Introduction to enumerative geometry via derived geometry

Marco Robalo - Jussieu

Plan

1. Derived geometry
2. Donaldson-Thomas invariants
3. Results

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2. Donaldson-Thomas invariants
3. Results

What is derived geometry?

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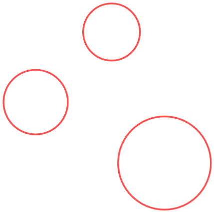
The Toën-Apollonius Example

Algebraic Geometry

The Apollonius Problem: How many circles simultaneously tangent to three fixed circles?



Algebraic Geometry

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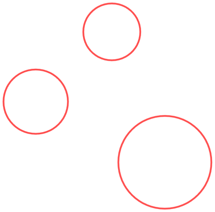
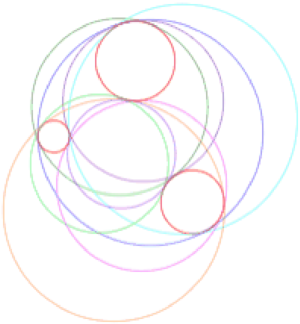


Algebraic Geometry



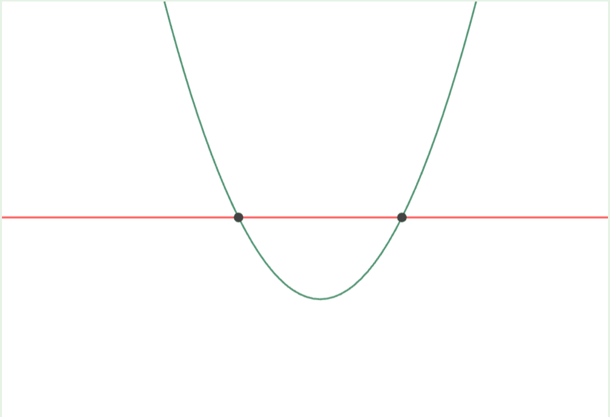
What happens when the radius go to zero?

Algebraic Geometry

Example

Algebraic Geometry

Example



Algebraic Geometry

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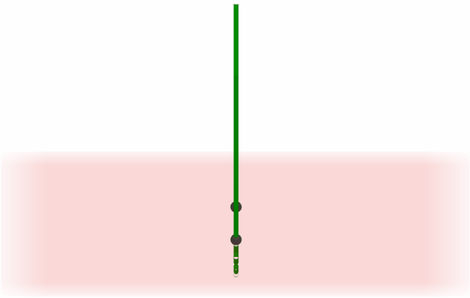


Always 2 (complex points) except when they collide!

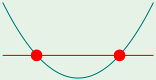
What happens at the collision?

Example

Use algebra to track the collision.

Before collision:

$$(Y = X^2) \cap (Y = 1)$$



$$\mathbb{C}[X,Y]/(Y-X^2) \otimes_{\mathbb{C}[X,Y]} \mathbb{C}[X,Y]/(Y-1) \simeq \mathbb{C}[X]/(X^2-1) = \mathbb{C}[X]/(X-1) \times \mathbb{C}[X]/(X+1) = \underbrace{\mathbb{C} \times \mathbb{C}}_{2\ \textit{rings}}$$

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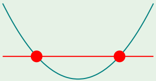

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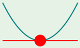
At collision:

$$(Y = X^2) \cap (Y = 0)$$



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Conclusion: The second became an algebraic infinitesimal

Algebraic Geometry

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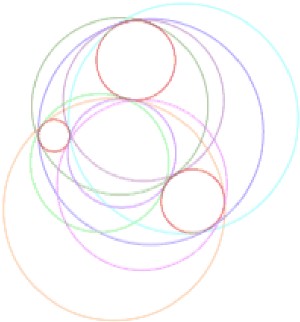
Apollonius: Fix three circles C_1, C_2, C_3. Understand the intersection

$$Z_{C_1} \cap Z_{C_2} \cap Z_{C_3} \qquad \text{inside } \mathbb{P}^3$$

Algebraic Geometry

7 became algebra:



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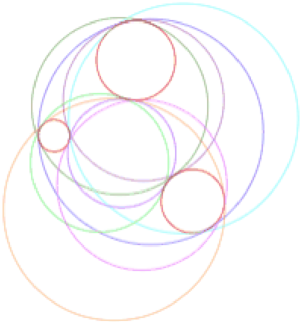

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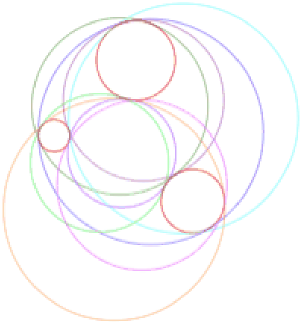

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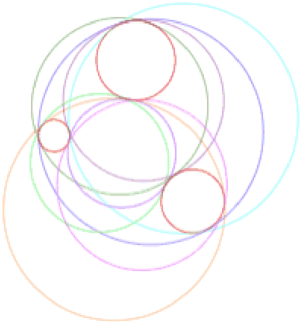

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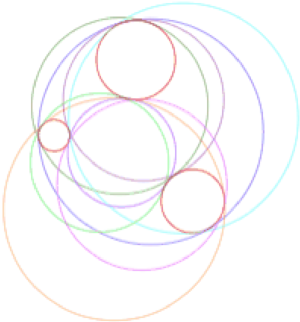
$$Z_{C_1} \cap^{Sch} Z_{C_2} \cap^{Sch} Z_{C_3} = \mathrm{Spec}(\mathbb{C}[\epsilon_x, \epsilon_y, \epsilon_z])$$

Algebraic Geometry

The **Toën-Appolonius** case: what if we collapse all to one point?



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How did 8 became infinitely many?

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How did 8 became infinitely many?

Answer

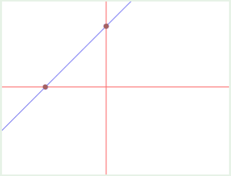
Redundancies!

Interlude I: Serre's formula (1957)

Example

Intersect the axis in 4-dimensions $R = \mathbb{C}[x, y, z, w]$, with the diagonal

$$\text{Axis} := \begin{cases} xz = 0 \\ xw = 0 \\ yz = 0 \\ yw = 0 \end{cases} \qquad \text{Diag} := \begin{cases} x - z = 0 \\ y - w = 0 \end{cases}$$



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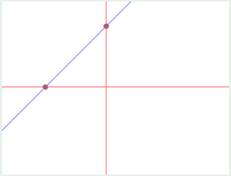

$$R/(xz, xw, yz, yw) \otimes_R R/(x - z, y - w) \simeq \mathbb{C}[x, y]/(x^2, xy, y^2)$$

$$= \underbrace{\mathbb{C} \oplus \mathbb{C}.\epsilon_x \oplus \mathbb{C}.\epsilon_y}_{3 \neq 2} \quad \text{too many!}$$

Example

Problem:

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Correct counting: $\underbrace{\mathbb{C} \oplus \mathbb{C}.\epsilon_x \oplus \mathbb{C}.\epsilon_y - \mathbb{C}.[f]}_{3-1=2}$

Example

$$\underbrace{\color{red}{R/I_{Axis}} \otimes_R \color{blue}{R/I_{Diag}}}_{\text{Solves the system}=\mathbb{C}+\mathbb{C}.\epsilon_x+\ \mathbb{C}.\epsilon_y}$$

0

Example

1

$$\underbrace{\frac{I_{Diag} \cap I_{Axis}}{I_{Diag}.I_{Axis}}}_{\text{Counts redundancies}} \simeq \mathbb{C}.[f]$$

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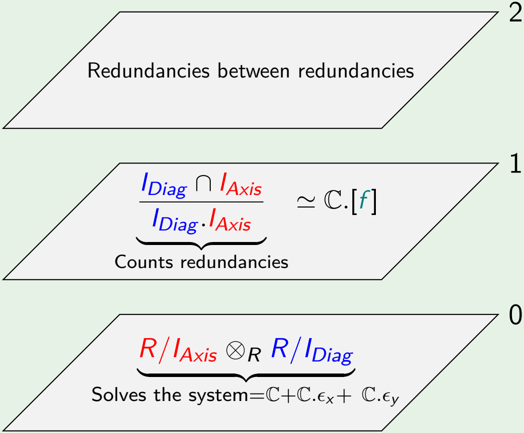
2
Redundancies between redundancies
1
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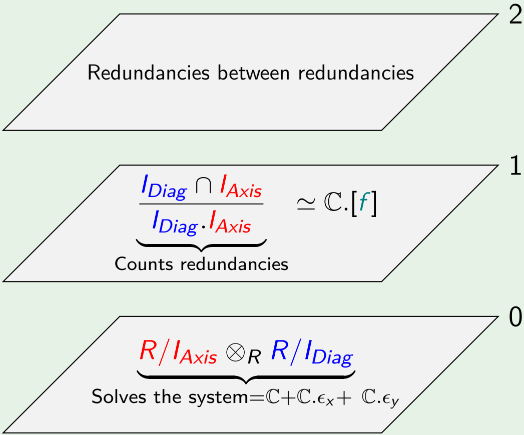

Chain complex

derived intersection

$$R/I_{Diag} \otimes_R^{\mathbb{L}} R/I_{Axis}$$

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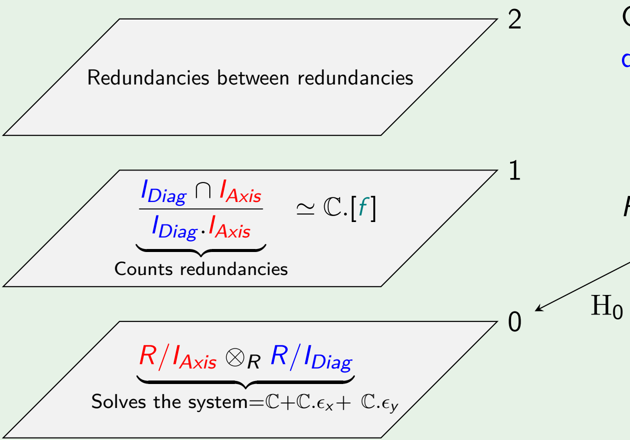

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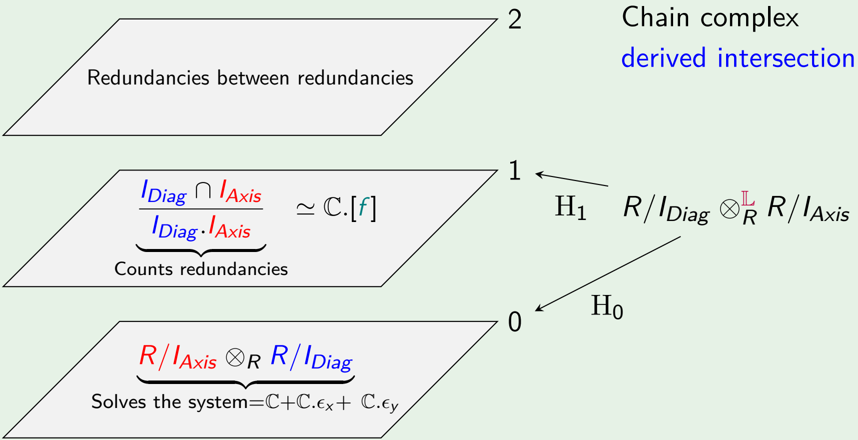
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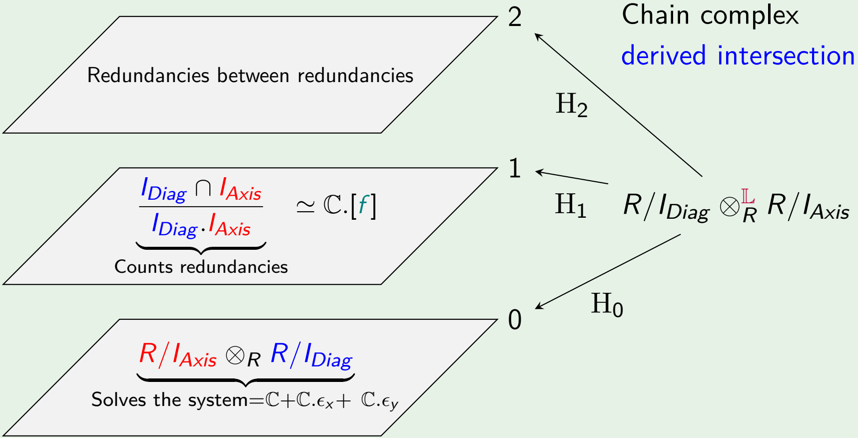

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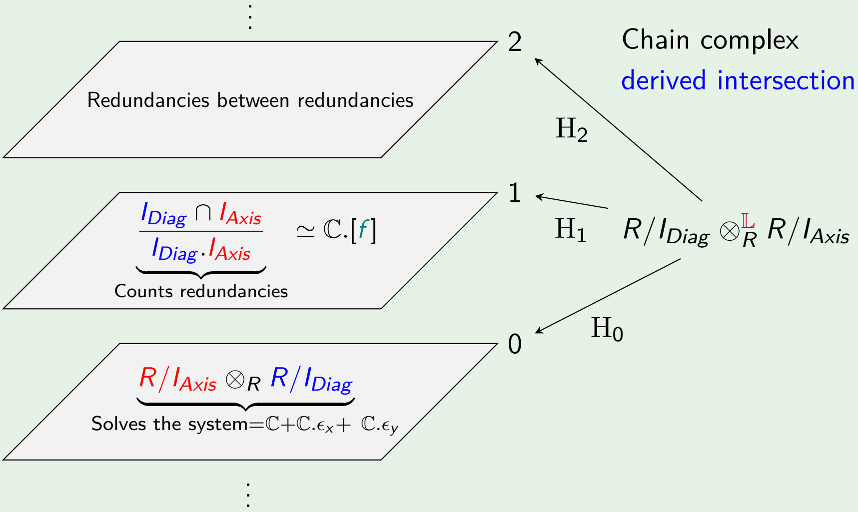

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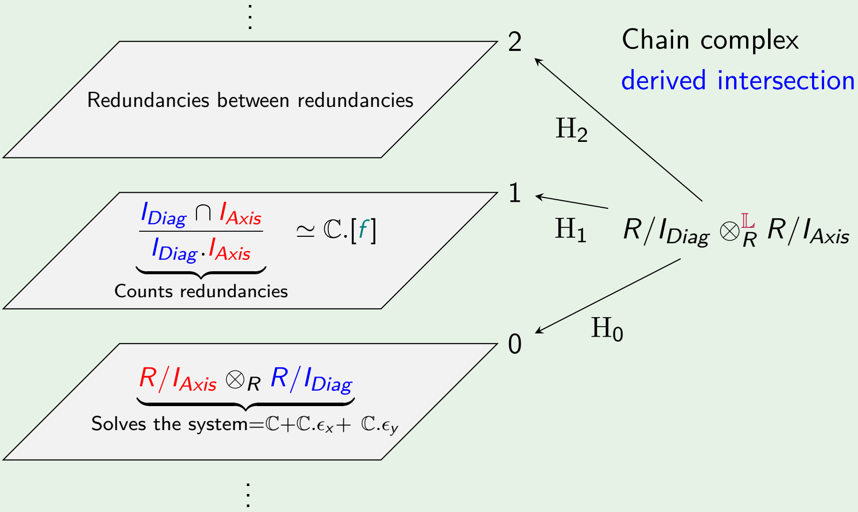

Serre's corrected excess = + dim. of even floors - dim. of odd floors

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End of Interlude I

Interlude II: Derived Geometry (2000)

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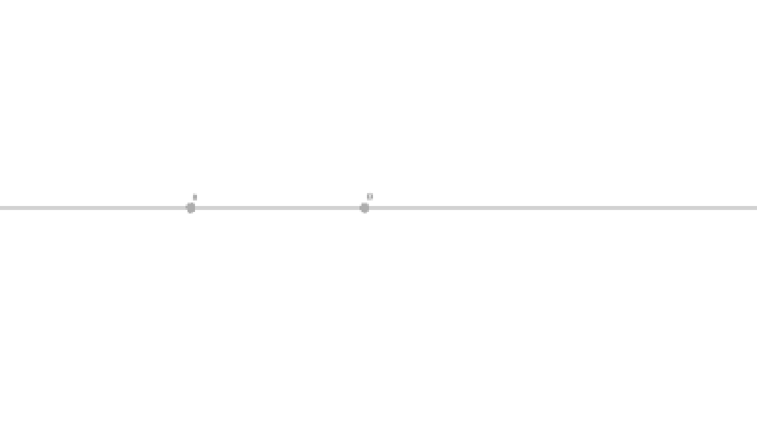
Intersections in **dSch** automatically account for Serre's formula.

Example

Intersection of two points $x = a$ and $x = 0$ in a line.



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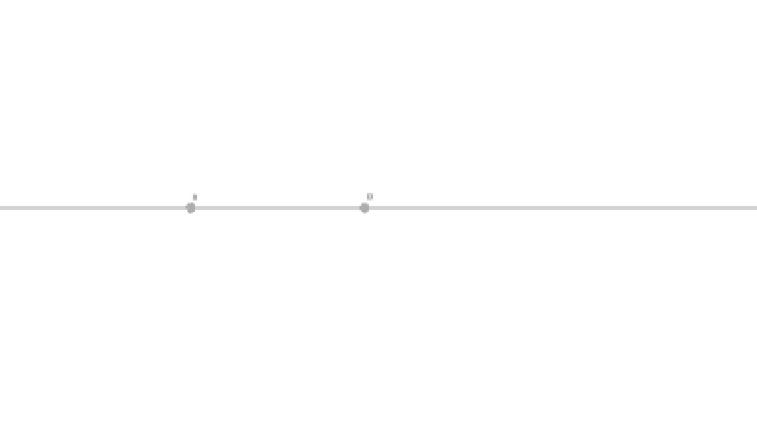

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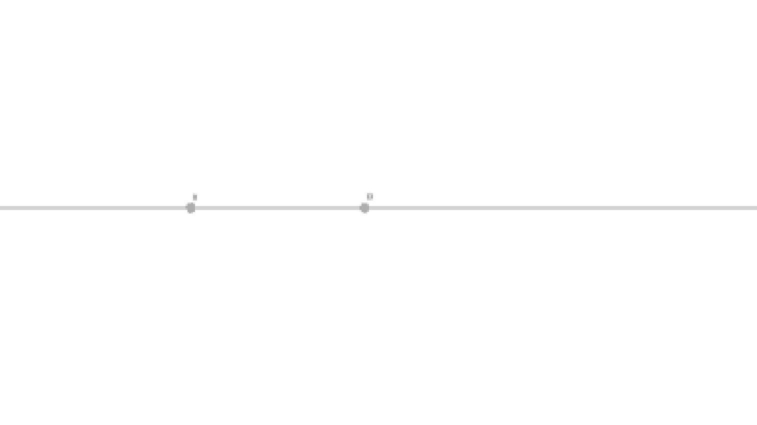

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- If $a = 0$, $\{x = 0\} \cap \{x = a\} = \{x = 0\}$ (multiplicity 1)

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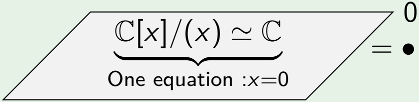

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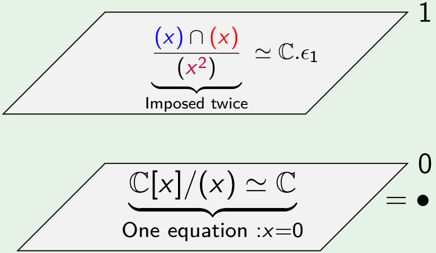

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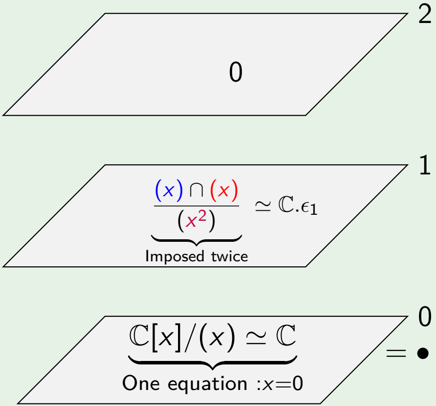

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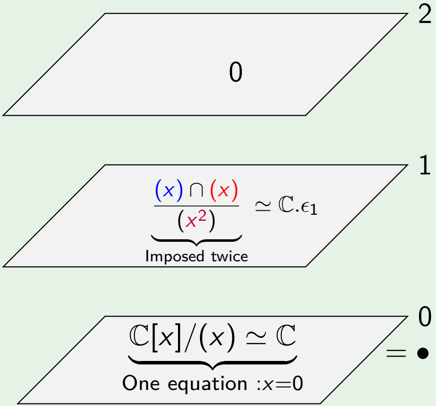

$$\mathbb{C} \otimes^{\mathbb{L}}_{\mathbb{C}[x]} \mathbb{C} \simeq \mathbb{C}[\epsilon_1], \quad |\epsilon_1| = 0, \ \ \epsilon_1^2 = 0$$

$$1 - 1 = 0$$

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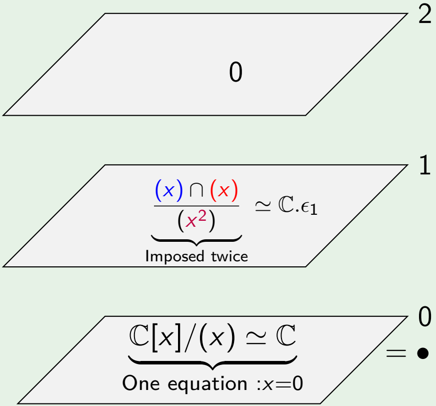

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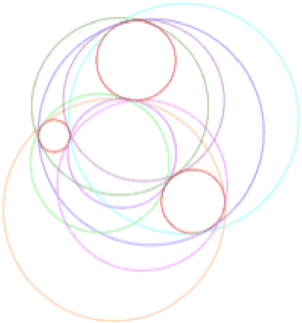
This is the basic block of derived geometry - a point with a redundancy.

End of the Interludes.

Back to the circles.

Algebraic Geometry

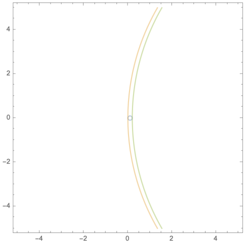


What is derived geometry?

Collapsed all circles C_1, C_2 and C_3 to a point $\rightsquigarrow$ jumped from 8 tangent circles to the whole plane of possibilities.

What is derived geometry?

In fact: two infinite planes of possibilities!



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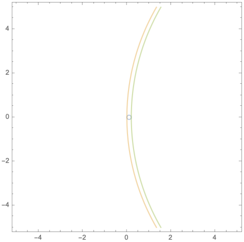

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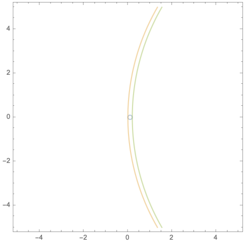


How to retrace 8?

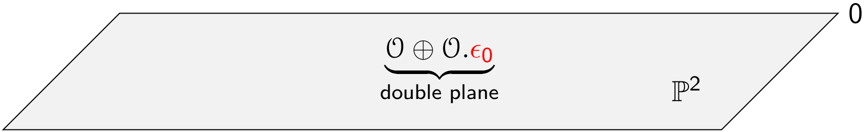
Answer

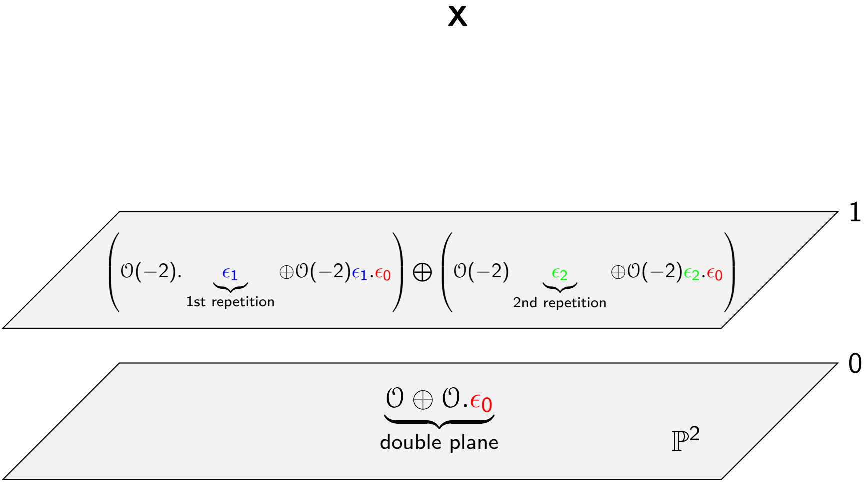
The ***Toën-Appolonius derived intersection***

$$\mathbf{X} = Z_C \cap^{\mathbf{dSch}} Z_C \cap^{\mathbf{dSch}} Z_C$$

is a derived projective plane. The derived structure subtracts the double infinity and retraces the 8 circles algebraically.

X




X
1
$\left(\mathcal{O}(-2).\underbrace{\epsilon_1}_{\text{1st repetition}} \oplus \mathcal{O}(-2)\epsilon_1.\epsilon_0\right) \oplus \left(\mathcal{O}(-2)\underbrace{\epsilon_2}_{\text{2nd repetition}} \oplus \mathcal{O}(-2)\epsilon_2.\epsilon_0\right)$
0
$\underbrace{\mathcal{O} \oplus \mathcal{O}.\epsilon_0}_{\text{double plane}}$
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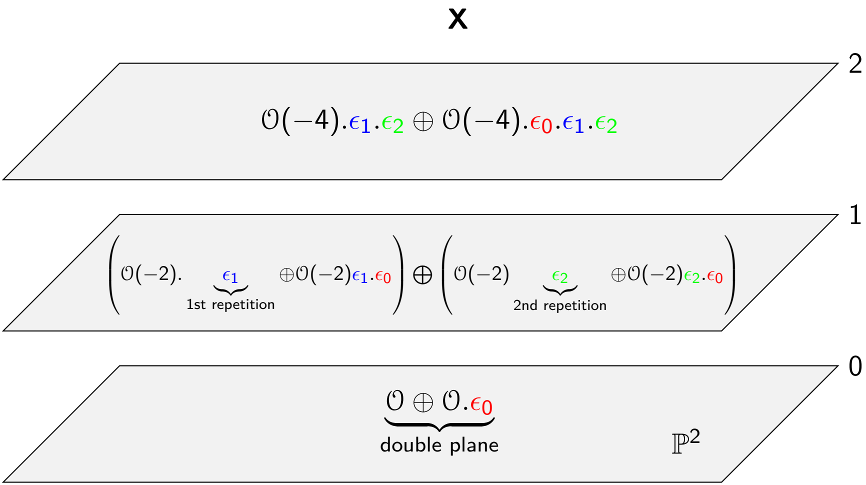
X
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$\mathcal{O}(-4).\epsilon_1.\epsilon_2 \oplus \mathcal{O}(-4).\epsilon_0.\epsilon_1.\epsilon_2$
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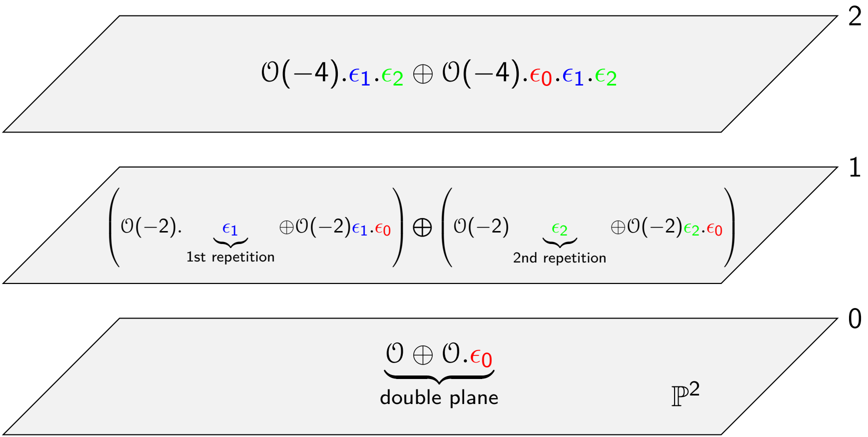

(Kontsevich 95, Fontanine-Kapranov 2009, Khan 2019)

$$\underbrace{[\mathbf{X}]}_{\textbf{Derived Fundamental Class}} := \mathrm{Ch}[2\mathcal{O} - 4\mathcal{O}(-2) + 2\mathcal{O}(-4)] \cap [\mathbb{P}^2] = 8.[pt] \in H_0(\mathbb{P}^2)$$

What is derived geometry?

The Toën-Appolonius Example

$\mathbf{X} = Z_C \cap^{\mathbf{dSch}} Z_C \cap^{\mathbf{dSch}} Z_C$ is a derived projective plane:

- of virtual dimension zero, which means it behaves like a point
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Upshot: The derived structure corrects the counting.

Plan

From Physics to Enumerative Geometry

Motivation: String theory unifies general relativity and quantum mechanics at a cost:

$$\text{Spacetime} = \mathbb{R}^{3+1} \times \underbrace{Y}_{\textbf{Extra dimensions}}$$

(Candelas-Horowitz-Strominger-Witten 85):

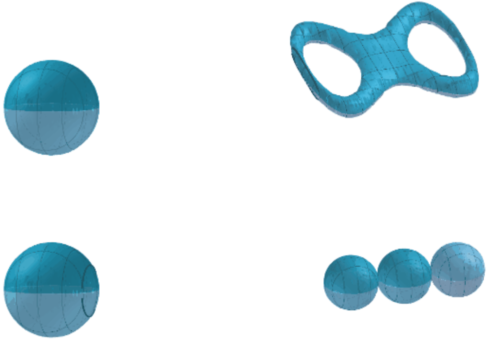
- Y is an algebraic variety of complex dimension 3;
- Calabi-Yau, ie, $\omega_Y \simeq \mathcal{O}_Y$;
- Example: the Fermat quintic $Y = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}^4_{\mathbb{C}}$

Principle

Physics in 4-dim = geometry of Y.

From Physics to Enumerative Geometry

Motivation: Paths of strings in Y define complex algebraic curves:



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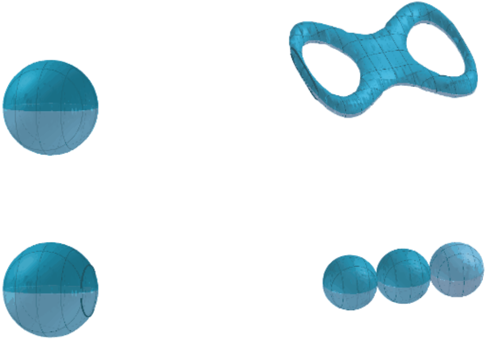

Path integrals for strings = Count algebraic curves in Y.

From Physics to Enumerative Geometry

Counting curves **in Y**:

From Physics to Enumerative Geometry

Counting curves **in Y**:three strategies

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- Donaldson-Thomas approach (2000):
  Replace curves by their functions (ideal sheaves I_C)

  $\mathcal{M}Coh^{\tau}(Y)$ = moduli space of coherent sheaves with stability τ

From Physics to Enumerative Geometry

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$$N_{ⓓ} = \sum_{\substack{d_A + d_B = ⓓ \\ d_A \geq 1;\, d_B \geq 1}} N_{d_A}.N_{d_B}.d_A^2.d_B \left(d_B \binom{3ⓓ - 4}{3d_A - 2} - d_A \binom{3ⓓ - 4}{3d_A - 1} \right)$$

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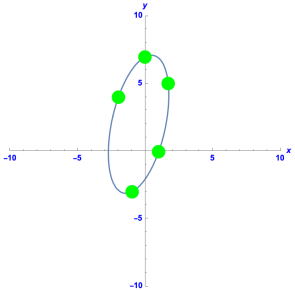

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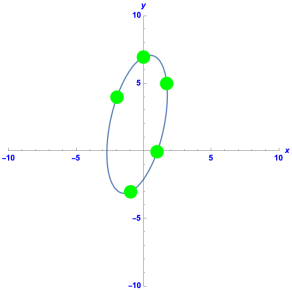

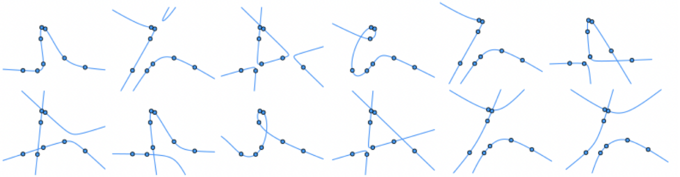

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Warning: Shifted forms cannot exist when the moduli is smooth!

Donaldson-Thomas and symplectic geometry

Example (Lagrangian intersections)

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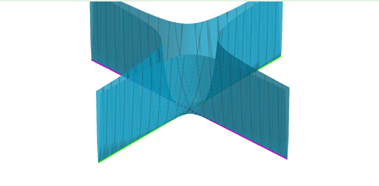

$-1 \qquad 0 \qquad 1$

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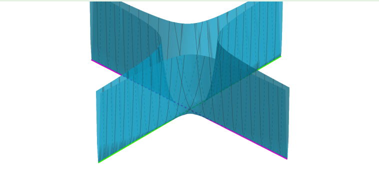

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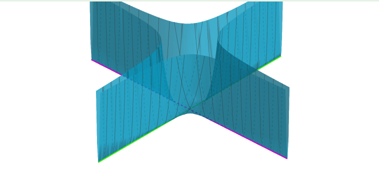

-1	0	1		
	$\mathbb{T}_U$	$\xrightarrow{\text{Hess(f)}} \mathbb{T}^*_U$	$=$	$\mathbb{T}$
$\mathbb{T}_U \xrightarrow{\text{Hess(f)}^\vee}$	$\mathbb{T}^*_U$		$=$	$\mathbb{T}^*$

Symmetry of the Hessian $\Rightarrow \mathbb{T} \simeq \mathbb{T}^*[-1]$ (-1)-shifted symplectic form

Donaldson-Thomas and symplectic geometry

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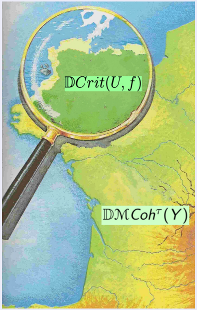

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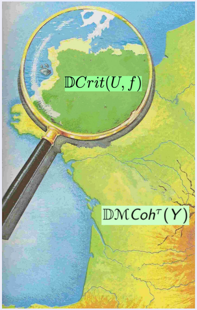

Local geometry of $\mathbb{DM}Coh^{\tau}(Y)$ = local geometry of the singularities of f.

Donaldson-Thomas and invariants of singularities

Capture the singularities of f:

Donaldson-Thomas and invariants of singularities

Capture the singularities of f:

- Milnor number, when f has isolated singularities

$$\mu(f) = \dim_{\mathbb{C}} \underbrace{\mathcal{O}_U/(\partial_1 f, \cdots \partial_n f)}_{\mathcal{O}_{Crit(f)}} \in \mathbb{Z}$$

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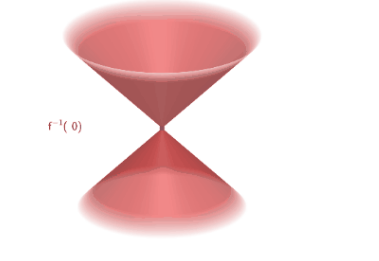
Example ($U = \mathbb{A}^2$, $f = x^2 + y^2$)



$$\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[x, y]/(2x, 2y) = 1$$

Donaldson-Thomas and invariants of singularities

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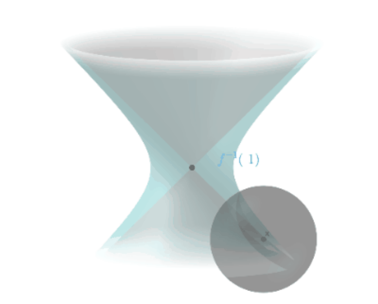

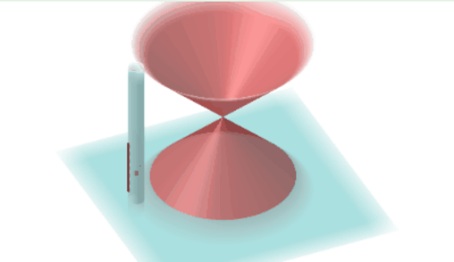

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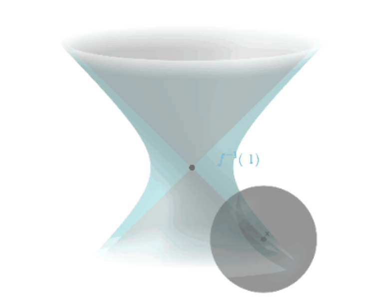

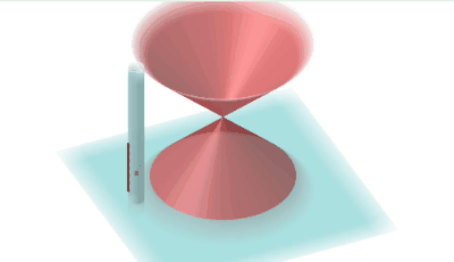

$$x \in f^{-1}(0) \mapsto H^{2-1}(\underbrace{B(x,\epsilon) \cap f^{-1}(\epsilon)}_{\text{Milnor Fiber at } x}) \simeq \begin{cases} \mathbb{Z}^{\mu(f)} & x = 0 \\ 0 & x \neq 0 \end{cases}$$

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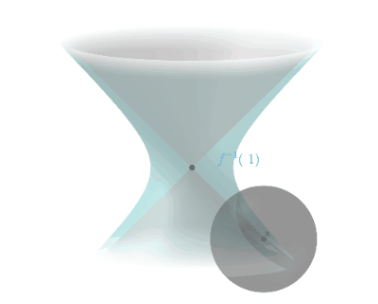

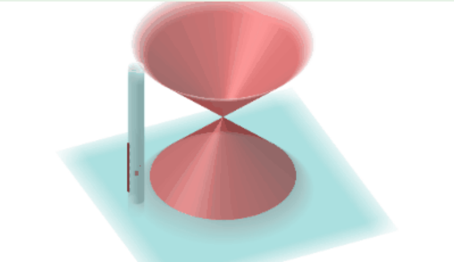

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are stalks of a **sheaf** $\mathbf{P_{U,f}}$ supported on critical points, with monodromy.

Result 2 - Matrix Factorizations

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Definition (Eisenbud-Orlov)

$$MF(U,f) := D^b_{Coh}(f^{-1}(0)) \underbrace{/}_{dg-quotient} Perf(f^{-1}(0))$$

Donaldson-Thomas and invariants of singularities

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$$\mu(f) \xleftarrow[\chi]{} P_{U,f} \xleftarrow[HP]{} MF(U, f)$$

Plan

1. Derived geometry
2. Donaldson-Thomas invariants
3. Results

Categorical Donaldson-Thomas Invariants

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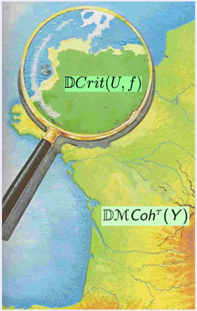

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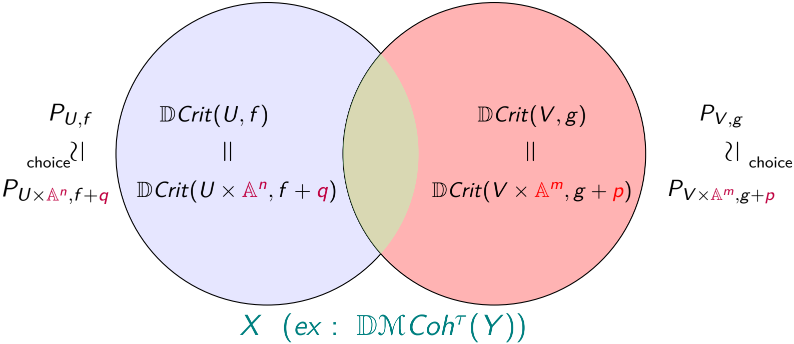

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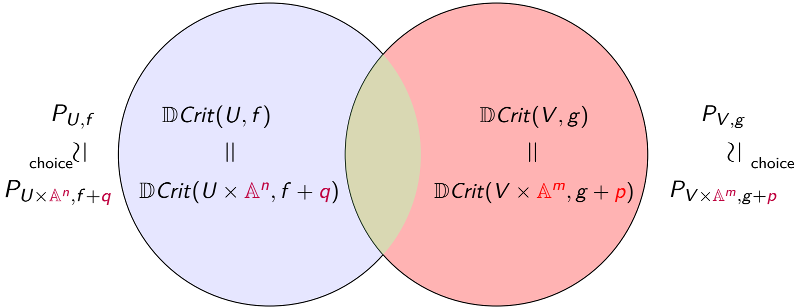


$X \quad (ex: \ \mathbb{D}\mathcal{M}Coh^{\tau}(Y))$

Theorem (Brav-Bussi-Dupont-Joyce-Szendroi 2015)

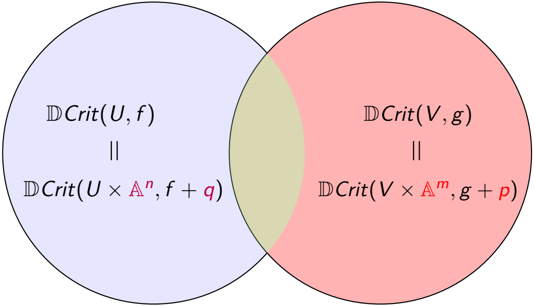
Assume there exists a square root of the canonical bundle of $\mathbb{D}\mathcal{M}Coh^{\tau}(Y)$. *Then the local ambiguities can be solved and the* $P_{U,f}$ *glued .*

Categorical Donaldson-Thomas Invariants

$MF(U, f)$

n mod $\mathbb{Z}/2$ $\simeq$

$MF(U \times \mathbb{A}^n, f + q)$



$MF(V, g)$

$\simeq$ m mod $\mathbb{Z}/2$

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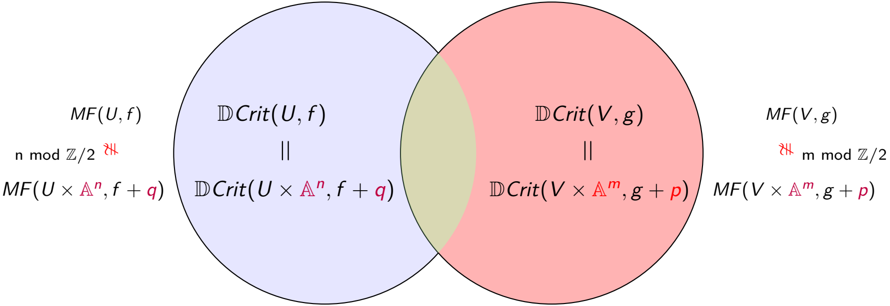
Categorical Donaldson-Thomas Invariants



Conjecture (Kontsevich-Soibelman, Toda): The local categories $MF(U,f)$ glue upon the prescription of an orientation data to be identified.

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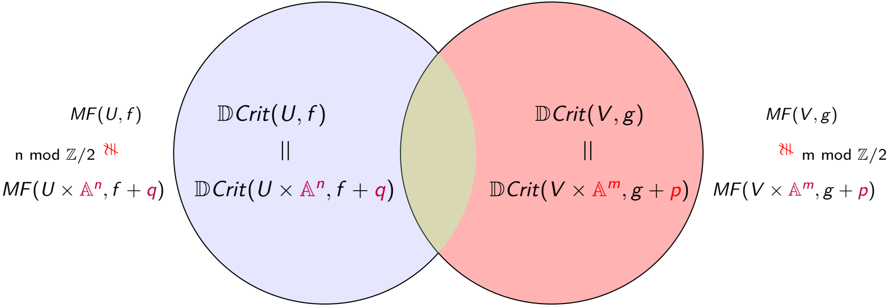

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Theorem (Hennion-Holstein-R., 2024)

The gluing is possible.

The orientation corresponds to the trivialization of three obstruction classes: $\alpha \in H^1(X,\mathbb{Z}/2)$, $\beta_{Joyce} \in H^2(X,\mathbb{Z}/2)$, $\gamma \in H^3(X,\mathbb{Z}/2)$

Categorical Donaldson-Thomas Invariants

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- γ is a delooping of the second Stiefel-Whitney class and corresponds to the gerbe classifying *Pin* structures.

Strategy:

Categorical Donaldson-Thomas Invariants

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$$\mathbf{Darb}_X/\mathbf{Quad} \xrightarrow{MF} \mathrm{dgCats}^{\nabla}, \qquad \mathbf{Darb}_X/\mathbf{Quad} \xrightarrow{P} \mathit{Perv}, \qquad \mathbf{Darb}_X/\mathbf{Quad} \xrightarrow{\mu} \mathbb{Z}$$

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Theorem (Hennion-Holstein-R., 2024)

$$(\mathbf{Darb}_X/\mathbf{Quad})/\text{isotopies} \simeq *$$

Thank you for your time.