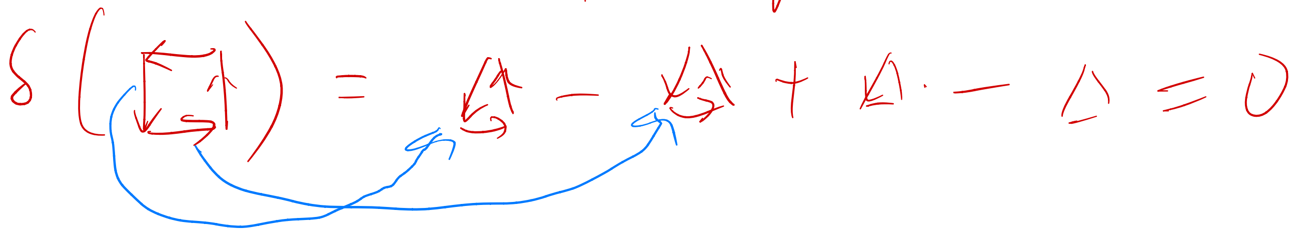


$= W(\delta\Gamma, -)$

de Rham diff e.

$\Gamma/e =$ 

$$\text{Graph complex} = \Big(\bigoplus_{\Gamma \in \text{directed graph}} \mathbb{R}\Gamma \, , \; \delta \Big)$$



$$\delta^2 = 0$$

$$\text{Graph complex} \longrightarrow \Omega^*(\mathrm{Conf}_n(\mathbb{R}^d))$$

$$\Gamma \longrightarrow W(\Gamma, -)$$

$$\mathrm{Conf}_2(\mathbb{R}^{d'}) \subseteq M^2$$

$$\Delta \subseteq M^2$$

$\widetilde{\mathrm{Conf}_2}(\mathbb{R}^{d'})$ = real blow up of M^2 along Δ

$$\simeq \mathrm{Conf}_2(\mathbb{R}^{d'}) \cup N(\Delta)/\mathbb{R}^+$$

Claim: $\bar{p}$ can be extended to

$\widetilde{\mathrm{Conf}_2}(\mathbb{R}^{d'})$ as smooth differential form!

3. Topological-holomorphic case

Spacetime: $M = \mathbb{C}^d \times \mathbb{R}^{d'}$ $\left(\Omega^*(M),\ \bar\partial + d_{\text{de Rham}}\right)$

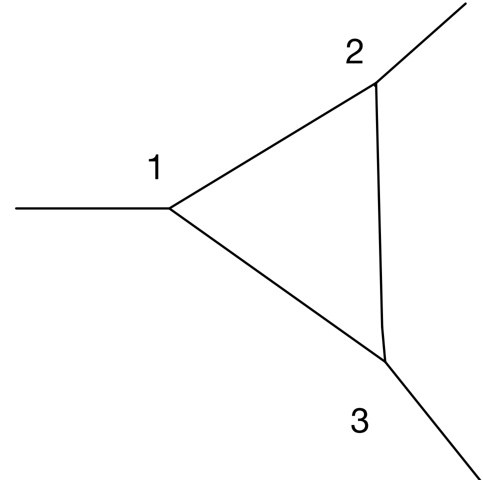
Propagator: $\tilde{P}(z-w, \overline{z-w}, x-y) =$

$$\frac{2^d\Gamma\left(d+\frac{d'}{2}\right)}{\pi^{d+\frac{d'}{2}}} \cdot \frac{1}{(2|z-w|^2+|x-y|^2)^{d+\frac{d'}{2}}} \cdot \left(\sum_{i=1}^{d}(-1)^{i-1}(\overline{z_i-w_i})\left(\prod_{j\neq i} d(\overline{z_j-w_j})\right)d^{d'}(x-y) + \sum_{i=1}^{d'}(-1)^{d+i-1}(x_i-y_i)d^d(\overline{z-w})\left(\prod_{j\neq i} d(x_j-y_j)\right)\right)$$

We have:

$$(\bar\partial + d)\tilde{P}(z-w, \overline{z-w}, x-y) = \delta(z-w, \overline{z-w}, x-y)d^d(\overline{z-w})d^{d'}(x-y)$$

Feynman graph integrals: $\Gamma =$



$s \in H^{1,0}(\Sigma)$

A

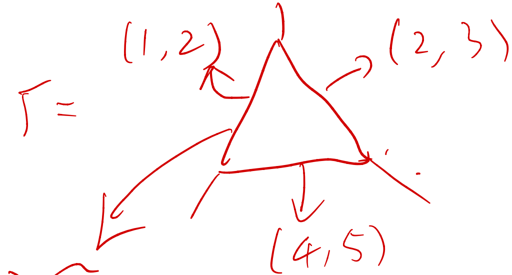
$\Sigma \xrightarrow{\varphi}$

$$W(\Gamma, \Phi) = \int_{Conf_3(\mathbb{C}^d \times \mathbb{R}^{d'})} \tilde{P}_{12}\tilde{P}_{13}\tilde{P}_{23}\Phi$$

$$\left\langle \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Sigma \times \cdots \times \Sigma} (\varphi^{*}A)^{n} \wedge s \wedge s \cdots \right\rangle$$

$M = \mathbb{C}^2$

More generally, we consider:



1. A decorated graph: $\partial_{z_1}\partial^2_{z_2}\tilde{P}$ $(\Gamma, \vec{n})$

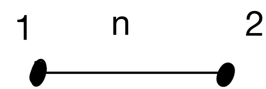
2. A Differential form with compact support: Φ

$$W((\Gamma, \vec{n}), \Phi) = \int_{Conf_{|\Gamma_0|}(\mathbb{C}^d \times \mathbb{R}^{d'})} (\prod_e \partial_{\vec{n}_e} \tilde{P}_e) \wedge \Phi$$

Theorem “finiteness” (M. Wang and B. Williams) $W((\Gamma,\vec{n}),\Phi) < +\infty.$

Difficulty:

$e.g.\ M = \mathbb{C}$ and $\Gamma =$



$$\tilde{P}(\vec{z}-\vec{w}, \overrightarrow{\bar{z}} - \overrightarrow{\bar{w}}) = \frac{1}{\pi}\cdot\frac{1}{z-w}$$

$$\partial_z^n \tilde{P} = c\,\frac{1}{(z-w)^{n+1}}$$

$$W((\Gamma,\vec{n}),\Phi) = constant \times \int_{\mathbb{C}^2} \frac{1}{(z-w)^{n+1}} \wedge \Phi$$

It is not absolutely convergent!

Theorem “anomalies vanishes” (M. Wang and B. Williams) If d’>1, the following result holds:

$$(\bar{\partial}+d)W((\Gamma,\vec{n}),-)=\sum_{edge\ e,\ \vec{n}'}C_{(e,\vec{n}')}W((\Gamma/e,\vec{n}'),D_{(e,\vec{n}')}-)$$

$$\Delta_{BV}\, e^{\frac{I}{\hbar}} = 0$$

where coefficients $C_{(e,\vec{n}')}$ are 0, -1 or 1.

"graph complex" $\longrightarrow (\Omega^*(\mathrm{Conf}_n(M)),\ \bar{\partial}+d)$

Theorem "holomorphic anomalies" (M. Wang) When d'=0, we have

$$\bar{\partial} W((\Gamma, \vec{n}), -) = \sum_{Laman\ subgraph\ \Gamma'} C_{(\Gamma', \vec{n}')} W((\Gamma \backslash \Gamma', \vec{n}'), D_{(\Gamma', \vec{n}')} -)$$

where coefficients $C_{(\Gamma', \vec{n}')}$ are periods.

$$\widetilde{W}(\Gamma, \Phi) = \int_{[0,+\infty)^{|\Gamma|}} \widetilde{\widetilde{W}}(\Gamma, \Phi)$$

$$Conf_n(M) \subseteq M^n \times [0,+\infty)^{|\Gamma|}$$

$$\downarrow$$

$$\int_{M^n} \widetilde{W}(\Gamma, \Phi)$$

4. Schwinger spaces.

For simplicity, let d'=0, so $M = \mathbb{C}^d$

Schwinger parameters:

Proposition: $$\tilde{P} = \int_0^{+\infty} dt \bar{\partial}^* H(t)$$

H(t) is the heat kernel. t is called Schwinger parameter.

Definition (propagator in Schwinger space):

$$P_t = -dt\bar{\partial}^* H(t) + H(t)$$

Lemma $\tilde{P} = -\int_0^{+\infty} P_t.$

Lemma 2 $(d_t + \bar{\partial}) P_t(\vec{z} - \vec{w}, \overrightarrow{\bar{z}} - \overrightarrow{\bar{w}}) = 0.$

Feynman graph integral:

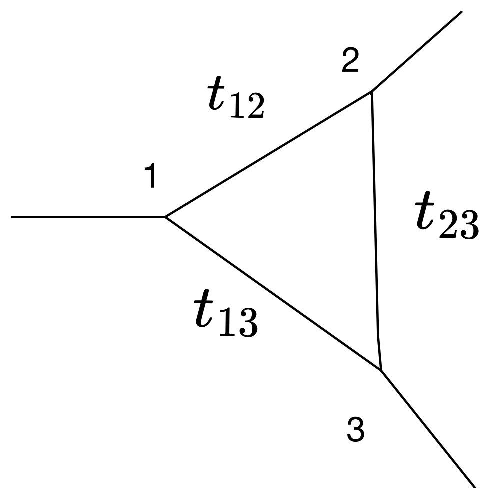
$$W((\Gamma, \vec{n}), \Phi) = \pm \int_{[0,\infty)^{|\Gamma_1|}} \int_{(\mathbb{C}^d)^{|\Gamma_0|}} (\prod_e \partial_{\vec{n}_e} P_{t_e}) \wedge \Phi.$$

Definition (integrand in Schwinger space):

$$\tilde{W}((\Gamma, \vec{n}), \Phi) = \int_{(\mathbb{C}^d)^{|\Gamma_0|}} \Big(\prod_e \partial_{\vec{n}_e} P_{t_e}\Big) \wedge \Phi.$$

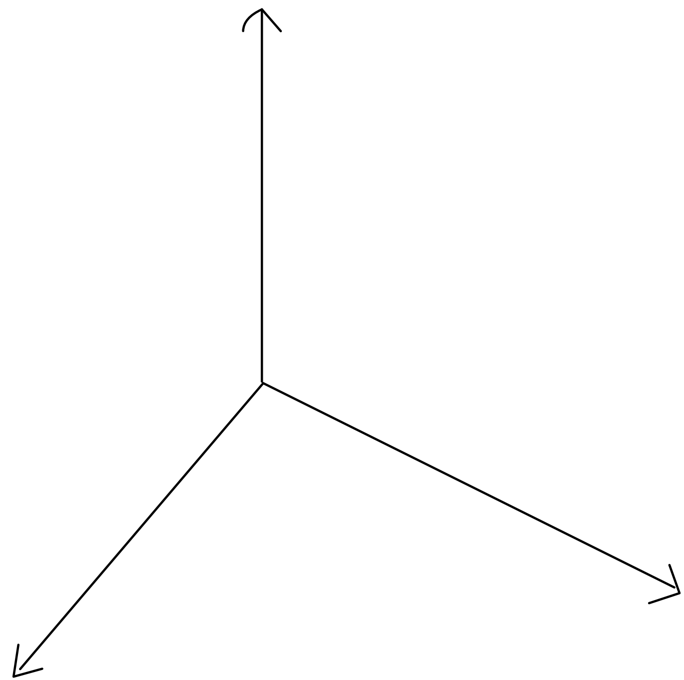
It is a differential form on Schwinger Space: $[0, \infty)^{|\Gamma_1|}$

$\tilde{W}((\Gamma, \vec{n}), \Phi)$ is singular at corners of Schwinger space!

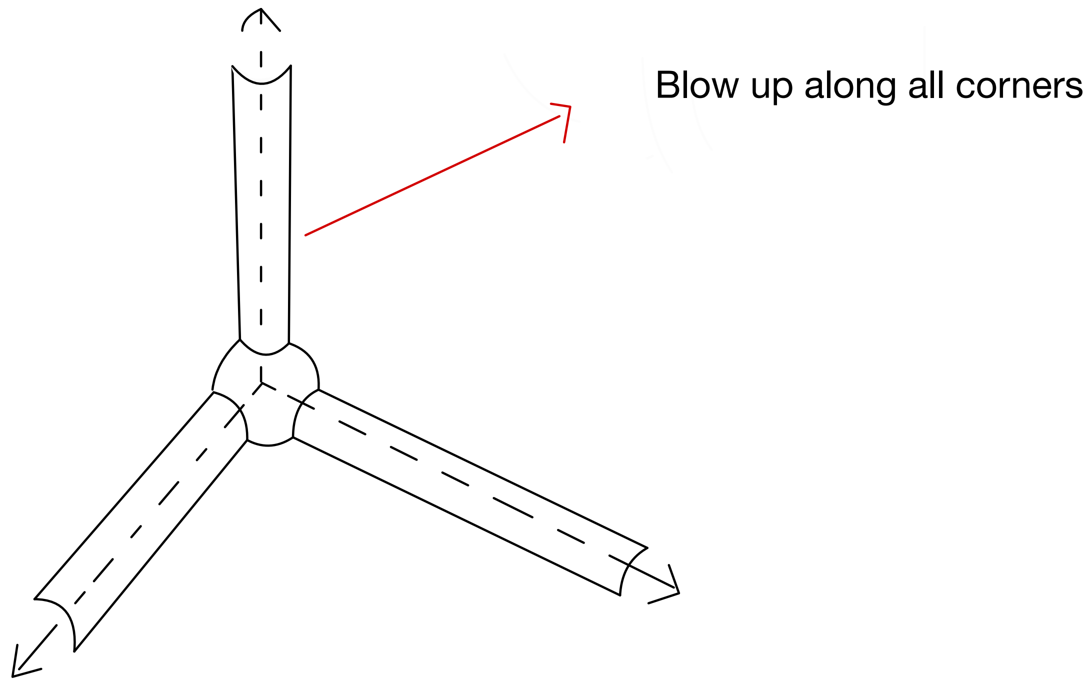


$t_{12} \to 0$: The collapse involving vertex 1 and vertex 2.

Naive Schwinger space: $[0, \infty)^3$



Compactified Schwinger Space:



Definition(Compactified Schwinger Space):

$\widetilde{[0,\infty)}^{|\Gamma_1|}$: Blow up of naive Schwinger space along all corners.

Main Theorem: $\tilde{W}((\Gamma,\vec{n}),\Phi)$ Can be extended to a smooth differential form on compactified Schwinger space.

Singularities disappeared!

5. Applications

- Factorization algebras on $M = \mathbb{C}^d \times \mathbb{R}^{d'}$

- Higher Chiral algebras(higher vertex algebras).

- BCOV theory and complex geometry.

potential applications:

- Twistor correspondence.
- BCOV theory and complex geometry.
- Holomorphic operad theory.
- Multi-theta functions.

6. Proof of main theorem.

Step 1: Write Feynman graph integrals as integrals of expressions in terms of graphic Green’s functions.

Step 2: Prove graphic Green’s functions can be extended to compactified Schwinger spaces.

Thank you!